Existence and Algorithm for the Systems of Hierarchical Variational Inclusion Problems

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1. Introduction

Let $H$ be a real Hilbert space with inner product and norm being $\langle \cdot , \cdot \rangle$ and $\| \cdot \|$, respectively, and let $C$ be a nonempty closed convex subset of $H$. A mapping $T : H \to H$ is called nonexpansive if

$$
\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in H. \tag{1}
$$

We use $F(T)$ to denote the set of fixed points of $T$; that is, $F(T) = \{x \in H : Tx = x\}$. It is well known that $F(T)$ is a closed convex set, if $T$ is nonexpansive mappings.

A variational inclusion problem [1–3] is the problem of finding a point $u \in H$ such that

$$
\theta \in A(u) + M(u), \tag{2}
$$

where $A : H \to H$ is a single-valued nonlinear mapping and $M : H \to 2^H$ is a multivalued mapping. We use $\Omega$ to denote the set of solutions of the variational inclusion (2).

On the other hand, a hierarchical fixed point problem [4–11] is the problem of finding a point $x^* \in F(T)$ such that

$$
\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in F(T). \tag{3}
$$

If the set $F(T)$ is replaced by the solution set of the variational inequality, then the hierarchical fixed point problems are called hierarchical variational inequality problems or hierarchical optimization problems. Many problems in mathematics, for example, the signal recovery [12], the power control problem [13], and the beamforming problem [14], can be considered in the framework of this kind of the hierarchical variational inequality problems.

Recently, Chang et al. [15] introduced bilevel hierarchical variational inclusion problems; that is, find $(x^*, y^*) \in \Omega_1 \times \Omega_2$ such that, for given positive real numbers $\rho$ and $\eta$, the following inequalities hold:

$$
\langle \rho F(y^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in \Omega_1, \tag{4}
$$

$$
\langle \eta F(x^*) + y^* - x^*, y - y^* \rangle \geq 0, \quad \forall y \in \Omega_2,
$$

where $F, A_1, A_2 : H \to H$ are mappings, $M_1, M_2 : H \to 2^H$ are multivalued mappings, and $\Omega_i$ is the set of solutions to variational inclusion problem (2) with $A = A_i, M = M_i$ for $i = 1, 2$. They solved the convex programming problems and quadratic minimization problems by using Maingés scheme.
In this paper, we consider the following system of hierarchical variational inclusion problem: find \((x^*, y^*, z^*) \in \Omega_1 \times \Omega_2 \times \Omega_3\) such that, for given positive real numbers \(\rho, \eta, \xi\), the following inequalities hold:

\[
\langle \rho F(y^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in \Omega_1,
\]

\[
\langle \eta F(z^*) + y^* - z^*, y - y^* \rangle \geq 0, \quad \forall y \in \Omega_2,
\]

\[
\langle \xi F(x^*) + z^* - x^*, z - z^* \rangle \geq 0, \quad \forall z \in \Omega_3.
\]

Some special cases of the system of hierarchical variational inclusion problem (5) are as follows.

(I) If \(M_i = 0, A_i = I - T_i\), where \(T_i : H \to H\) is a nonlinear mapping for each \(i = 1, 2, 3\), in (5), then \(\Omega_i = F(T_i)\) and the system of hierarchical variational inclusion problem (5) reduces to the following system of optimization problem:

\[
\langle \rho F(y^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in F(T_1),
\]

\[
\langle \eta F(z^*) + y^* - z^*, y - y^* \rangle \geq 0, \quad \forall y \in F(T_2),
\]

\[
\langle \xi F(x^*) + z^* - x^*, z - z^* \rangle \geq 0, \quad \forall z \in F(T_3),
\]

which was studied by Li [16].

(II) If \(T_i = P_{K_i}\) for each \(i = 1, 2, 3\), where \(P_{K_i}\) is the metric projection from \(H\) onto a nonempty closed convex subset \(K_i\) in (6), then it is clear that \(\Omega_i = F(T_i) = K_i\) and the system of hierarchical optimization problem (6) reduces to the following system of optimization problem:

\[
\langle \rho F(y^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in K_1,
\]

\[
\langle \eta F(z^*) + y^* - z^*, y - y^* \rangle \geq 0, \quad \forall y \in K_2,
\]

\[
\langle \xi F(x^*) + z^* - x^*, z - z^* \rangle \geq 0, \quad \forall z \in K_3,
\]

(III) If \(K_1 = K_2 = K_3\), then the system of optimization problem (7) reduces to the following system of variational inequality problem:

\[
\langle \rho F(y^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in K_1,
\]

\[
\langle \eta F(z^*) + y^* - z^*, y - y^* \rangle \geq 0, \quad \forall y \in K_1,
\]

\[
\langle \xi F(x^*) + z^* - x^*, z - z^* \rangle \geq 0, \quad \forall z \in K_1.
\]

(IV) If \(\xi = \eta = 0, \Omega_1 = \Omega_3, \) and \(x^* = z^*\) in (5) then the system of hierarchical variational inclusion problem (5) reduces to the following bilevel hierarchical variational inclusion problem:

\[
\langle \rho F(y^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in \Omega_1,
\]

\[
\langle \eta F(x^*) + y^* - x^*, y - y^* \rangle \geq 0, \quad \forall y \in \Omega_2,
\]

which was studied by Chang et al. [15].

