Ramsey-type properties of relational structures

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Abstract


Let \( \mathcal{L} \) be a relational language and \( \mathfrak{A} \) be a set of \( \mathcal{L} \)-structures. \( \mathfrak{A} \) is indivisible if for each \( A \in \mathfrak{A} \) there is a relational structure \( R(A) \in \mathfrak{A} \) such that for every partition of \( R(A) \) into two classes \( C \) and \( D \), there is an embedding of \( A \) into \( C \) or into \( D \). (If Folkman’s Theorem (1970) hold in \( \mathfrak{A} \)).

We will investigate this property of indivisibility in the case where \( \mathfrak{A} = \text{age } S \) for some countable relational structure \( S \) (age \( S \) is the set of all finite substructures of \( S \) up to isomorphism). In particular, if \( S \) is homogeneous, the divisibility or indivisibility of age \( S \) is related to the way in which the elements of age \( S \) amalgamate.

Introduction

We fix some relational language \( \mathcal{L} \) and concern ourselves only with countably infinite or finite models of \( \mathcal{L} \). We also restrict ourselves to languages \( \mathcal{L} \) and models \( S \) with the property that if \( x, y \in S \) then the restriction of \( S \) to \( x \) is isomorphic to the restriction of \( S \) to \( y \). (The 1-profile is 1.) In particular we assume that \( \mathcal{L} \) does not contain unary relational symbols. The age of a relational structure \( S \) is the set of all finite substructures of \( S \) considered up to isomorphism.

If \( S \) is an infinite relational structure then:

(i) \( S \) is indivisible (negation divisible) if for every partition of \( S \) into two classes one of the classes contains an isomorphic copy of \( S \).

(ii) \( S \) is weakly indivisible if for every partition of \( S \) into two classes \( C \) and \( D \) with age \( C \neq \text{age } S \) there is an embedding from \( S \) into \( D \).

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(iii) \( S \) is strongly inexhaustible if for every finite subset \( F \subseteq S \), \( S - F \) is isomorphic to \( S \).

(iv) \( S \) is inexhaustible if for every finite subset \( F \subseteq S \) there is an embedding from \( S \) into \( S - F \).

(v) \( S \) is age inexhaustible if for every finite subset \( F \subseteq S \), \( \text{age}(S - F) = \text{age} S \).

(vi) The age of \( S \) is indivisible (\( S \) is age indivisible) if for every partition of \( S \) into two classes \( C \) and \( D \), \( \text{age} C \neq \text{age} S \) implies that \( \text{age} D = \text{age} S \).

It is clear that (i) implies (ii) implies (iv) implies (v), (ii) implies (vi) and (vi) implies (v). Even if \( S \) is a homogeneous structure (ii) does not imply (i) [10], and we will construct homogeneous structures to show that (v) and (iii) are unrelated by implication. We will also prove that (vi) is equivalent to the statement that every \( A \in \text{age} S \) has a Ramsey structure \( R(A) \in \text{age} S \). \( B \) is a Ramsey structure for \( A \), if for every partition of \( B \) into two classes \( C \) and \( D \) there is an embedding of \( A \) into \( C \) or an embedding of \( A \) into \( D \). This notion has been extensively studied by Pouzet [12] where he anticipated most of Theorem 1. Folkman [2] has proven that the set of \( K_n \)-free graphs is indivisible. Nösgtril and Rödl [13] have given a different proof of Folkman’s Theorem and generalised it to the case of edge partitions. It would be very interesting to find a suitable generalisation of the theory presented here to the case of edge-colourings. The notions of indivisible and inexhaustible originate in [1]. Henson [3] has shown that the \( K_n \)-free homogeneous graphs \( H_n \) are weakly indivisible. Rödl and Komjath [8] have shown that \( H_3 \) is indivisible. We have shown that the graphs \( H_n \) for all \( n \) are indivisible [9]. In [10] we established the indivisibility or weak indivisibility for certain classes of directed homogeneous graphs. We do not know whether age \( S \) indivisibility implies \( S \) is weakly indivisible, but feel that this implication might only be true under some additional condition, as for example the homogeneity of \( S \).

