Chromatic Ramsey Theory

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Let $\mathcal{G}$ be a countable graph which has infinite chromatic number. If $\gamma$ is a coloring of $[G]^k$ with two colors, is there then a subset $H \subseteq G$ such that $\gamma$ is constant on $[H]^k$ and $\mathcal{G}|_H$, the graph induced by $\mathcal{G}$ on $H$, has infinite chromatic number? As edges and non-edges can be colored with different colors this will be the case if $\mathcal{G}$ contains an infinite clique. It turns out that if the clique size of $\mathcal{G}$ is unbounded but $\mathcal{G}$ does not contain an infinite clique then for every coloring of $[G]^k$ with $\tau$ colors, there are some two of the $\tau$ colors such that there is an infinite chromatic subgraph of $\mathcal{G}$ the vertex set of which forms only pairs colored in those two colors; and this is best possible, because one can always distinguish between edges and non-edges. In the case in which the graphs do not contain the complete graph on $n$ vertices the situation is much more complicated. We will show that for every $3 \leq n \leq \omega$ there is a graph $\mathcal{G}$, which does not embed the complete graph on $n$ vertices, with the property that for every positive number $\tau$ there exists a coloring of $[G]^k$ with $\tau$ colors such that the vertex set of every infinite chromatic subgraph of $\mathcal{G}$ forms pairs in each of the $\tau$ colors. On the other hand there is a graph $\mathcal{G}$, which does not embed the complete graph on $n$ vertices, and which has the property that for every positive number $\tau$ and every coloring of $[G]^k$ with $\tau$ colors there is an infinite chromatic subgraph of $\mathcal{G}$ the pairs of which use at most 3 colors. We will generalize to the case of colorings of $k$-element subsets.

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1. Introduction

We consider a graph $\mathcal{G}$ to be a set of two-element sets. Hence our graphs do not contain multiple edges or loops. We will usually denote the set of vertices $\bigcup \mathcal{G}$ of $\mathcal{G}$ by $G$. If $H$ is a subset of $G$, then the induced subgraph $\mathcal{G}|_H$ of $\mathcal{G}$ is the graph $\mathcal{G}_H = \mathcal{G} \cap [H]^2$; where, as usual, $[H]^2$ denotes the set of $\tau$-element subsets of $H$. The family of graphs which do not contain a complete graph on $n$ vertices will be denoted by $\mathcal{N}_n$. For cardinals $\kappa$, $\rho$, $\sigma$ and $\nu$ we write

$$G \leftrightarrow_{\nu} (\kappa)^{\nu}$$

to mean that for every coloring $\gamma: [G]^\nu \rightarrow \rho$ there exists a subset $S \subseteq \rho$ with $|S| \geq \sigma$, a subset $H \subseteq G$ such that $\gamma([H]^\nu) \subseteq S$ and the chromatic number of the graph $\mathcal{G}|_H$ is at least $\kappa$. The negation of $G \leftrightarrow_{\nu} (\kappa)^{\nu}$ is $G \not\leftrightarrow_{\nu} (\kappa)^{\nu}$. In this paper we are interested in the relation $G \leftrightarrow_{\nu} (N_0^\kappa)^{\nu}$ for finite cardinals $\tau$, $s$ and $k$. For $c \geq d$ two positive numbers, we obtain

$$G \leftrightarrow_{\nu} (N_0^\kappa)^{\nu} \quad \text{implies} \quad G \leftrightarrow_{\nu+c} (N_0^\kappa)^{\nu+d}$$  \quad (1)

and

$$G \leftrightarrow_{\nu} (N_0^\kappa)^{\nu+d} \quad \text{implies} \quad G \leftrightarrow_{\nu+c} (N_0^\kappa)^{\nu+d}$$  \quad (2)

To see (1), just combine $d+1$ colors of a coloring with $\tau+d$ colors into one color to obtain a coloring with $\tau$ colors. Equation (2) follows from the observation that every coloring with $\tau$ colors can be interpreted to be a coloring with $\tau+d$ colors. If $\mathcal{G}$ contains an infinite complete subgraph $\mathcal{S}$ and $\gamma$ is a coloring of $[G]^k$ with $\tau$ colors, then apply the ordinary Ramsey Theorem [$9$] to the restriction of $\gamma$ to $[H]^\nu$, to deduce that $H$ contains an infinite subset $S$ such that $\gamma$ is the constant function on $[S]^\nu$. Hence we obtain the easy Theorem 1,
Theorem 1. If \( \emptyset \) contains an infinite clique and \( 1 \leq k, \tau \in \omega \), then \( \emptyset \leftrightarrow_2 (N_0)_k^2 \).

We will denote by \( p(k) = 2^{k-1} \) the number of ordered partitions of the number \( k \). That is, \( p(k) \) is equal to the number of sequences of positive integers of the form \((\tau_1, \tau_2, \ldots, \tau_w)\) such that \( \tau_1 + \tau_2 + \cdots + \tau_w = k \). We will prove the following.

Theorem 2. If, for every \( n \in \omega \), \( \emptyset \notin N_n \), then, for every \( \tau \in \omega \), \( \emptyset \leftrightarrow_{p(k)-1} (N_0)_k^2 \).

Theorem 3. If, for every \( n \in \omega \), \( \emptyset \notin N_n \) but \( \emptyset \) does not contain an infinite complete subgraph, then \( \emptyset \leftrightarrow_{p(k)-1} (N_0)_k^2 \).

Note that the above three theorems together with equations (1) and (2) provide a complete answer for graphs with unbounded clique sizes. The situation seems to be much more complicated in the case of the families \( N_n \) of graphs, as the following two theorems indicate.

Theorem 4. There exists a triangle-free graph \( \emptyset \) such that, for every \( 1 \leq \tau \in \omega \), \( \emptyset \leftrightarrow_2 (N_0)_\tau^2 \).