(V) In (9), if \(M_i = 0, A_i = I - T_i\), for each \(i = 1, 2\), then bilevel hierarchical variational inclusion problem (9) reduces to the following bilevel hierarchical optimization problem:

\[
\langle \rho F(y^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in F(T_1),
\]

\[
\langle \eta F(x^*) + y^* - x^*, y - y^* \rangle \geq 0, \quad \forall y \in F(T_2),
\]

which was studied by Maingé [17] and Kraikaew and Saejung [18].

(VI) In (10), if \(T_i = P_{K_i}\) for each \(i = 1, 2\), then bilevel hierarchical optimization problem (10) reduces to the following problem [19–21]: find \((x^*, y^*) \in K_1 \times K_2\) such that

\[
\langle \rho F(y^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in K_1,
\]

\[
\langle \eta F(x^*) + y^* - x^*, y - y^* \rangle \geq 0, \quad \forall y \in K_2.
\]

(VII) In (11), if \(K_1 = K_2\) then the problem (II) reduces to the following problem: find \((x^*, y^*) \in K_1 \times K_1\) such that

\[
\langle \rho F(y^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in K_1,
\]

\[
\langle \eta F(x^*) + y^* - x^*, y - y^* \rangle \geq 0, \quad \forall y \in K_1.
\]

(VIII) In (12), if \(\xi = \eta = 0, \Omega_1 = \Omega_2 = \Omega_3, \) and \(x^* = y^* = z^*\) then the system of hierarchical variational inclusion problem (5) reduces to the following hierarchical variational inclusion problem: find \(x^* \in \Omega_1\) such that

\[
\langle F(y^*), x - x^* \rangle \geq 0, \quad \forall x \in \Omega_1.
\]

(IX) In (13), if \(M_i = 0, A_i = I - T_i\), then the hierarchical variational inclusion problem (13) reduces to the following hierarchical fixed point problem: find \(x^* \in F(T_1)\) such that

\[
\langle F(y^*), x - x^* \rangle \geq 0, \quad \forall x \in F(T_1).
\]

(X) In (15), if \(T_1 = P_{K_1}\), then the hierarchical fixed point problem (15) reduces to the following classic variational inequality problem: find \(x^* \in K_1\) such that

\[
\langle F(y^*), x - x^* \rangle \geq 0, \quad \forall x \in K_1.
\]

Motivated and inspired by Chang et al. [15], we introduce the system of a hierarchical variational inclusion problem (5) and investigate a more general variant of the scheme proposed by Chang et al. [15] to solve the system of a hierarchical variational inclusion problem. Our analysis and method allow us to prove the existence and approximation of solutions to the system of a hierarchical variational inclusion problem (5). The results presented in this paper extend and improve the results of Chang et al. [15], Maingé [17], Kraikaew and Saejung [18], and some authors.
Definition 1. Let \( A, T, F : H \to H \) be a mapping and let \( M : H \to 2^H \) be a multivalued mapping.

(1) A mapping \( T \) is called \( \text{nonexpansive} \) if
\[
\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in H.
\] (16)

(2) A mapping \( T \) is called \( \text{quasinonexpansive} \) if \( F(T) \neq \emptyset \) and
\[
\|Tx - p\| \leq \|x - p\|, \quad \forall x \in H, \ p \in F(T).
\] (17)

It should be noted that \( T \) is quasinonexpansive if and only if for all \( x \in H, \ p \in F(T) \)
\[
\langle x - Tx, x - p \rangle \geq \frac{1}{2} \|x - Tx\|^2.
\] (18)

(3) A mapping \( T \) is called \( \text{strongly quasinonexpansive} \) if \( T \) is quasinonexpansive and \( x_n - Tx_n \to 0 \), whenever \( \{x_n\} \) is a bounded sequence in \( H \) and \( \|x_n - p\| - \|Tx_n - p\| \to 0 \) for some \( p \in F(T) \).

(4) A mapping \( F \) is called \( \mu \text{-Lipschitzian} \) if there exists \( \alpha > 0 \) such that
\[
\|Fx - Fy\| \leq \mu \|x - y\|, \quad \forall x, y \in H.
\] (19)

(5) A mapping \( F \) is called \( r \text{-strongly monotone} \) if there exists \( r > 0 \) such that
\[
\langle Fx - Fy, x - y \rangle \geq r \|x - y\|^2, \quad \forall x, y \in H.
\] (20)

It is easy to prove that if \( F : H \to H \) is a \( \mu \text{-Lipschitzian} \) and \( r \text{-strongly monotone} \) mapping and if \( \rho \in (0, 2r/\mu^2) \), then the mapping \( I - \rho F \) is a contraction.

(6) A mapping \( A \) is called \( \alpha \text{-inverse-strongly monotone} \) if there exists \( \mu > 0 \) such that
\[
\langle Ax - A y, x - y \rangle \geq \alpha \|Ax - A y\|^2, \quad \forall x, y \in H.
\] (21)

(7) A multivalued mapping \( M \) is called \( \text{monotone} \) if for all \( x, y \in H, u \in Mx \) and \( v \in My \) imply that
\[
\langle u - v, x - y \rangle \geq 0.
\] (22)

(8) A multivalued mapping \( M \) is called \( \text{maximal monotone} \) if it is monotone and for any \( (x, u) \in H \times H \),
\[
\langle u - v, x - y \rangle \geq 0
\] (23)

for every \( (y, v) \in \text{Graph}(M) \) (the graph of mapping \( M \)) implies that \( u \in Mx \).