A relational structure \( S \) is homogeneous if it has the embedding property, that is, if for every \( A \in \text{age} S \) and \( x \in A \) and embedding \( \varphi : A - x \rightarrow S \) there exists an extension \( \varphi' \) of \( \varphi \) which is an embedding from \( A \rightarrow S \). See [1, p. 313] for the alternate definition: The countable set \( \mathfrak{A} \) of relational structures is the age of a countable homogeneous structure \( S \) if and only if \( \mathfrak{A} \) is closed under substructures and \( \mathfrak{A} \) is amalgamable. \( \mathfrak{A} \) is amalgamable iff for \( A, B, C \in \mathfrak{A} \) and embeddings \( \alpha_0 : C \rightarrow A \) and \( \beta_0 : C \rightarrow B \) there exists \( D \in \mathfrak{A} \) and embeddings \( \alpha : A \rightarrow D \) and \( \beta : B \rightarrow D \) such that \( \alpha \circ \alpha_0 = \beta \circ \beta_0 \). Observe that the embedding property implies that if \( A \in \text{age} S \) and \( \varphi \) an embedding of some subset of \( A \) into \( S \) then \( \varphi \) can be extended to an embedding of \( A \) into \( S \). Homogeneous structures have been widely studied. See [4–7] for classifications under various restrictive conditions. Apart from the general Theorem 1 we will be mainly interested in properties (ii) to (vi) for homogeneous structures. We will define certain stronger versions of amalgamation and relate them to the various divisibility properties of the resultant homogeneous structures. If \( S \) is a homogeneous structure and \( F \subseteq S \) finite and \( R \) an orbit of the stabiliser subgroup of the automorphism group of \( S \).
which fixes \( F \), we say that \( R \) is a finitely induced orbit of \( S \). It follows from [10] and [11] that the age indivisible finitely induced orbits of \( S \) have to be investigated in order to determine whether \( S \) is indivisible. The finitely induced orbits of \( S \) are again homogeneous structures, but in an expanded language. It is therefore our aim to find the most general stronger versions of amalgamation, which assure age indivisibility and which also assure that the finitely induced orbits satisfy the same stronger amalgamation properties.

1. Divisible age of relational structures

Let \( S \) be a relational structure. We define the following properties of \( S \):

- \( P(1, n) \) For every partition of \( S \) into \( n \) classes \( C_1, C_2, \ldots, C_n \), the age of one of the classes is equal to the age of \( S \).

- \( P(2, n) \) For every partition of \( S \) into \( n \) classes \( C_1, C_2, \ldots, C_n \),

\[
\bigcup_{i=1}^{n} \text{age } C_i = \text{age } S.
\]

- \( P(3, n) \) For each \( A \in \text{age } S \) and every partition of each element \( B \in \text{age } S \) into \( n \) classes \( c_1(B), c_2(B), \ldots, c_n(B) \), there exists an element \( D \in \text{age } S \) and an \( i \) with \( 1 \leq i \leq n \) such that \( A \in \text{age } C_i(D) \).

- \( P(4, n) \) The age of \( S \) is indivisible, that is, for each \( A \in \text{age } S \) there is an element \( R(A) \in \text{age } S \) such that for every partition of \( R(A) \) into \( n \) classes, \( A \) is an element of the age of one of the classes. \( R(A) \) is called a Ramsey element of \( A \).

**Theorem 1.** For all \( n \geq 2 \), \( P(1, n) \) implies \( P(2, n) \) implies \( P(3, n) \) implies \( P(4, n) \) implies \( P(4, 2) \) implies \( P(4, n) \) implies \( P(1, n) \).

**Proof.** Clearly, \( P(1, n) \) implies \( P(2, n) \).

\( P(2, n) \) implies \( P(3, n) \): Choose \( A \in \text{age } S \) and let for each \( B \in \text{age } S \) a colouring \( c_1(B), c_2(B), \ldots, c_n(B) \) can be given. Let \( s_1, s_2, s_3, \ldots \) be an enumeration of \( S \). Let \( I_k = (s_1, s_2, \ldots, s_k) \) be the \( k \)th initial segment of \( S \). Let \( F \) be an ultrafilter over the natural numbers \( \mathbb{N} \) which contains all of the cofinal sets. We consider the ultraproduct \( \prod_F I_k \) and observe that \( S \) is isomorphic to the set of constant sequences \( \prod_{i} \). We partition \( S \) into \( n \) classes \( C_1, C_2, \ldots, C_n \), by stipulating that \( s \in C_i \) just in case when the set of indices \( k \) for which \( s \in c_i(I_k) \) is an element of \( F \). Because \( S \) has properly \( P(2, n) \) there is some \( i, 1 \leq i \leq n \), with \( A \in \text{age } C_i \). Clearly then there is some \( I_k \) such that \( A \in \text{age } C_i(I_k) \).