Theorem 5. There exists a triangle-free graph \( \emptyset \) such that, for every \( 2 \leq \tau \in \omega \), \( \emptyset \leftrightarrow_2 (N_0)_\tau^2 \).

Let \( \Pi_n \) be the homogeneous graph which embeds every finite member of \( N_n \). It is well known [5] that every countable graph in \( N_n \) can be embedded into \( \Pi_n \). Hence it follows from Theorem 4 that for \( 3 \leq n \) and \( 1 \leq \tau \in \omega \), \( \Pi_n \leftrightarrow_2 (N_0)_\tau^2 \). This is best possible, since we can show the following.

Theorem 6. For \( 3 \leq n \in \omega \) and \( 3 \leq \tau \in \omega \), \( \Pi_n \leftrightarrow_{p_2(k)-1} (N_0)_\tau^2 \).

The reader will note that in the case in which clique sizes are bounded, we do not do as well as the partition number, since \( p(2) = 2 < 3 \).

Let \((C_i; i \in \omega)\) be a sequence of pairwise disjoint finite sets such that, for all \( i \in \omega \), \((|C_i| < |C_{i+1}|)\). A stepgraph \( \mathcal{C} \) with partition \( C_0, C_1, C_2, \ldots \) is a graph such that \( C = \bigcup_{i < \omega} C_i \), there are no edges between different \( C_i \)'s and different \( C_i \)'s are pairwise disjoint. The stepgraph is said to be connected if each \( C_i \) is a connected component of \( \mathcal{C} \). The stepgraph \( \mathcal{C} \) is complete if each of the \( C_i \) induces a clique of \( \mathcal{C} \). For \( i \in \omega \) we will put \( P_i(\mathcal{C}) = C_i \). A refinement of the stepgraph \( \mathcal{C} \) is a subgraph of \( \mathcal{C} \) which is itself a stepgraph. For a finite subset \( K \) of \( C \), the number \( i \in \omega \) is significant if \( K \cap C_i \neq \emptyset \). Let \( i_1 < i_2 < \cdots < i_w \) be the significant numbers for \( K \). Then the ordered partition \((\mid K \cap C_{i_1}, K \cap C_{i_2}, \ldots, K \cap C_{i_w})\) of \( \mid K \) is the partition type of \( K \), and the partition of \( K \) induced by \( \mathcal{C} \) is \( P_1(K) = K \cap C_{i_1}, P_2(K) = K \cap C_{i_2}, \ldots, P_w(K) = K \cap C_{i_w} \). If \( \tau = (s_1, s_2, \ldots, s_w) \) is an ordered partition of the number \( k \geq 1 \), we will denote by \( \mathcal{T}_\tau(\mathcal{C}) \) the set of all subsets of \( C \) the partition type of which is \( \tau \). If \( \gamma: \mathcal{T}_\tau(\mathcal{C}) \rightarrow \tau \) is a coloring of the elements of \( \mathcal{T}_\tau(\mathcal{C}) \) with \( \tau \) colors, then we will say the following.

The coloring \( \gamma \) is uniform if any two subsets \( K \) and \( L \) in \( \mathcal{T}_\tau(\mathcal{C}) \) with \( P_i(K) = P_i(L) \) for all \( 1 \leq i \leq w-1 \) have the same color.

The coloring \( \gamma \) is uniform up to \( n \), for \( n \in \omega \), if any two subsets \( K \) and \( L \) in \( \mathcal{T}_\tau(\mathcal{C}) \) for which \( P_i(K) \cup P_i(L) \subseteq \bigcup_{j>n} C_j \) and which have the property that \( P_i(K) = P_i(L) \subseteq \bigcup_{i<n} C_i \), for all \( 1 \leq i \leq w-1 \), have the same color.

For \( \tau, k, t \geq 1 \), we will denote by \( R(\tau, k, t) \) the ordinary Ramsey number. That is, \( R(\tau, k, t) \) is the smallest number such that for every set \( S \) with \( |S| \geq R(\tau, k, t) \) and
every coloring $\gamma: [S]^k \rightarrow \tau$, there is a subset $T \subseteq S$ with $|T| \geq t$ such that $\gamma$ is constant on $[T]^k$ (see [9]).

2. The Results for Graphs with Unbounded CLIQUE Sizes

**Lemma 1.** Let $C$ be a stepgraph with partition $(C_0, C_1, C_2, \ldots)$, $\tau = (\tau_1, \tau_2, \ldots, \tau_n)$ an ordered partition of the number $k$ and $\gamma: \mathcal{T}_d(C) \rightarrow \tau$ a coloring of the $k$-element subsets of $C$, which have partition type $\tau$ with $\tau_1 \geq 1$ colors. Then there is, for every $n \in \omega$, a refinement $\mathcal{D}$ of $\mathcal{C}$ with $P(\mathcal{D}) = C_i$ for all $i < n$ and such that the restriction of $\gamma$ from $\mathcal{T}_d(\mathcal{C})$ to $\mathcal{T}_d(\mathcal{D})$ is uniform up to $n$. Furthermore, if $w = 1$ and $n = 0$, then the restriction of $\gamma$ from $\mathcal{T}_d(\mathcal{C})$ to $\mathcal{T}_d(\mathcal{D})$ is the constant function.