Lemma 2 (see [22]). Let \( A : H \to H \) be an \( \alpha \text{-inverse-strongly monotone} \) mapping. Then

(1) \( A \) is an \( 1/\alpha \text{-Lipschitz} \) continuous and \( \text{monotone mapping} \).

(2) for any constant \( \lambda > 0 \), one has
\[
\| (I - \lambda A) x - (I - \lambda A) y \|^2 \\
\leq \| x - y \|^2 + \lambda \left( \frac{\lambda}{\lambda - 2 \alpha} \right) \| Ax - Ay \|^2; \quad (24)
\]

(3) if \( \lambda \in (0, 2\alpha] \), then \( I - \lambda A \) is a nonexpansive mapping, where \( I \) is the identity mapping on \( H \).

Lemma 3. Let \( x \in H \) and \( z \in C \) be any points. Then one has the following.

(1) That \( z = P_C[x] \) if and only if there holds the relation:
\[
\langle x - z, y - z \rangle \leq 0, \quad \forall y \in C.
\] (25)

(2) That \( z = P_C[x] \) if and only if there holds the relation:
\[
\| x - z \|^2 \leq \| x - y \|^2 - \| y - z \|^2, \quad \forall y \in C.
\] (26)

(3) There holds the relation:
\[
\langle P_C[x] - P_C[y], x - y \rangle \geq \| P_C[x] - P_C[y] \|^2, \quad \forall x, y \in H.
\] (27)

Consequently, \( P_C \) is nonexpansive and \( \text{monotone} \).

Definition 4. Let \( M : H \to 2^H \) be a multivalued maximal monotone mapping. Then the mapping \( J_{M,\lambda} : H \to H \) defined by
\[
J_{M,\lambda}(u) = (I + \lambda M)^{-1}(u), \quad u \in H
\] (28)
is called the \( \text{resolvent operator associated with} \ M \), where \( \lambda \) is any positive number and \( I \) is the identity mapping.

Proposition 5 (see [22]). Let \( M : H \to 2^H \) be a multivalued maximal monotone mapping, and let \( A : H \to H \) be an \( \alpha \text{-inverse-strongly monotone} \) mapping. Then the following conclusions hold.

(1) The resolvent operator \( J_{M,\lambda} \) associated with \( M \) is single-valued and \( \text{nonexpansive} \) for all \( \lambda > 0 \).

(2) The resolvent operator \( J_{M,\lambda} \) is \( 1 \text{-inverse-strongly monotone} \); that is,
\[
\| J_{M,\lambda}(x) - J_{M,\lambda}(y) \|^2 \leq \langle x - y, J_{M,\lambda}(x) - J_{M,\lambda}(y) \rangle, \quad \forall x, y \in H.
\] (29)

(3) \( u \in H \) is a solution of the variational inclusion (2) if and only if \( u = J_{M,\lambda}(u - \lambda Au) \), for all \( \lambda > 0 \); that is, \( u \) is a fixed point of the mapping \( J_{M,\lambda}(I - \lambda A) \). Therefore one has
\[
\Omega = F(J_{M,\lambda}(I - \lambda A)), \quad \forall \lambda > 0,
\] (30)
where $\Omega$ is the set of solutions of variational inclusion problem (2).

(4) If $\lambda \in (0,2\alpha]$, then $\Omega$ is a closed convex subset in $H$.

Lemma 6 (see [23]). For $x, y \in H$ and $\omega \in (0,1)$, the following statements hold:

1. $\|x + y\|^2 \leq \omega \|x\|^2 + 2 \omega x \cdot y$;
2. $\|x + \omega y\|^2 = (1 - \omega) \|x\|^2 + \omega \|y\|^2 - \omega(1 - \omega) \|x - y\|^2$.

Lemma 7 (see [24]). Let $\{a_n\}$ be a sequence of real numbers, and there exists a subsequence $\{a_{m_k}\}$ of $\{a_n\}$ such that $a_{m_k} < a_{m_{k+1}}$ for all $j \in N$, where $N$ is the set of all positive integers. Then there exists a nondecreasing sequence $\{a_k\}$ of $N$ such that $\lim_{k \to \infty} a_k = \infty$ and the following properties are satisfied by all (sufficiently large) number $k \in N$:

$$a_n \leq a_{n+1}, \quad a_k \leq a_{n_k}.$$  \hspace{1cm} (31)

In fact, $a_k$ is the largest number $n$ in the set $\{1, 2, \ldots, k\}$ such that $a_n < a_{n+1}$ holds.