\( P(3, n) \) implies \( P(4, n) \): Choose \( A \in \text{age } S \). Assume that for every \( D \in \text{age } S \) there exists a partition of \( D \) into classes \( c_1(D), c_2(D), \ldots, c_n(D) \) with \( A \not\in \cup c_i(D) \). But this means that \( P(3, n) \) does not hold for \( S \).

Clearly \( P(4, n) \) implies \( P(4, 2) \).
P(4, 2) implies P(4, n): We show that P(4, n) implies P(4, 2n). Let A ∈ age S, R1 a Ramsey structure for A using two colours and R2 a Ramsey structure for R1 using n colours. Observe that R2 is a Ramsey structure for A using 2n colours.

P(4, n) implies P(1, n): Let C1, C2, . . . , Cn, be a partition of S into n classes. Assume that for each i, 1 ≤ i ≤ n, there exists Ai ∈ age S such that A, â ∈ age Ci. Let \( A = \bigcup_{i=1}^{n} A_i \). Choose for each B ∈ age S a particular embedding \( \varphi_B : B \to S \) and partition B accordingly into n classes \( c_1(B), \ldots, c_n(B) \), using the induced colouring of \( \varphi_B(B) \). Let D be the Ramsey structure for A. Then \( A \in \text{age } c_l(D) \) implies \( A \in \text{age } C_l \), a contradiction. □

2. Strongly inexhaustible homogeneous structures

In this chapter we will define an amalgamation which is equivalent to the property of strong inexhaustibility for the resultant homogeneous structure. The following notation is not really necessary for this chapter but will be used extensively in later chapters. As it is also useful in this chapter we may as well introduce it here.

Let \( \mathcal{L} \) be a relational language and \( \mathcal{L}_n = \mathcal{L} \cup \{ \Pi_1, \Pi_2, \ldots, \Pi_n \} \) where each \( \Pi_i \) is a unary relation symbol. We are only interested in those models \( A \) of \( \mathcal{L}_n \) in which there exists for each \( x \in A \) exactly one \( i \) such that \( \Pi_i(x) \) holds. In other words, if we denote by \( \Pi_i(A) \subset A \) the set of all elements for which \( \Pi_i \) holds, then \( \Pi_1(A), \Pi_2(A), \ldots, \Pi_n(A) \) is a partition of \( A \) into \( n \) classes. If \( S \) is a model of \( \mathcal{L} \) then \( \mathcal{P}_n(S) \) denotes the set of all \( \mathcal{L}_n \)-structures \( A \) which are elements of age \( S \) if viewed as \( \mathcal{L} \)-structures. So, \( \mathcal{P}_n(S) = \text{age}(S) \). If \( A \in \mathcal{P}_n(S) \) and \( 1 \leq i_1 < i_2 < \cdots < i_s \leq n \) then \( \Pi_{i_1}, \Pi_{i_2}, \ldots, \Pi_{i_s}(A) \) is that substructure of \( A \) in \( \mathcal{P}_r(S) \) such that for \( 1 \leq s \leq r \), \( \Pi_s(b) \) is isomorphic to \( \Pi_s(A) \). Also, if \( \sigma \) is a permutation of the numbers from 1 to \( n \) then \( \Pi_{\sigma(1)}, \Pi_{\sigma(2)}, \ldots, \Pi_{\sigma(n)} \in (B) \in \mathcal{P}_n(S) \) is obtained from \( B \) by permuting the assignment of the unary relations \( \Pi_1, \Pi_2, \ldots, \Pi_n \). So for example if \( A \in \mathcal{P}_2(S) \) then \( \Pi_{2,1}(A) \in \mathcal{P}_2(S) \) is such that there is an isomorphism \( \alpha \) from \( \Pi_1(\Pi_{2,1}(A)) \) to \( \Pi_2(A) \) and there is an isomorphism \( \beta \) from \( \Pi_2(\Pi_{2,1}(A)) \) to \( \Pi_1(A) \) such that the union of \( \alpha \) and \( \beta \) is an \( \mathcal{L} \)-isomorphism from \( \Pi_{2,1}(A) \) to \( A \). For \( A, B \in \mathcal{P}_n(S) \) we understand that isomorphism from \( A \) to \( B \) embedding from \( A \) into \( B \) means \( \mathcal{L}_r \)-isomorphism and \( \mathcal{L}_r \)-embedding. \( \equiv \) means: is isomorphic to.