**Proof.** For $w \geq 2$, let the ordered partition $\sigma$ of $k - \tau_n$ be given by $\sigma = (\tau_1, \tau_2, \ldots, \tau_{n-1})$ and the set $S$ by

$$S = \mathcal{T}_d(\mathcal{C}) \cap \left[ \bigcup_{i=n}^k C_i \right]^{k - \tau_n}.$$ 

We define the number $c$ to be

$$c = \begin{cases} \tau_1, & \text{if } w = 1; \\ |\tau|, & \text{otherwise.} \end{cases}$$

Note that $S$ is the set of all $(k - \tau_n)$-element subsets of $\bigcup_{i=n}^k C_i$ which have partition type $\sigma$. Hence $|\tau|$ is the number of colorings with $\tau$ colors of the set of all $(k - \tau_n)$-element subsets of $\bigcup_{i=n}^k C_i$, which have partition type $\sigma$. Let $j \geq n$ and $T \subseteq C_j$, with $|T| = \tau_n$. Note that if $S \in S$ then $S \cup T \in \mathcal{T}_d(\mathcal{C})$. We can therefore define a coloring $\gamma_T$ of $S$, by putting $\gamma_T(S) = \gamma(S \cup T)$ for $S \in S$. Then $\gamma_T \in [S]^\tau$. The association of $T$ with $\gamma_T$ is a coloring $\Gamma$ of $\bigcup_{i=n}^k [C_i]^{\tau_n}$ with $|\Gamma^*| = c$ colors.

Choose a strictly increasing sequence of numbers $(s_i; i \in \omega)$ such that, for $i < n$, $s_i = |C_i|$. There exists then a sequence $t_i$ such that, for $i < n$, $t_i = i$ and, for all $i \geq n$, $|C_i| \geq R(c, w, s_i)$. For $i \geq n$, using the definition of the number $R(c, w, s_i)$, let $B_i$ be a subset, $B_i \subseteq C_i$, such that $\Gamma$ is constant on $|B_i|^\tau$. Put $B_i = C_i$ for $i < n$ and $\mathcal{B} = \mathcal{C} \left[ \bigcup_{i=n}^\omega B_i \right]$. Because the coloring $\Gamma$ is finite, there is a refinement $\mathcal{D}$ of $\mathcal{B}$ such that $\Gamma$ is constant on $\bigcup_{i=n}^\omega |P(\mathcal{D})|^\tau$. Clearly, $\mathcal{D}$ is the desired refinement of $\mathcal{C}$ for which the restriction of $\gamma$ from $\mathcal{T}_d(\mathcal{C})$ to $\mathcal{T}_d(\mathcal{D})$ is uniform up to $n$.

**Lemma 2.** Let $C$ be a stepgraph, $\tau$ an ordered partition of the number $k$ and $\gamma: \mathcal{T}_d(\mathcal{C}) \rightarrow \tau$ a coloring of the k-element subsets of $C$ which have partition type $\tau$ with $\tau_1 \geq 1$ colors. Then there is a refinement $\mathcal{D}$ of $\mathcal{C}$ such that the restriction of $\gamma$ from $\mathcal{T}_d(\mathcal{C})$ to $\mathcal{T}_d(\mathcal{D})$ is uniform.

**Proof.** Let $\mathcal{C}_0 = \mathcal{C}$ and, for $0 < n \in \omega$, let $\mathcal{C}_n$ be a refinement of $\mathcal{C}_{n-1}$ such that the restriction of $\gamma$ from $\mathcal{T}_d(\mathcal{C})$ to $\mathcal{T}_d(\mathcal{C}_n)$ is uniform up to $n$, and such that, for all $i < n$, $P_i(\mathcal{C}_n) = P_i(\mathcal{C}_{n-1})$. Note that, for all $n, m \in \omega$ and $i \leq n$, $P_i(\mathcal{C}_n) = P_i(\mathcal{C}_{n+m})$. Therefore,

$$D = \bigcap_{i \in \omega} C_i \neq \emptyset,$$

and $\mathcal{D} = \mathcal{C}|_D$ is a stepgraph, and hence a refinement of $\mathcal{C}$. Also, of course, the restriction of $\gamma$ from $\mathcal{T}_d(\mathcal{C})$ to $\mathcal{T}_d(\mathcal{D})$ is uniform.

**Lemma 3.** Let $C$ be a stepgraph, $\tau = (\tau_1, \tau_2, \ldots, \tau_n)$ an ordered partition of the
number \( k \) and \( \gamma: \mathcal{G}(\tau) \to \tau \) a coloring of the \( k \)-element subsets of \( C \), which have partition type \( \tau \), with \( \tau \geq 1 \) colors. Then there is a refinement \( \mathcal{D} \) of \( \mathcal{G} \) such that the restriction of \( \gamma \) from \( \mathcal{G}(\mathcal{D}) \) to \( \mathcal{G}(\mathcal{D})' \) is the constant function.

**Proof.** We proceed by induction on \( w \). For \( w = 1 \) we use Lemma 1 with \( n = 0 \). It follows from Lemma 2 that we may assume that the coloring \( \gamma \) is uniform on \( \mathcal{G}(\mathcal{D}) \). Let \( \sigma \) be the ordered partition of \( k - \tau_w \) given by \( \sigma = (\tau_1, \tau_2, \ldots, \tau_{w-1}) \). Note, then, that if \( S \in \mathcal{G}(\mathcal{D}) \) and \( R \) and \( T \) are any two subsets of \( C \) such that \( S \subseteq R \subseteq \mathcal{G}(\mathcal{D}) \) and \( S \cup T \subseteq \mathcal{G}(\mathcal{D}) \), then \( \gamma(S \cup R) = \gamma(S \cup T) \). Hence we can associate with any such \( S \in \mathcal{G}(\mathcal{D}) \) the color \( \gamma^*(S) = \gamma(S \cup T) \) for any \( T \) such that \( S \cup T \subseteq \mathcal{G}(\mathcal{D}) \). Using induction, let \( \mathcal{D} \) be a refinement of \( \mathcal{G} \) such that \( \gamma^* \) is constant. Clearly, then, the restriction of \( \gamma \) from \( \mathcal{G}(\mathcal{D}) \) to \( \mathcal{G}(\mathcal{D})' \) is constant. \( \square \)

**Lemma 4.** If \( \mathcal{D} \) is a complete stepgraph and \( 0 < k \in \omega \), then, for all \( 1 \leq \tau \in \omega \),

\[
\mathcal{D} \leftrightarrow \mathcal{D}(\mathcal{D})^k \cap \tau.
\]

**Proof.** The proof follows easily from Lemma 3. \( \square \)

**Lemma 5.** Every graph \( \mathcal{G} = (G, E) \) in which the clique sizes are unbounded but which does not contain an infinite complete subgraph contains a complete stepgraph as an induced subgraph.