Lemma 8 (see [18]). Let $\{a_n\} \subset [0, \infty)$, $\{a_n\} \subset [0, 1]$, $\{b_n\} \subset (-\infty, +\infty)$, and $h \in (0, 1)$ be such that

1. $\{a_n\}$ is a bounded sequence;
2. $a_{n+1} \leq (1 - a_n) \alpha_n + 2a_n h \sqrt{a_n} \sqrt{a_{n+1}} + \alpha_n b_n$ for all $n \geq 1$;
3. whenever $\{a_n\}$ is a subsequence of $\{a_n\}$ satisfying
   $$\lim_{k \to \infty} \frac{a_{n_k} - a_{n_k}}{n_k} \geq 0,$$
   it follows that $\lim_{k \to \infty} a_k = \infty$ and $\sum_{n=1}^{\infty} a_n = \infty$.
4. Then $\lim_{n \to \infty} a_n = 0$.

Lemma 9 (see [15]). Let $M : H \to 2^H$ be a multivalued maximal monotone mapping, let $A : H \to H$ be an $\alpha$-inverse-strongly monotone mapping, and let $\Omega$ be the set of solutions of variational inclusion problem (2) and $\Omega \neq 0$. Then the following statements hold.

1. If $\lambda \in (0, 2\alpha]$, then the mapping $K : H \to H$ defined by
   $$K = J_{M \lambda} (I - \lambda A)$$
   is quasinonexpansive, where $I$ is the identity mapping and $J_{M \lambda}$ is the resolvent operator associated with $M$.
2. The mapping $I - K : H \to H$ is demiclosed at zero; that is, for any sequence $\{x_n\} \subset H$, if $x_n \to x$ and $(I - K)x_n \to 0$, then $x = Kx$.
3. For any $\beta \in (0, 1)$, the mapping $K_\beta$ defined by
   $$K_\beta = (1 - \beta) I + \beta K$$
   is a strongly quasinonexpansive mapping and $F(K_\beta) = F(K)$.
4. $I - K_\beta$, $\beta \in (0, 1)$ is demiclosed at zero.

3. Main Results

Throughout this section, we always assume that the following conditions are satisfied:

(C1) $M_i : H \to 2^H$ is a multivalued maximal monotone mapping, $A_i : H \to H$ is an $\alpha_i$-inverse-strongly monotone mapping, and $\Omega_i$ is the set of solutions to variational inclusion problem (2) with $A = A_i$, $M = M_i$, and $\Omega_i \neq 0$, for all $i = 1, 2, 3$;

(C2) $K_i$ and $K_{i,\beta}$, $\beta \in (0, 1)$, $i = 1, 2, 3$, are the mappings defined by

$$K_i := J_{M_i \lambda} (I - \lambda A_i), \quad \lambda \in (0, 2\alpha_i],$$

$$K_{i,\beta} := (1 - \beta) I + \beta K_i, \quad \beta \in (0, 1),$$

respectively.

Next, there are our main results.

3.1. An Existence Theorem

Theorem 10. Let $A_i$, $M_i$, $\Omega_i$, $K_i$, and $K_{i,\beta}$ satisfy conditions (C1) and (C2), and let $f_i : H \to H$ be contractions with a contractive constant $h_i \in (0, 1)$, for all $i = 1, 2, 3$. Then there exists a unique element $(x^*, y^*, z^*) \in \Omega_1 \times \Omega_2 \times \Omega_3$ such that the following three inequalities are satisfied:

$$\langle x^* - f_1(y), x - x^* \rangle \geq 0, \quad \forall x \in \Omega_1,$$

$$\langle y^* - f_2(z), y - y^* \rangle \geq 0, \quad \forall y \in \Omega_2,$$

$$\langle z^* - f_3(x), z - z^* \rangle \geq 0, \quad \forall z \in \Omega_3.$$

Proof: The proof is a consequence of Banach’s contraction principle but it is given here for the sake of completeness. By Proposition 5 and Lemma 9, $\Omega_1$, $\Omega_2$, and $\Omega_3$ are nonempty closed and convex. Therefore the metric projection $P_{\Omega_i}$ is well defined for each $i = 1, 2, 3$.

Since $f_i$ is a contraction mapping for each $i = 1, 2, 3$, then we have $P_{\Omega_i} f_i$ which is a contraction and also have

$$P_{\Omega_1} f_1 \circ P_{\Omega_2} f_2 \circ P_{\Omega_3} f_3$$

which is a contraction. Hence there exists a unique element $x^* \in H$ such that

$$x^* = (P_{\Omega_1} f_1 \circ P_{\Omega_2} f_2 \circ P_{\Omega_3} f_3) x^*.$$  \hspace{1cm} (38)

Putting $z^* = P_{\Omega_3} f_3(x^*)$ and $y^* = P_{\Omega_2} f_2(z^*)$, then $z^* \in \Omega_3$, $y^* \in \Omega_2$, and $x^* \in P_{\Omega_1} f_1(y^*)$.