Let \( S \) denote a homogeneous structure. If \( A, B \in \mathcal{P}_2(S) \) with \( \Pi_2(A) \equiv \Pi_2(B) \), then \( \text{amal}(A, B) = \{ D \in \mathcal{P}_2(S) : \exists H \in \mathcal{P}_2(S) (A \equiv \Pi_{1,3}(H) \text{ and } B \equiv \Pi_{2,3}(H) \text{ and } D \equiv \Pi_{1,2}(H) \} \). If \( H \in \mathcal{P}_2(S) \) with \( A \equiv \Pi_{1,3}(H) \) and \( B \equiv \Pi_{2,3}(H) \) we say that \( H \) has been constructed by amalgamating \( A \) with \( B \) over \( \Pi_2(A) \).

**Definition.** \( S \) has the disjoint amalgamation property, or equivalently the elements of age \( S \) can be disjointly amalgamated, if for all \( A, B \in \mathcal{P}_2(S) \) with \( \Pi_2(A) \equiv \Pi_2(B) \) the set \( \text{amal}(A, B) \) is not empty.
Remark. This type of amalgamation is called strong amalgamation in [1, p. 315]. We prefer this change of name because we wish to say that \( \Pi_1(A) \) and \( \Pi_1(B) \) remain disjoint when being amalgamated over \( \Pi_2(A) \equiv \Pi_2(B) \). In a way, disjoint amalgamation expresses the fact that there is no "hidden" functional relationship from the elements of \( \Pi_2(A) \) to the elements of \( \Pi_1(A) \) or \( \Pi_1(B) \) respectively, which has to be taken care of in the amalgamation. So, in any theory of homogeneous algebraic structures or if we allow function symbols in \( \mathcal{L} \) we can not expect disjoint amalgamation to play a role.

Definition. \( S \) has the strong embedding property if for all \( A \in \text{age} \, S \) and \( x \in A \) and embeddings \( \varphi : A - x \rightarrow S \) there are infinitely many different extensions of \( \varphi \) to embeddings of \( A \) into \( S \).

Theorem 2. If \( S \) is a homogeneous structure, then the following are equivalent:

(i) \( S \) has disjoint amalgamation.
(ii) \( S \) has the strong embedding property.
(iii) \( S \) is strongly inexhaustable.

Proof. (i) implies (ii): Assume not, then there is \( A \in \text{age} \, S \) and \( x \in A \) and an embedding \( \varphi : A - x \rightarrow S \) which has only finitely many different extensions \( \varphi_1, \varphi_2, \varphi_3, \ldots, \varphi_n \). \( n \neq 1 \) because \( S \) is homogeneous. Let \( B \subseteq S \) be the substructure induced by \( \bigcup_{i=1}^{n} \varphi_i(A) \). Let \( H \) be a structure constructed from \( A \) and \( B \) by disjoint amalgamation over \( A - x \). \( B \) has an embedding into \( S \) which can be extended to an embedding of \( H \) which then provides an extension of \( \varphi \) different from \( \varphi_1, \varphi_2, \ldots, \varphi_n \).

(ii) implies (iii): Let \( F \subseteq S \) be finite. To prove that \( S - F \) is isomorphic to \( S \), we have to show that for every \( A \in \text{age} \, S \) and \( x \in A \) and every embedding \( \varphi \) from \( A - x \) into \( S - F \), \( \varphi \) can be extended to an embedding of \( A \) into \( S - F \). But this then clearly follows from the strong embedding property.

(iii) implies (i): Let \( A, B \in \mathcal{P}_2(S) \) with \( \Pi_2(A) \equiv \Pi_2(B) \) be given. Let \( \varphi \) be an embedding of \( A \), as an \( \mathcal{L} \)-structure, into \( S \). Then \( S - \varphi(\Pi_1(A)) \) is isomorphic to \( S \). Hence, the restriction of \( \varphi \) to \( \Pi_2(A) \) has an extension to an embedding \( \varphi \) from \( B \), as an \( \mathcal{L} \)-structure, into \( S - \varphi(\Pi_1(A)) \). The image of the union of \( \varphi \) and \( \psi \) is the desired disjoint amalgamation of \( A \) with \( B \) over \( \Pi_2(A) \).