**Proof.** There exists a sequence of pairwise disjoint subsets \( (C_i; i \in \omega) \) such that the sequence \( (|C_i|; i \in \omega) \) is strictly increasing, and for each \( i \in \omega \) the graph \( \mathcal{G}
\] is complete. This follows from the fact that every finite such sequence can be extended.

Let \( \mathcal{D} \) be the stepgraph such that \( \mathcal{D}(\mathcal{D}) = C_i \). Let \( \tau \) be the ordered partition \( (1, 1) \) of \( 2 \) and \( \gamma \) the coloring of \( \mathcal{G}(\mathcal{D}) \) given by

\[
\gamma(K) = \begin{cases} 0, & \text{if } K \text{ is not an edge of } \mathcal{G}; \\ 1, & \text{otherwise.} \end{cases}
\]

According to Lemma 3, there is a refinement \( \mathcal{D} \) of \( \mathcal{G} \) such that \( \gamma \) is constant on \( \mathcal{G}(\mathcal{D})' \). If \( \gamma \) would be the function constant to \( 1 \), then \( \mathcal{G} \) would contain an infinite complete subgraph. Hence \( \gamma \) is constantly \( 0 \), which implies that \( \mathcal{D} \) is an induced subgraph of \( \mathcal{G} \). \( \square \)

**Theorem 2.** If, for every \( n \in \omega \), \( \mathcal{G} \notin \mathcal{N}_n \), then, for every \( \tau \in \omega \),

\[
\mathcal{G} \leftrightarrow \mathcal{D}(\mathcal{D})^k \cap \tau.
\]

**Proof.** If \( \mathcal{G} \) contains an infinite complete subgraph, the theorem follows from Theorem 1. If \( \mathcal{G} \) does not contain an infinite complete subgraph, then we deduce Theorem 2 from Lemmas 5 and 4. \( \square \)

**Lemma 6.** For every \( 1 \leq n \in \omega \), a graph in \( \mathcal{N}_n \) with infinite chromatic number contains an infinite induced path or it contains an induced stepgraph with infinite chromatic number.

**Proof.** The lemma is obviously true for \( n = 0 \). We proceed by induction on \( n \). Let \( \mathcal{G} \in \mathcal{N}_{n+1} \) be a \( K_{n+1} \)-free graph. If \( \mathcal{G} \) has infinite chromatic number then either \( \mathcal{G} \) contains a connected stepgraph with infinite chromatic number as induced subgraph or
one of the connected components of \( \Phi \) has infinite chromatic number. Hence we may assume that \( \Phi \) is connected. We will either find an infinite path or a connected stepgraph with infinite chromatic number as induced subgraph of \( \Phi \), or we will be able to construct a sequence \( ((x_l, \Theta_i) ; l \in \omega) \) such that, for each \( l \in \omega \):

(i) \( \Theta_i \) restricted to \( \{x_0, x_1, \ldots, x_l\} \) is an induced path of \( \Theta_i \);

(ii) \( x_l \) is adjacent to \( x_{l+1} \);

(iii) \( \Theta_i \cup \{x_l\} \) is a connected infinite chromatic subgraph of \( \Theta_i \); and

(iv) for all \( l < i \), \( x_i \) is not adjacent to any vertex in \( \Theta_i \).

Let \( x_0 \) be some vertex of \( \Theta \) and put \( \Theta_0 = \emptyset \). Assume that \( l \in \omega \) and, for all \( i < l \), \((x_i, \Theta_i) \) has already been chosen. Let \( N \) be the set of those vertices of \( \Theta_i \) which are adjacent to \( x_l \). The graph \( \Theta_i | N \) is in \( N_n \), for otherwise \( x_l \) would be contained in a \( K_{n+1} \). Using induction, \( \Theta_i | N \) either contains an infinite path or a connected stepgraph with infinite chromatic number or has finite chromatic number. If it has finite chromatic number, the graph \( \Theta_i | N \) has finite chromatic number. If \( \Theta_i | N \) does not contain a connected stepgraph with infinite chromatic number, then one of the connected components of \( \Theta_i | N \) must have infinite chromatic number. Let \( \Sigma \) be a connected component of \( \Theta_i | N \), which has infinite chromatic number. Because \( \Theta \) is connected, one of the vertices in \( N \), say \( x_{l+1} \), must be adjacent to some vertex in \( \Sigma \). Let \( \Theta_{l+1} \) be the subgraph of \( \Theta_l \) induced by the vertices of \( \Sigma \). It is now easy to check that the sequence \( ((x_l, \Theta_l); i \leq l + 1) \) satisfies the four conditions listed above.

\( \square \)

**Theorem 3.** If for every \( n \in \omega \), \( \Theta \notin N_n \) but \( \Theta \) does not contain an infinite complete subgraph, then \( \Theta \leq (\prod_{k=1}^{n} (N_n)^{k}) \).