Suppose that there is an element $(\tilde{x}, \tilde{y}, \tilde{z}) \in \Omega_1 \times \Omega_2 \times \Omega_3$ such that the following three inequalities are satisfied:

$$\langle \tilde{x} - f_1(y), x - \tilde{x} \rangle \geq 0, \quad \forall x \in \Omega_1,$$

$$\langle \tilde{y} - f_2(\tilde{z}), y - \tilde{y} \rangle \geq 0, \quad \forall y \in \Omega_2,$$

$$\langle \tilde{z} - f_3(\tilde{x}), z - \tilde{z} \rangle \geq 0, \quad \forall z \in \Omega_3.$$
Then
\[
\tilde{x} = P_{\Omega_1} f_1 (\tilde{y}),
\]
\[
\tilde{y} = P_{\Omega_2} f_2 (\tilde{z}),
\]
\[
\tilde{z} = P_{\Omega_3} f_3 (\tilde{x}).
\]
Therefore
\[
\tilde{x} = (P_{\Omega_1} f_1 * P_{\Omega_2} f_2 * P_{\Omega_3} f_3) \tilde{x}.
\]
This implies that \(\tilde{x} = x^*, \tilde{y} = y^*, \) and \(\tilde{z} = z^*\). This completes the proof. □

3.2. A Convergence Theorem

**Theorem 11.** Let \(A_i, M_i, \Omega_i, K_i, \) and \(K_{i,\beta}\) satisfy conditions (C1) and (C2), and let \(f_i : H \to H\) be contractions with a contractive constant \(h_i \in (0, 1)\), for all \(i = 1, 2, 3\). Let \(\{x_n\}, \{y_n\}, \) and \(\{z_n\}\) be three sequences defined by
\[
x_0, y_0, z_0 \in H,
\]
\[
x_{n+1} = (1 - \alpha_n) K_{1,\beta} x_n + \alpha_n f_1 (K_{2,\beta} y_n),
\]
\[
y_{n+1} = (1 - \alpha_n) K_{2,\beta} y_n + \alpha_n f_2 (K_{3,\beta} z_n),
\]
\[
z_{n+1} = (1 - \alpha_n) K_{3,\beta} z_n + \alpha_n f_3 (K_{1,\beta} x_n),
\]
where \(\alpha_n\) is a sequence in \((0, 1)\) satisfying \(\alpha_n \to 0\) and \(\sum_{n=0}^\infty \alpha_n = \infty\). Then the sequences \(\{x_n\}, \{y_n\}, \) and \(\{z_n\}\) generated to be (42) converge to \(x^*, y^*,\) and \(z^*,\) respectively, where \((x^*, y^*, z^*)\) is the unique element in \(\Omega_1 \times \Omega_2 \times \Omega_3\) verifying (36).

**Proof.** (i) First we prove that sequences \(\{x_n\}, \{y_n\}, \) and \(\{z_n\}\) are bounded.

From Lemma 9, it follow that \(K_{i,\beta}\) is strongly quasinonexpansive and \(F(K_{i,\beta}) = F(K_i) = \Omega_i\) for each \(i = 1, 2, 3\). Since \(f_i\) is contraction with the coefficient \(h_i\) for each \(i = 1, 2, 3\) and \(x^* \in F(K_{1,\beta}), y^* \in F(K_{2,\beta}), \) and \(z^* \in F(K_{3,\beta}),\) it follows that
\[
\|x_{n+1} - x^*\| \leq (1 - \alpha_n) \|K_{1,\beta} x_n - x^*\| + \alpha_n \|f_1 (K_{2,\beta} y_n) - x^*\|
\]
\[
\leq (1 - \alpha_n) \|x_n - x^*\| + \alpha_n \|f_1 (K_{2,\beta} y_n) - f_1 (y^*)\|
\]
\[
+ \alpha_n \|f_1 (y^*) - x^*\|
\]
\[
\leq (1 - \alpha_n) \|x_n - x^*\| + \alpha_n h_1 \|K_{2,\beta} y_n - y^*\|
\]
\[
+ \alpha_n \|f_1 (y^*) - x^*\|
\]
\[
\leq (1 - \alpha_n) \|x_n - x^*\| + \alpha_n h_1 \|y_n - y^*\|
\]
\[
+ \alpha_n \|f_1 (y^*) - x^*\|
\]
\[
\leq (1 - \alpha_n) \|x_n - x^*\| + \alpha_n h \|y_n - y^*\|
\]
\[
+ \alpha_n \|f_1 (y^*) - x^*\|, \tag{43}
\]
where \(h = \max\{h_1, h_2, h_3\}.\) Similarly, we can also compute that
\[
\|y_{n+1} - y^*\| \leq (1 - \alpha_n) \|y_n - y^*\| + \alpha_n h \|z_n - z^*\|
\]
\[
+ \alpha_n \|f_2 (z^*) - y^*\|,
\]
\[
\|z_{n+1} - z^*\| \leq (1 - \alpha_n) \|z_n - z^*\| + \alpha_n h \|x_n - x^*\|
\]
\[
+ \alpha_n \|f_3 (x^*) - z^*\|. \tag{44}
\]
This implies that
\[
\|x_{n+1} - x^*\| + \|y_{n+1} - y^*\| + \|z_{n+1} - z^*\|
\]
\[
\leq (1 - \alpha_n) (1 - h) (\|x_n - x^*\| + \|y_n - y^*\| + \|z_n - z^*\|)
\]
\[
+ \alpha_n (1 - h) \|f_1 (y^*) - x^*\| + \|f_2 (z^*) - y^*\| + \|f_3 (x^*) - z^*\| \tag{45}
\]
\[
\leq \max \left\{ \|x_n - x^*\| + \|y_n - y^*\| + \|z_n - z^*\|, \right.
\]
\[
\lim (\|f_1 (y^*) - x^*\| + \|f_2 (z^*) - y^*\| + \|f_3 (x^*) - z^*\|) \times (1 - h)^{-1}\right\}.
\]
By induction, we have
\[
\|x_{n+1} - x^*\| + \|y_{n+1} - y^*\| + \|z_{n+1} - z^*\|
\]
\[
\leq \max \left\{ \|x_0 - x^*\| + \|y_0 - y^*\| + \|z_0 - z^*\|, \right.
\]
\[
\lim (\|f_1 (y^*) - x^*\| + \|f_2 (z^*) - y^*\| + \|f_3 (x^*) - z^*\|) \times (1 - h)^{-1}\right\}, \tag{46}
\]
for all \(n \geq 1.\)