Let \( A \subseteq S \) be a finite subset and \( T \subseteq S \) be an orbit of the subgroup of the automorphism group of \( S \) which stabilises \( A \), element by element. We will then call \( T \) an orbit induced by \( A \). A finitely induced orbit of \( S \) is an orbit induced by some finite subset of \( S \). If \( T \) is an orbit induced by \( A \), then there exists an expansion of \( \mathcal{L} \) to some language \( \mathcal{L}^T \) such that \( T \) is a homogeneous structure \( T^* \) in the language \( \mathcal{L}^T \). The additional elements of \( \mathcal{L}^T \) describe the different \( n \)-types in \( T \) over \( A \). Observe that if \( \mathcal{L} \) is finite, then \( \mathcal{L}^T \) is finite and if \( \mathcal{L} \) contains only binary relational symbols, then \( \mathcal{L} = \mathcal{L}^T \).
**Theorem 3.** If $S$ has the disjoint amalgamation property, then $T^*$ has the disjoint amalgamation property for every finitely induced orbit $T$ of $S$.

**Proof.** Let $T$ be an orbit induced by the finite subset $A \subseteq S$. We will show that $T^*$ has the strong embedding property. So, pick $B \in \text{age } T^*$ and $x \in B$ and $\varphi$ an embedding of $B - x$ into $T^*$. Then, because $T^*$ is homogeneous, there is an embedding $\psi$ from $B$ into $T^*$. The identity map $i$ on $A \cup \varphi(B - x)$ is an embedding of $A \cup \varphi(B - x)$ into $S$ and clearly $A \cup \varphi(B) \in \text{age } S$. Hence $i$ has infinitely many different extensions $i_1, i_2, i_3, \ldots$ to embeddings of $A \cup \varphi(B)$ into $S$ each the identity map on $A \cup \varphi(B - x)$. Every partial automorphism of a homogeneous structure has an extension to an automorphism. But this means that for each $j \in \omega$ the element $i_j((\psi(x)) \in T$ and because $i_j$ is the identity on $\varphi(B - x)$ the product $i_1 \circ \psi$ is an embedding of $B$ into $T^*$ extending $\varphi$. □

**Theorem 4.** (a) There is a homogeneous structure with disjoint amalgamation whose age is divisible. (b) There is an indivisible homogeneous structure which does not have disjoint amalgamation. (D. MacPherson, oral communication).

**Proof.** (a) The infinite graph which consists of two disjoint copies of $K_\omega$ has the required properties. $K_\omega$ is the complete graph on countably many vertices.

(b) Let $T$ be a countably infinite set and $S$ the set of all two-element subsets of $T$. $\mathcal{L}$ consists of a binary relation $B(.,.)$ and a ternary relation $D(.,.,.)$. For $a, b, c \in S$ we write $B(a,b)$ iff $a$ and $b$ have exactly one element in common. We write $D(a,b,c)$ iff $a, b, c$ form a triangle. The resultant relational structure $S$ is homogeneous which can be directly checked from the definitions. It is also one of the well-known structures described in [4]. Now then, the amalgamation is not disjoint because two triangles which have two edges in common must also have the third edge in common. On the other hand, if we colour the elements of $S$ with two colours there is then by Ramsey's Theorem an infinite subset of $T$ all of its pairs have the same colour. Hence $S$ is indivisible. □

3. Consistent amalgamation

Let $A, B \in \mathcal{P}_2(S)$ with $\Pi_2(A) \equiv \Pi_2(B)$ and $F \in \text{age } (S)$. We say that $F$ can be consistently amalgamated with $A$ respecting $B$ if the following holds: $\exists D \in \mathcal{P}_2(S)$ with $\Pi_1(D) \equiv \Pi_1(A)$ and $\Pi_2(D) \equiv F$ such that for all $H$ which can be embedded into $D$ for which $\Pi_1(H) \equiv \Pi_1(A)$ and $\Pi_2(H) \equiv \Pi_1(B)$ we have $H \in \text{amal } (A, B)$. We call $D$ a consistent amalgam of $A$ with $F$ respecting $R$. $S$ has the consistent amalgamation property if for all, $A, B \in \mathcal{P}_2(S)$ and $F \in \text{age } (S)$, $\Pi_2(A) \equiv \Pi_2(B)$ implies that $F$ can be consistently amalgamated with $A$ respecting $B$. Observe that if $S$ has the consistent amalgamation property then $S$ has the
disjoint amalgamation property. The age of $S$ has the weak consistent amalgamation property if for $A, B \in \mathcal{P}_2(S)$ and $F \in \text{age}(S)$ and $\Pi_2(A)$ and $\Pi_2(B)$ a singleton, then $F$ can be consistently amalgamated with $A$ respecting $B$. We will prove, Theorem 5, that weak consistent amalgamation implies weak indivisibility and hence age indivisibility. Even so weak consistent amalgamation in the weakest form of amalgamation we know to imply a Folkman type theorem. We do not know whether the converse is true. Next we need the following technical lemma.