**Proof.** Fix a bijection \( \alpha \) from \( G \) to \( \omega \). If \( A \) and \( B \) are two induced subgraphs of \( \Theta \), we write \( A \prec B \) if for every vertex \( a \in A \) and vertex \( b \in B \), \( \alpha(a) < \alpha(b) \). Let \( S \) be a \( k \)-element subset of \( G \). The set \( S \) is a partitioned subset of \( G \) if the connected components of \( \Theta | S \) are totally ordered by \( \prec \). If \( P_1(S) < P_2(S) < \cdots < P_n(S) \) are the connected components of the partitioned subset \( S \) of \( G \), then we associate with \( S \) the ordered partition \( \tau = (P_1(S), P_2(S), \ldots, P_n(S)) \). If \( S \) is not a partitioned subset of \( G \) we associate arbitrarily some ordered partition of \( k \) with \( S \). This provides a coloring \( \gamma \) of all \( k \)-element subsets of \( G \) with the ordered partitions of \( k \).

Next we prove that if \( \Sigma \) is an induced subgraph of \( \Theta \) which has infinite chromatic number, then the restriction of \( \gamma \) to \( [H]^{\omega} \) is a surjection onto all ordered partitions of \( k \). (This of course will imply the theorem.)

Assume that \( \Sigma \) contains a connected stepgraph \( \mathcal{C} \) as an induced subgraph. Let \( \tau = (\tau_1, \tau_2, \ldots, \tau_v) \) be an ordered partition of \( k \). Choose an index \( i \), such that \( |P_i(\mathcal{C})| > \tau_i \), and a subset \( S_i \subseteq P_i(\mathcal{C}) \) such that \( |S_i| = \tau_i \) and \( \mathcal{C} | S_i \) is connected. Assume that \( S_1, S_2, \ldots, S_v \) have already been chosen, so that, for all \( 1 \leq i \leq v \), \( |S_i| = \tau_i \), \( \mathcal{C} | S_i \) is connected and, for all \( 1 \leq i < v \), \( S_i < S_{i+1} \). Because there are only finitely many numbers which are not larger than all of the numbers \( \alpha(x) \) with \( x \in \bigcup_{1 \leq i \leq v} S_i \), there is an index \( j \) such that

\[
\bigcup_{1 \leq i \leq v} S_i < \bigcup_{j = i \omega} P_i(\mathcal{C}) \quad \text{and} \quad |P_j(\mathcal{C})| > \tau_{v+1}.
\]

We choose \( S_{v+1} \subseteq P_j(\mathcal{C}) \) such that \( |S_{v+1}| = \tau_{v+1} \) and \( \mathcal{C} | S_{v+1} \) is connected. This means that the sequence \( (S_1, S_2, \ldots, S_v) \) can be extended, and hence there exists, for every ordered partition \( \tau \) of \( k \), a \( k \)-element subset \( K \subseteq C \) with \( \gamma(K) = \tau \). We conclude that if \( \Sigma \) contains a stepgraph, then \( \gamma \) restricted to \( [H]^{\omega} \) is onto, and hence we are done.

It now follows from Lemma 5 that we may assume that the clique sizes of \( \Sigma \) are bounded and from Lemma 6 that we may assume that \( \Sigma \) contains an infinite induced
path $\mathfrak{m}$. Let $\tau = (\tau_1, \tau_2, \ldots, \tau_n)$ be an ordered partition of $k$. We wish to construct a partitioned subset $K$ of $W$ such that $\gamma(K) = \tau$ by successively constructing $P(K) < P_t(K)$ and so on. Choose for $P_t(K)$ any connected induced subgraph of $\mathfrak{m}$ containing $\tau_i$ vertices. Assume that, for some $v < w$ and all $i \leq v$, the sets $P_i(K)$ have already been found. The function $\alpha$ takes on a maximal finite value $m$ on the sets $P_i(K)$ and hence there are only finitely many vertices $x$ in $\mathfrak{m}$ such that $\alpha(x) = m$. That implies that we can find a subset $P_{v+1}(K)$ having the required properties.

3. The Results for Graphs with Bounded Clique Sizes

Here we have to use a result from finite Ramsey theory. First, we present some notation. The set of all induced subgraphs of a graph $\mathcal{G}$ which are isomorphic to the graph $\mathfrak{m}$ is denoted by $\left(\mathcal{G}\right)^\mathfrak{m}_\tau$.

\[
\mathcal{G} \mapsto (\mathcal{G})^\mathfrak{m}_\tau
\]

means that, for every coloring $\gamma: \left(\mathcal{G}\right)^\mathfrak{m}_\tau \rightarrow \tau$, there is an induced subgraph $\mathcal{G}'$ isomorphic to $\mathcal{G}$ of $\mathcal{G}$ such that $\gamma$ is constant on $\left(\mathcal{G}\right)^\mathfrak{m}_\tau$. An ordered graph $(\mathfrak{m}, \leq)$ is a graph together with a total order on $G$. Isomorphisms and embeddings of ordered graphs must also respect the order. The set of all induced subgraphs of an ordered graph $(\mathfrak{m}, \leq)$ which are isomorphic to the ordered graph $(\mathfrak{m}, \leq)$ is denoted by $\left(\mathfrak{m}, \leq\right)^{(\mathfrak{m}, \leq)}$. The expression

\[
(\mathfrak{m}, \leq) \mapsto (\mathcal{G})^{(\mathfrak{m}, \leq)}_\tau
\]

means that, for every coloring $\gamma: \left(\mathfrak{m}, \leq\right)^{(\mathfrak{m}, \leq)} \rightarrow \tau$, there is an induced subgraph $\mathcal{G}'$ isomorphic to $(\mathfrak{m}, \leq)$ of $\left(\mathfrak{m}, \leq\right)^{(\mathfrak{m}, \leq)}$ such that $\gamma$ is constant on $\left(\mathfrak{m}, \leq\right)^{(\mathfrak{m}, \leq)}$. The following theorem is due to Nešetřil and Rödl [7,8] and independently, Abramson and Harrington (1).