Hence \(\{x_n\}, \{y_n\},\) and \(\{z_n\}\) are bounded. Consequently, \(\{K_{1,\beta} x_n\}, \{K_{2,\beta} y_n\},\) and \(\{K_{3,\beta} z_n\}\) are bounded.

(ii) Next we prove that for each \(n \geq 1\) the following inequality holds:
\[
\|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2 + \|z_{n+1} - z^*\|^2
\]
\[
\leq (1 - \alpha_n)^2 (\|x_n - x^*\|^2 + \|y_n - y^*\|^2 + \|z_n - z^*\|^2)
\]
\[
+ 2 \alpha_n h (\|x_n - x^*\| \|y_n - y^*\| + \|y_{n+1} - y^*\| \|z_n - z^*\| \|x_n - x^*\|)
\]
\[
+ 2 \alpha_n (\langle f_1 (y^*) - x^*, x_{n+1} - x^* \rangle
\]
\[
+ \langle f_2 (z^*) - y^*, y_{n+1} - y^* \rangle
\]
\[
+ \langle f_3 (x^*) - z^*, z_{n+1} - z^* \rangle). \tag{47}
\]
Similarly, we can also prove that
\[
\|y_{n+1} - y^*\|^2 \leq (1 - \alpha_n)^2 \|y_n - y^*\|^2 \\
+ 2\alpha_n \langle f_1 (K_{2, \beta} y_n) - f_1 (y^*) \rangle \\
+ 2\alpha_n \langle f_2 (z^*) - y^*, y_{n+1} - y^* \rangle,
\]
(49)

Adding up inequalities (48) and (49), inequality (47) is proved.

(iii) Next, we prove that if there exists a subsequence 
\( \{n_k\} \subset \{n\} \) such that
\[
\liminf_{k \to \infty} \left( \|x_{n_{k+1}} - x^*\|^2 + \|y_{n_{k+1}} - y^*\|^2 + \|z_{n_{k+1}} - z^*\|^2 \right) \\
- \left( \|x_{n_k} - x^*\|^2 + \|y_{n_k} - y^*\|^2 + \|z_{n_k} - z^*\|^2 \right) \geq 0,
\]
(50)

then
\[
\limsup_{k \to \infty} \left\{ \langle f_1 (y^*) - x^*, x_{n_{k+1}} - x^* \rangle \\
+ \langle f_2 (z^*) - y^*, y_{n_{k+1}} - y^* \rangle \\
+ \langle f_3 (x^*) - z^*, z_{n_{k+1}} - z^* \rangle \right\} \leq 0.
\]
(51)

Since the norm \( \cdot \|^2 \) is convex and \( \lim_{n \to \infty} \alpha_n = 0 \), by (42), we have
\[
0 \leq \liminf_{k \to \infty} \left\{ \|x_{n_{k+1}} - x^*\|^2 + \|y_{n_{k+1}} - y^*\|^2 \\
+ \|z_{n_{k+1}} - z^*\|^2 \right\} \\
- \left( \|x_{n_k} - x^*\|^2 + \|y_{n_k} - y^*\|^2 + \|z_{n_k} - z^*\|^2 \right) \]
(52)

\[
= \liminf_{k \to \infty} \left\{ \|K_{1, \beta} x_{n_k} - x^*\|^2 - \|x_{n_k} - x^*\|^2 \right\} \\
+ \left( \|K_{2, \beta} y_{n_k} - y^*\|^2 - \|y_{n_k} - y^*\|^2 \right) \\
+ \left( \|K_{3, \beta} z_{n_k} - z^*\|^2 - \|z_{n_k} - z^*\|^2 \right) \leq 0.
\]

This implies that
\[
\lim_{k \to \infty} \left( \|K_{1, \beta} x_{n_k} - x^*\|^2 - \|x_{n_k} - x^*\|^2 \right) \\
= \lim_{k \to \infty} \left( \|K_{2, \beta} y_{n_k} - y^*\|^2 - \|y_{n_k} - y^*\|^2 \right) \\
= \lim_{k \to \infty} \left( \|K_{3, \beta} z_{n_k} - z^*\|^2 - \|z_{n_k} - z^*\|^2 \right) = 0.
\]
(53)
Since the sequences \( \{\|K_{1,\beta}x_{n_k} - x^*\| + \|x_{n_k} - x^*\|\} \), \( \{\|K_{2,\beta}y_{n_k} - y^*\| + \|y_{n_k} - y^*\|\} \), and \( \{\|K_{3,\beta}z_{n_k} - z^*\| + \|z_{n_k} - z^*\|\} \) are bounded, we have

\[
\lim_{k \to \infty} \left( \|K_{1,\beta}x_{n_k} - x^*\| - \|x_{n_k} - x^*\| \right) = \lim_{k \to \infty} \left( \|K_{2,\beta}y_{n_k} - y^*\| - \|y_{n_k} - y^*\| \right) = \lim_{k \to \infty} \left( \|K_{3,\beta}z_{n_k} - z^*\| - \|z_{n_k} - z^*\| \right) = 0.
\]