**Lemma 1.** Given $A$, $B \in \mathcal{P}_2(S)$ with $\Pi_2(A) \equiv \Pi_2(B)$. If $A$ can be consistently amalgamated with every $F \in \text{age}(S)$ respecting $B$ and $\alpha: \Pi_1(A) \to S$ is an embedding, then there exists an embedding $\varphi: S \to S$ such that $\varphi(S) \cap \alpha(\Pi_1(A)) = \emptyset$ and such that for every embedding $\beta: \Pi_1(B) \to \varphi(S)$ there exists an embedding $\gamma: \Pi_2(A) \to S$ such that the induced substructure $(\alpha(\Pi_1(A)), \gamma(\Pi_2(A)))$ is isomorphic to $A$ and the induced substructure $(\beta(\Pi_1(B)), \gamma(\Pi_2(B)))$ is isomorphic to $B$.

**Proof.** We assume $S$ to be ordered as the natural numbers and that $I_0, I_1, I_2, I_3, \ldots$ is the sequence of initial segments of $S$. We denote by $\mathcal{D}_i$ the set of consistent amalgams (up to isomorphism) of $A$ with $I_i$ respecting $B$. We will put a directed edge from an element $X \in \mathcal{D}_{i+1}$ to an element $Y \in \mathcal{D}_i$ if $Y$ has an embedding into $X$ as $\mathcal{L}_2$-structures. Clearly, this directed graph satisfies all of the conditions for König's Lemma and so there exists a infinite path. We denote the elements of this infinite path by $D_0, D_1, D_2, \ldots$ with the understanding that $D_i \in \mathcal{D}_i$ for all $i \in \omega$. Now, because of the mapping property the $D_i$'s can be so embedded into $S$ that the base of $D_i$ is a subset of the base of $D_{i+1}$. This clearly will give the desired result. $\square$

**Theorem 5.** If $S$ has the weak consistent amalgamation property, then $S$ is weakly indivisible.

**Proof.** Let $B, R$ be a partition of $S$ into a set of blue ($B$) and a set of red ($R$) elements. Assume $\text{age}(R) \neq \text{age}(S)$. Let $G \in \text{age}(S)$ have a minimal number of elements under the condition that $G \notin \text{age}(R)$. We proceed by induction on the number of elements in $G$. Choose $A \in \text{age}(S)$ and an embedding $\alpha: (A - x) \to B$. If we can show that $\alpha$ always extends to an embedding of $A$, then $B$ would be an isomorphic copy of $S$. Let $y \in G$ be some element. We consider the element $\tilde{A} \in \mathcal{P}_2(S)$ which is as an $\mathcal{L}$-structure isomorphic to $A$ and for which $\Pi_1(\tilde{A}) = A - x$ and $\Pi_2(\tilde{A})$ is the singleton $x$. Further let $\tilde{G} \in \mathcal{P}_2(S)$ be such that as an $\mathcal{L}$-structure it is isomorphic to $G$ and such that $\Pi_1(\tilde{G}) = G - y$ and $\mathcal{P}_2(\tilde{G}) = \{ y \}$. By the foregoing lemma there is an embedding $\varphi: S \to S$ such that for every embedding $\beta: G - y \to \varphi(S)$ there is an embedding $\gamma: x \to S$ such that $(\alpha(A - x), \gamma(x)) \equiv \tilde{A}$ and $(\beta(G - y), \gamma(x)) \equiv \tilde{G})$. The colouring $B, R$ induces a colouring of $\varphi(S)$. It follows from the induction hypothesis that if $B$ does not contain an
isomorphic copy of $S$, then there exists an embedding $\beta : G - y \to R \cap \varphi(S)$. That means, by the above described property of $\varphi(S)$ that there exists $\gamma : x - S$ with $(\alpha(A - x), \gamma(x)) \cong \tilde{A}$ and $(\beta(G - y), \gamma(x)) \cong \tilde{G}$. So $\gamma(x) \not\in R$, otherwise $G \in \text{age}(R)$. Hence $\gamma(x) \in B$, giving the appropriate extension of the embedding $\alpha$. □