**Theorem.** Given $\tau \in \omega$ and finite ordered graphs $(\mathcal{G}, \leq)$ and $(\mathfrak{m}, \leq)$ such that $\mathcal{G}, \mathfrak{m} \in \mathcal{N}_\omega$, there exists a finite ordered graph $(\mathfrak{m}, \leq)$, with $\mathfrak{m} \in \mathcal{N}_\omega$, such that

\[
(\mathfrak{m}, \leq) \mapsto (\mathcal{G})^{(\mathfrak{m}, \leq)}_\tau
\]

Note that this theorem implies the following, as has been observed by the above authors.

**Statement.** If the group of automorphisms of $\mathfrak{m}$ is the full symmetric group on $K$, that is if $\mathfrak{m}$ is either complete or does not contain any edges and

\[
(\mathfrak{m}, \leq) \mapsto (\mathcal{G})^{(\mathfrak{m}, \leq)}_\tau
\]

then

\[
\mathfrak{m} \mapsto (\mathcal{G})^{\mathfrak{m}}_\tau.
\]

(3)

In particular, (3) holds if $\mathfrak{m}$ is a single edge or consists of two non adjacent vertices. If $\mathfrak{m} \in \mathcal{N}_\omega$ we can also require that $\mathfrak{m} \in \mathcal{N}_\omega$.

The symbol

\[
\mathfrak{m} \mapsto (\mathcal{G})^{\mathfrak{m}}_{\tau, \nu}
\]

means that, for every $\gamma: \mathfrak{m} \rightarrow \kappa$ and for every $\delta: ([G]^2 - \mathfrak{m}) \rightarrow \nu$, there is an induced subgraph $\mathcal{G}'$ isomorphic to $\mathcal{G}$ such that $\gamma$ is constant on $\mathcal{G}'$ and $\nu$ is constant on $\mathcal{G}'$ (the edge set of the complement of $\mathcal{G}'$). We need the following result:

\[
\forall \mathfrak{m} \in \omega \forall \tau \in \omega \forall \mathcal{G}((\mathfrak{m})^\mathfrak{m} \leq \omega \land \mathcal{G} \in \mathcal{N}_\omega) \Rightarrow \exists \mathcal{G}(\mathfrak{m} \in \mathcal{N}_\omega \land [\mathfrak{m}] < \omega \land \mathcal{G} \mapsto (\mathcal{G})^{(\mathfrak{m}, \leq)}_{\tau, \nu}).
\]

(4)
Chromatic Ramsey theory

It is easy to see that (4) follows from statement * by ‘Ramseying twice’: that is, by first finding a graph $\mathcal{G} \in \mathcal{N}_n$ such that $\mathcal{G} \to (\mathcal{G})^2_2$, where $\mathcal{G}$ is the graph on two vertices without an edge, and then by finding a graph $\mathcal{G} \in \mathcal{N}_n$ such that $\mathcal{G} \to (\mathcal{G})^5_2$, where $\mathcal{G}$ is the graph on two adjacent vertices. Clearly, $\mathcal{G} \in \mathcal{N}_n$ and $\mathcal{G} \to (\mathcal{G})^5_2$. Another interesting way of proving (4) has recently been found by Shelah in [6]. Komjáth and Shelah prove a consistency result which was then generalized by Shelah at the Banff conference on ‘Finite and Infinite Combinatorics in Sets and Logic’, in April 1991 to:

The following is consistent for every $\lambda \geq \beta$:

$$\forall x \forall y \forall z \exists \mathcal{G} \exists \mathcal{H} \mathcal{G} \to (\mathcal{H})^2_2$$

(5)

Using compactness and absoluteness, one obtains (4) from (5) (Shelah).

**Lemma 7.** If $\mathcal{G}$ is a finite graph $\mathcal{G} \leftrightarrow (sk)^{\gamma-1(i)}_{\gamma-1(i)}$ and $\gamma$ is a coloring of $G$ with $l$ colors, then there exists an $i < l$ such that $\mathcal{G}|_{\gamma-1(i)} \leftrightarrow (k)^{\gamma}_{\gamma}$.

**Proof.** Otherwise,

$$\forall i < l, \quad \mathcal{G}|_{\gamma-1(i)} \not\leftrightarrow (k)^{\gamma}_{\gamma}$$

(6)

For $i < l$, fix a coloring $\delta_i$ witnessing (6). We may assume that different color sets are used for distinct values of $i$. We define a coloring $\delta$ of $[G]^2$ as follows. If, for $i < \tau$, $e \in \gamma^{-1}(i)$, then $\delta(s) = \delta_i(e)$. For $i \neq j$, $x \in \gamma^{-1}(i)$ and $y \in \gamma^{-1}(j)$, $\delta\{x, y\} = \{i, j\}$. If $T \subseteq G$ and $|\delta([T]^2)| = s$, then there are at most $s$ different numbers $i$ such that $T \cap \gamma^{-1} \neq \emptyset$. But then it follows from (6) that the chromatic number of $\mathcal{G}|_{\tau}$ is less than $sk$. Hence $\delta$ is a witness to

$$\mathcal{G} \not\leftrightarrow (sk)^{\gamma-1(i)}_{\gamma-1(i)}$$

and we have arrived at a contradiction.

We will use Lemma 7 in the following form.