By Lemma 9, \( K_{1,\beta}, K_{2,\beta}, \) and \( K_{3,\beta} \) are strongly quasinonexpansive. We have

\[
K_{1,\beta}x_{n_k} - x_{n_k} \to 0, \quad K_{2,\beta}y_{n_k} - y_{n_k} \to 0, \quad K_{3,\beta}z_{n_k} - z_{n_k} \to 0.
\]

Consequently, we obtain that

\[
x_{n_k} - x_{n_k+1} \to 0, \quad y_{n_k} - y_{n_k+1} \to 0, \quad z_{n_k} - z_{n_k+1} \to 0.
\]

It follows from the boundedness of \( \{x_{n_k}\} \) and \( H \) which is reflexive that there exists a subsequence \( \{x_{n_{i_k}}\} \) of \( \{x_{n_k}\} \) such that \( x_{n_{i_k}} \to p \) and

\[
\lim_{l \to \infty} \left( f_1(y^*) - x^*, x_{n_l} - x^* \right) = \lim_{k \to \infty} \left( f_1(y^*) - x^*, x_{n_k} - x^* \right) = \lim_{k \to \infty} \left( f_1(y^*) - x^*, x_{n_k+1} - x^* \right).
\]

By Lemma 9, \( I - K_{1,\beta} \) is demiclosed at zero, and so \( p \in F(K_{1,\beta}) = \Omega_1 \). Hence from (36) we have

\[
\lim_{l \to \infty} \left( f_1(y^*) - x^*, x_{n_l} - x^* \right) = \left( f_1(y^*) - x^*, p - x^* \right) \leq 0.
\]

Therefore

\[
\lim_{k \to \infty} \left( f_1(y^*) - x^*, x_{n_k+1} - x^* \right) = \lim_{l \to \infty} \left( f_1(y^*) - x^*, x_{n_l} - x^* \right) \leq 0.
\]

Similarly, we can also prove that

\[
\lim_{k \to \infty} \left( f_2(z^*) - y^*, y_{n_k} - y^* \right) \leq 0,
\]

\[
\lim_{k \to \infty} \left( f_3(x^*) - z^*, z_{n_k} - z^* \right) \leq 0.
\]

Hence, we have the desired inequality.

(iv) Finally, we prove that the sequences \( \{x_{n_k}\}, \{y_{n_k}\} \), and \( \{z_{n_k}\} \) generated to be (42) converge to \( x^*, y^* \), and \( z^* \), respectively.

It is clear that

\[
\|x_{n+1} - x^*\| + \|y_{n+1} - y^*\| + \|z_{n+1} - z^*\| \leq \left( \|x_n - x^*\|^2 + \|y_n - y^*\|^2 + \|z_n - z^*\|^2 \right)^{1/2}  
\]

\[
\times \left( \|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2 + \|z_{n+1} - z^*\|^2 \right)^{1/2}.
\]

Substituting (61) into (47), we have

\[
\|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2 + \|z_{n+1} - z^*\|^2  
\]

\[
\leq (1 - \alpha_n)^2 \left( \|x_n - x^*\|^2 + \|y_n - y^*\|^2 + \|z_n - z^*\|^2 \right) + 2\alpha_n h \left( \|x_n - x^*\|^2 + \|y_n - y^*\|^2 + \|z_n - z^*\|^2 \right)  
\]

\[
\times \left( \|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2 + \|z_{n+1} - z^*\|^2 \right)^{1/2}  
\]

\[
+ 2\alpha_n (\left( f_1(y^*) - x^*, x_{n+1} - x^* \right)  
\]

\[
+ \left( f_2(z^*) - y^*, y_{n+1} - y^* \right)  
\]

\[
+ \left( f_3(x^*) - z^*, z_{n+1} - z^* \right).
\]

Set

\[
a_n := \|x_n - x^*\|^2 + \|y_n - y^*\|^2 + \|z_n - z^*\|^2,  
\]

\[
b_n := 2 \left( \left( f_1(y^*) - x^*, x_{n+1} - x^* \right)  
\]

\[
+ \left( f_2(z^*) - y^*, y_{n+1} - y^* \right)  
\]

\[
+ \left( f_3(x^*) - z^*, z_{n+1} - z^* \right).
\]

Then, we have the following statements.

(i) From (i), \( \{a_n\} \) is bounded sequence.

(ii) From (62), \( \alpha_{n+1} \leq (1 - \alpha_n)^2 a_n + 2\alpha_n h \sqrt{a_n} \sqrt{a_n} + \alpha_n b_n, \) for all \( n \geq 1 \).

(iii) From (iii), whenever \( \{a_{n_k}\} \) is a subsequence of \( \{a_n\} \) satisfying

\[
\lim_{k \to \infty} (a_{n_{k+1}} - a_{n_k}) \geq 0,
\]

it follows that \( \limsup_{k \to \infty} b_{n_k} \leq 0 \).