4. Independent amalgamation

The age of $S$ has the independent amalgamation property if for all $A, B \in \text{age} S$ there exists $D \in \mathcal{P}_2(S)$ with $A \equiv \Pi_1(D)$ and $B \equiv \Pi_2(D)$ such that whenever $A', B' \in \mathcal{P}_2(S)$ with $\Pi_1(A') \equiv A$ and $\Pi_1(B') \equiv B$ and $\Pi_2(A') \equiv \Pi_2(B')$ are given, then there exists $H \in \Pi_3(S)$ such that $\Pi_1,2(H) \equiv D$ and $\Pi_1,3(H) \equiv A'$ and $\Pi_2,3(H) \equiv B'$. $D$ is called independent amalgam of $A$ and $B$. $H$ is called independent amalgam of $A'$ with $B'$ over $\Pi_2(A')$. (The independent amalgam $D$ of $A$ with $B$ is independent of $\Pi_2(A')$)

Lemma 2. If age $S$ has the independent amalgamation property, then age $S$ has the consistent amalgamation property.

Proof. Let $A, B \in \mathcal{P}_2(S)$ with $\Pi_2(A) \equiv \Pi_2(B)$ and $F \in \text{age}(S)$ be given. Let $D$ be an independent amalgam of $\Pi_1(A)$ and $F$. We will argue that $D$ is also a consistent amalgam of $A$ with $F$ respecting $B$. So assume that $H$ with $\Pi_1(H) \equiv \Pi_1(A)$ and $\Pi_2(H) \equiv \Pi_1(B)$ has an embedding $\varphi$ into $D$. Consider the structure $K \in \mathcal{P}_2(S)$ with $\Pi_1(K) \equiv \Pi_2(D) - \varphi(\Pi_2(H))$ and $\Pi_2(K) \equiv \varphi(\Pi_2(H))$. Now $\varphi(\Pi_2(H)) \equiv \Pi_2(H) \equiv \Pi_1(B)$ and hence by the disjoint amalgamation property with respect to $K$ and $\Pi_2(B)$ there exists $L \in \mathcal{P}_2(S)$ with $\Pi_1(L) \equiv \Pi_2(D) - \varphi(\Pi_2(H))$ and $\Pi_2(L) \equiv \varphi(\Pi_2(H))$ and $\Pi_3(L) \equiv \Pi_3(B)$. Hence there is an $M \in \mathcal{P}_2(S)$ with $\Pi_2(M) \equiv F$, $\Pi_3(M) \equiv \Pi_2(B)$ and an embedding of $B$ into $M$. Because $D$ is an independent amalgam of $\Pi_1(A)$ and $F$ there exists $N \in \Pi_3(S)$ with $\Pi_1,3(N) \equiv A$, $\Pi_2,3(N) \equiv M$, $\Pi_1,2(n) \equiv D$ and an embedding of $B$ into $\Pi_2,3(N)$. Looking back at the definition of $H$ we observe that this means that $H \in \text{amal}(A, B)$ □

Lemma 3. If $S$ is a homogeneous structure which allows independent amalgamation and $T$ is an orbit induced by the finite subset $C \subset S$ then $T^*$ as a homogeneous structure for the language $\mathcal{L}^T$ also allows for independent amalgamation.

Proof. Given $A, B \in \text{age} T^*$ let $\bar{A}, \bar{B}$ denote the reduction of $A, B$ as models for the language $\mathcal{L}^T$ to models for $\mathcal{L}$. Let $\bar{D} \in \mathcal{P}_2(S)$ be an independent amalgam of $\bar{A}$ and $\bar{B}$ and $D \in \mathcal{P}_2(T)$ an appropriate expansion of $\bar{D}$ to $\mathcal{L}^T$ such that $\Pi_1(D) \equiv A$ and $\Pi_2(D) \equiv B$.

Let $A', B' \in \mathcal{P}_2(T^*)$ be such that $\Pi_1(A') \equiv A$, $\Pi_1(B') \equiv B$ and $\Pi_2(A') \equiv \Pi_2(B')$. Also $\bar{A}', \bar{B}' \in \mathcal{P}_2(S)$ is such that $\Pi_1(\bar{A}') \equiv \bar{A}$, $\Pi_1(\bar{B}') \equiv \bar{B}$, $\Pi_2(\bar{A}') \equiv$
And, $\Pi_2(\tilde{A}^{'})$ partitions into a structure isomorphic to $\Pi_2(A^{'})$ and a structure $C^*$ isomorphic to $C$ such that there is an embedding from $\tilde{A}^{' }$ into $C \cup T$ which maps $C^*$ onto $C$ and which maps $\Pi_1(\tilde{A}^{'})$ to a subset isomorphic to $\Pi_1(A^{'})$.