**Corollary 1.** For every three integers $2 \leq s, k, l \in \omega$ there is a number $N = N(s, k, l)$ such that whenever $\mathcal{G}$ is a graph with $\mathcal{G} \leftrightarrow (N)^{\gamma}_{\gamma}$ and $\gamma$ is a function from $G$ into $l$, then, for some $i \in l$,

$$\mathcal{G}|_{\gamma-1(i)} \leftrightarrow (k)^{\gamma}_{\gamma}$$

Let $(C_i : i \in \omega)$ be a sequence of pairwise disjoint finite sets. An $s$-chromatic stepgraph $\mathcal{G} = (C, E)$ with partition $C_0, C_1, C_2, \ldots$ is a graph such that $C = \bigcup_{i \in \omega} C_i$ and, for all $i \in \omega$, $(\mathcal{G}|_{C_i}) \leftrightarrow (i)^{\gamma}_{\gamma}$. We define the partition type and other notions as for ordinary stepgraphs. A refinement of an $s$-chromatic stepgraph $\mathcal{G}$ is an induced subgraph of $\mathcal{G}$ which is also an $s$-chromatic stepgraph. We obtain the following immediately from Corollary 1.

**Lemma 8.** If $\gamma$ is a coloring of the vertices of an $s$-chromatic stepgraph $\mathcal{G}$ with finitely many colors, then there is a refinement $\mathcal{D}$ of $\mathcal{G}$ such that $\gamma$ is constant on the vertices of $\mathcal{D}$.

**Lemma 9.** Let $\mathcal{G} = (C, E)$ be an $s$-chromatic stepgraph with partition $(C_0, C_1, C_2, \ldots)$, $\tau$ the ordered partition $(1, 1)$ of 2 and $\gamma : \mathcal{F}(\mathcal{G}) \to \tau$ a coloring of the 2-element subsets of $C$ which have partition type $\tau$ with $\tau \geq 1$ colors. Then, for every
n ∈ ω, there is a refinement $\mathcal{D}$ of $G$ with $P(\mathcal{D}) = C_i$ for $i < n$ and such that the restriction of $\gamma$ from $\mathcal{T}_i(G)$ to $\mathcal{T}_i(\mathcal{D})$ is uniform up to $n$.

**Proof.** We define the number $c$ to be
\[ c = |{\tau}|_{i<n} C_i. \]
Note that $|{\tau}|_{i<n} C_i$ is the number of colorings of the set of all elements of $S = \bigcup_{i<n} C_i$, with $\tau$ colors. Let $j > n$ and $x \in C_j$. Note that if $\gamma(i, x)$. Then $\gamma_i \in \tau$. Now consider the association of $x$ with $\gamma_i$ to be coloring $\Gamma$ of $\bigcup_{i<n} C_i$ with $c$ colors.

There exists a sequence $t_i$ such that
\[ \forall i < n, t_i = i \land \forall i \geq n, \quad \mathcal{C} ig| C_i \leftrightarrow (N(s, i, c))_{\mathcal{T}_i(s, i, c)}. \]
By the definition of the number $N(s, i, c)$, for each $i \geq n$, there is a subset $B_i \subseteq C_i$ such that $\Gamma$ is constant on $[B_i]^\times$. Put $B_i = C_i$ for $i < n$ and $\mathcal{B} = \mathcal{C} \cup_{i<n} B_i$. Because the coloring $\Gamma$ is finite, there is a refinement $\mathcal{D}$ of $\mathcal{B}$ such that $\Gamma$ is constant on $\bigcup_{i<n} P_i(\mathcal{D})$. Clearly, $\mathcal{D}$ is the desired refinement of $G$ for which the restriction of $\gamma$ from $\mathcal{T}_i(G)$ to $\mathcal{T}_i(\mathcal{D})$ is uniform up to $n$. \hfill $\square$

**Lemma 10.** If $G$ is an s-chromatic stepgraph and $\gamma : \mathcal{T}_{(1,1)}(G) \to \tau$ is a coloring of the pairs that connect different parts of $G$ with $\tau$ colors, then there is a refinement $\mathcal{D}$ of $G$ such that $\gamma$ is constant on $\mathcal{T}_{(1,1)}(\mathcal{D})$.

**Proof.** Using Lemma 9 repeatedly, we can construct as in the proof of Lemma 2 a refinement $\mathcal{B}$ of $G$ such that $\gamma$ is $n$-uniform on $\mathcal{B}$ for every $n \in \omega$, which of course means that $\gamma$ is uniform on $\mathcal{B}$. We obtain that there exists a function $\rho : B \to \tau$ such that
\[ \forall i \in \omega \quad \forall x \in P_i, \forall i < j \in \omega \quad \forall y \in P_j(\mathcal{B}), \quad \gamma([i, x]) = \rho(y). \]
According to Lemma 9, there is a restriction $\mathcal{D}$ of $\mathcal{B}$ on which the function $\rho$ is constant. Clearly, then, $\gamma$ is constant on $\mathcal{T}_{(1,1)}(\mathcal{D})$. \hfill $\square$

**Lemma 11.** If $G$ is an s-chromatic stepgraph, then, for all $2 \leq \tau \in \omega$, $\mathcal{C} \leftrightarrow_{\tau+1} (N_0)^	au$.

**Proof.** If $\gamma : [C]^2 \to \tau$ is a coloring of the two-element subsets of $C$, there exists, according to Lemma 10, a refinement $\mathcal{B}$ of $G$ such that $\gamma$ is constant on $\mathcal{T}_{(1,1)}(\mathcal{B})$. Because $\mathcal{C} \big| C_i \leftrightarrow (i)_i^2$, there is an infinite chromatic subgraph $\mathcal{D}$ of $\mathcal{B}$ such that $|\gamma([S]^2)| = s + 1$. Hence, for every $1 \leq \tau \in \omega$, $\mathcal{B} \leftrightarrow_{\tau+1} (N_0)^	au$. \hfill $\square$

**Theorem 4.** There exists a triangle-free graph $\mathcal{B}$ such that, for each $1 \leq \tau \in \omega$, $\mathcal{B} \leftrightarrow_{\tau} (N_0)^	au$.