And similarly for $\tilde{B}^{' }$, that is $\Pi_2(\tilde{B}^{'})$ partitions into a structure isomorphic to $\Pi_2(B^{'})$ and a structure $C^{**}$ isomorphic to $C$ such that there is an embedding from $B^{' }$ into $C \cup T$ which maps $C^{**}$ onto $C$ and which maps $\Pi_1(\tilde{B}^{'})$ to a subset isomorphic to $\Pi_1(B^{'})$.

Because $D$ is an independent amalgam of $A$ and $B$ there exists $H \in \mathcal{P}_3(S)$ such that $\Pi_1,3(H) \equiv \tilde{A}^{' }$, $\Pi_2,3(H) \equiv \tilde{B}^{' }$ and $\Pi_1,3(H) \equiv \tilde{D}$. But this means that $D$ is an independent amalgam for $A$ and $B$ in $T$.

**Theorem 6.** If $S$ has the independent amalgamation property then $S$ and every finitely induced orbit for $S$ are weakly indivisible.

**Proof.** Theorem 6 is an immediate consequence of Theorem 5, Lemma 2 and Lemma 3.

**Lemma 4.** $S$ has the independent amalgamation property if and only if for every $A \in \text{age } S$ and embedding $\alpha:A \to S$ there exists an embedding $\varphi:S \to S$ such that for every $B \in \text{age } S$ and $\beta:B \to \varphi(S)$ the structure $(\alpha(A), \beta(B)) \in \mathcal{P}_2(S)$ is an independent amalgam of $A$ and $B$.

The proof of this lemma is a simpler version of the proof of Lemma 1.

The following 'free' amalgamation is a special important case of independent amalgamation. We say the age of the homogeneous structure $S$ has the free amalgamation property if for all $A, B \in \text{age } S$ the disjoint union of $A$ and $B$ without adding any additional realisations of the relations in $L$ is an independent amalgam of $A$ and $B$. A finite model $M$ of $L$ is 2-covering if every 2-element subset $\{a, b\}$ of $M$ is 'covered' by one of the relations in $L$, that is if there is a relation $R$ in $L$ and elements $x_1, x_2, \ldots, x_n$ in $M$ with $\{a, b\} \subset \{x_1, x_2, \ldots, x_n\}$ and such, that $R(x_1, x_2, \ldots, x_n)$ holds. (See [14].) A 2-covering model is an irreducible relational system there). If $\mathcal{J}$ is a set of 2-covering models of $L$ then the set $\mathcal{A}$ of all finite models $A$ of $L$ which do not embed any element of $\mathcal{J}$ has the free amalgamation property. The homogeneous structure $S$ with age $S = \mathcal{A}$ is then called the $\mathcal{J}$-free homogeneous structure $H_\mathcal{J}$. We will agree that the phase $\mathcal{J}$-free homogeneous structure implies that $\mathcal{J}$ is a 2-covering set of structures. As a special case of Theorem 6 we get the following.

**Theorem 7.** Every $\mathcal{J}$-free homogeneous structure $H_\mathcal{J}$ and all of its finitely induced orbits are weakly indivisible. (This result is essentially contained in [13])

**Example.** Let $L$ be the language which consists of five binary relation symbols, $R_1, R_2, R_3, R_4, R_5$. We will restrict our attention to models $M$ of $L$ in which $R_1, R_2, R_3, R_4, R_5$ are symmetric and antireflexive. Furthermore, we require that
for every pair \( \{x, y\} \subset M \) with \( x \neq y \) exactly one of the relations \( R_i \), \( 1 \leq i \leq 5 \), holds. Let \( \mathcal{T} \) be the set of triangles \( T \) in which for every pair \( \{x, y\} \subset T \) with \( x \neq y \) exactly one of the relations \( R \), \( 1 \leq i \leq 4 \), holds. We further require that for different pairs of \( T \) different relations hold. Then the set of all \( \mathcal{S} \)-free \( \mathcal{S} \)-structures allows free amalgamation. (It is obviously quite irrelevant whether \( R_5 \) indicates the existence of a non-edge or is viewed as a relation in its own right.) The well-defined homogeneous structure \( H_{\mathcal{S}} \) is weakly indivisible. This then means that to every graph \( G \) whose edges are coloured with four colours and which does not contain any of the triangles in \( \mathcal{S} \) there exists another such graph \( R(G) \) such that for any partition of \( R(G) \) into finitely many classes, \( G \) can be embedded into one of those classes.

Clearly, there is an abundance of free amalgamation classes and hence of classes of structures for which Folkman’s Theorem holds.

References