**Proof.** Let $(\mathcal{D}_i : i \in \omega)$ be a family of finite graphs such that the chromatic number of $\mathcal{D}_i$ is larger than $i$. We use (4) to obtain a family $(\mathcal{B}_i : i \in \omega)$ of finite graphs such that $\mathcal{E}_i \leftrightarrow (\mathcal{D}_i)_{i}^2$. Note that, for each $i \in \omega$, $\mathcal{E}_i \leftrightarrow (i)_i^2$ and hence it is obvious that the graph $G$ with $P_i(G) = C_i$ is a 2-chromatic stepgraph. It now follows from Lemma 11 that, for every $1 \leq \tau \in \omega$, $\mathcal{B}_i \leftrightarrow_{\tau+1} (N_0)^	au$. \hfill $\square$

**Theorem 5.** There exists a triangle-free graph $\mathcal{B}$ such that, for every $2 \leq \tau \in \omega$, $\mathcal{B} \leftrightarrow_{\tau} (N_0)^	au$.

**Proof.** Let $\mathcal{B}$ be a stepgraph such that, for every $i \in \omega$, the part $\mathcal{B}_i$ has chromatic number at least $i$ and girth at least $i$. Such graphs exist, by [8]. Denote by $d(x, y)$ the
distance between the vertices \(x\) and \(y\) in \(\mathcal{G}\). Color the elements of \([G]^2\) with \(\tau\) colors according to distance modulo \(\tau\); that is,

\[
\forall x \neq y \land x, y \in G, \quad \gamma(x, y) = d(x, y) \mod \tau
\]

Every infinite chromatic subgraph of \(\mathcal{G}\) must contain subgraphs of large chromatic numbers of infinitely many of the graphs \((\mathcal{G})\), and hence pairs in all of the \(\tau\) colors. \(\square\)

**Theorem 6.** For \(\exists \leq n \in \omega\) and \(\exists \leq \tau \in \omega\), \(\Pi_n \not\rightarrow (\aleph_0)^\tau\).

**Proof.** Let \(\Pi_n\) be the homogeneous universal \(K_n\)-free graph and suppose that < well orders \(|U_n|\) in order type \(\omega\). We fix a coloring \(\gamma\) of \([U_n]^2\) as follows: edges are colored black; if \(\{a, b\} \not\in \Pi_n\) and \(a < b\), then \(\{a, b\}\) is colored red just in case there is an \(x < a\) with \(\{x, b\} \in \Pi_n\); otherwise \(\{a, b\}\) is colored blue.

We prove that any infinite chromatic subgraph \(\mathcal{H}\) of \(\mathcal{G}\) must use all three colors. Note that this argument also proves that the complement of \(\Pi_n\) is not weakly edge indivisible.

Specifically, we prove by induction on \(m \leq n\) that if \(\mathcal{H} \subseteq \Pi_n\) is homogeneous in two colors and \(K_m\)-free, then \(\chi(\mathcal{H})\) is finite. It is clear that unless one of the colors used is black, the chromatic number of a homogeneous subgraph is one, as the graph is edge-free. Let \(\mathcal{H}\) be \(K_m\)-free and homogeneous in two colors. If \(m < 2\), \(\mathcal{H}\) is edge-free, and colorable with one color. We are led to two cases.

**Case 1:** \(\mathcal{H}\) is homogeneous with black edges and red non-edges. Let \(h_0\) be the least element of \(\mathcal{H}\). Partition \(\mathcal{H}\) into finitely many classes: \(X_i = \{h_0\}\) and, for \(h_0\) be the least element of \(\mathcal{H}\). Partition \(\mathcal{H}\) into finitely many classes: \(X_i = \{h_0\}\) and, for \(0 \leq i \leq h_0\), \(X_i = \{h \in \mathcal{H}\) such that \(i = \min \{x \in \Pi_n : (x, h) \in \Pi_n\}\). This is a partition of \(\mathcal{H}\), since either \(h_0 = h, h_0 < h \in \{h_0, h\} \in \Pi_n\) or \(h_0 < h\) and \(\gamma(\{h_0, h\}) = \text{red}\), so \(\exists i < h_0\). Now each \(X_i\) induces a subgraph of \(\Pi_n\), which is \(K_{\Pi_n, i}\)-free, and homogeneous in black and red. By the induction hypothesis, \(\chi(X_i) = \eta_i < \infty\), so \(\chi(\mathcal{H}) \leq \sum_{i=0}^{\infty} \eta_i < \infty\), as required.

**Case 2:** \(\mathcal{H}\) is homogeneous in black and blue. We argue that, in fact, \(\chi(\mathcal{H}) < m\). Otherwise there is a finite \(\mathcal{H} \subseteq \mathcal{H}\) which induces a subgraph of chromatic number of at least \(m\). It must be the case that some member of \(\mathcal{H}\), \(h_0\) say, has at least \(m - 1\) neighbors \(h \in \mathcal{H}\) with \(h_0 < h\); otherwise, it has a coloring number of at most \(m - 1\). So let \(h_0, h_1, \ldots, h_{m-1} \subseteq \mathcal{H}\) with \(\{h_0, h_i\} \in \Pi_n\) for \(0 < i < m - 1\) and with \(h_0 < h_1 < \cdots < h_{m-1}\). Observe that, for \(0 < i < j \leq m - 1\), we must have \(\{h_i, h_j\} \in \Pi_n\); for otherwise the fact that \(\{h_0, h_i\} \in \Pi_n\) would entail that the non-edge \(\{h_i, h_j\}\) is colored red. But then \(\{h_0, h_1, \ldots, h_{m-1}\}\) induces a complete graph \(K_m\) in \(\mathcal{H}\), contrary to assumption.

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**References**


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