By Lemma 8, we have

\[
\lim_{n \to \infty} (\|x_n - x^*\|^2 + \|y_n - y^*\|^2 + \|z_n - z^*\|^2) = 0.
\]
Hence, we obtain that
\[
\lim_{n \to \infty} \|x_n - x^*\| = \lim_{n \to \infty} \|y_n - y^*\| = \lim_{n \to \infty} \|z_n - z^*\| = 0.
\] (66)
This completes the proof.

3.3. Consequence Results. Using Theorem II, we can prove the following results.

**Theorem 12.** Let \( A_1, M_1, \Omega_1, K_1, \) and \( K_{1,\beta} \) satisfy conditions (C1) and (C2), and let \( F : H \to H \) be a \( \mu \)-Lipschitzian and \( r \)-strongly monotone mapping. Let \( \{x_n\}, \{y_n\}, \) and \( \{z_n\} \) be three sequences defined by
\[
x_0, y_0, z_0 \in H,
\]
x_{n+1} = (1 - \alpha_n) K_{1,\beta} x_n + \alpha_n f_1 (K_{2,\beta} y_n),
y_{n+1} = (1 - \alpha_n) K_{2,\beta} y_n + \alpha_n f_2 (K_{3,\beta} z_n),
z_{n+1} = (1 - \alpha_n) K_{3,\beta} z_n + \alpha_n f_3 (K_{1,\beta} x_n),
\]
where \( f_1 := I - \rho F, f_2 := I - \eta F, f_3 := I - \xi F \) with \( \rho, \eta, \xi \in (0, 2r/\mu^2) \), and \( \{\alpha_n\} \) is a sequence in \( (0, 1) \) satisfying \( \alpha_n \to 0 \) and \( \sum_{n=0}^{\infty} \alpha_n = \infty \). Then the sequences \( \{x_n\}, \{y_n\}, \) and \( \{z_n\} \) converge to \( x^*, y^*, \) and \( z^* \), respectively, where \( (x^*, y^*, z^*) \) is the unique element in \( \Omega_1 \times \Omega_2 \times \Omega_3 \) such that the following three inequalities are satisfied:
\[
\langle \rho F (y^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in \Omega_1,
\]
\[
\langle \eta F (z^*) + y^* - z^*, y - y^* \rangle \geq 0, \quad \forall y \in \Omega_2,
\]
\[
\langle \xi F (x^*) + z^* - x^*, z - z^* \rangle \geq 0, \quad \forall z \in \Omega_3.
\] (67)

**Proof.** It is easy to see that \( f_1, f_2, \) and \( f_3 \) are contraction mappings and all the conditions in Theorem II are satisfied. By Theorem II, we have the sequences \( \{x_n\}, \{y_n\}, \) and \( \{z_n\} \) which converge to \( (x^*, y^*, z^*) \in \Omega_1 \times \Omega_2 \times \Omega_3 \) such that the following three inequalities are satisfied:
\[
\langle x^* - f_1 (y^*), x - x^* \rangle \geq 0, \quad \forall x \in \Omega_1,
\]
\[
\langle y^* - f_2 (z^*), y - y^* \rangle \geq 0, \quad \forall y \in \Omega_2,
\]
\[
\langle z^* - f_3 (x^*), z - z^* \rangle \geq 0, \quad \forall z \in \Omega_3.
\] (68)
Substituting \( f_1 := I - \rho F, f_2 := I - \eta F, \) and \( f_3 := I - \xi F \) into (68), we obtain that the sequences \( \{x_n\}, \{y_n\}, \) and \( \{z_n\} \) converge to \( (x^*, y^*, z^*) \in \Omega_1 \times \Omega_2 \times \Omega_3 \) such that the following three inequalities are satisfied:
\[
\langle \rho F (y^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in \Omega_1,
\]
\[
\langle \eta F (z^*) + y^* - z^*, y - y^* \rangle \geq 0, \quad \forall y \in \Omega_2,
\]
\[
\langle \xi F (x^*) + z^* - x^*, z - z^* \rangle \geq 0, \quad \forall z \in \Omega_3.
\] (69)
This completes the proof.

Setting \( A_1 = A_2 = A_3 \), \( f_1 = f_2 = f_3 \), and \( x_0 = y_0 = z_0 \) in Theorem II, we obtain the following corollary.

**Corollary 13.** Let \( A_1, M_1, \Omega_1, K_1, \) and \( K_{1,\beta} \) satisfy conditions (C1) and (C2), and let \( f_1 : H \to H \) be contractive with a contractive constant \( h_i \in (0, 1) \) for all \( i = 1, 2, 3 \). Let \( \{x_n\}, \{y_n\}, \) and \( \{z_n\} \) be three sequences defined by
\[
x_{n+1} = (1 - \alpha_n) K_{1,\beta} x_n + \alpha_n f_1 (K_{1,\beta} y_n),
y_{n+1} = (1 - \alpha_n) K_{1,\beta} y_n + \alpha_n f_2 (K_{1,\beta} z_n),
z_{n+1} = (1 - \alpha_n) K_{1,\beta} z_n + \alpha_n f_3 (K_{1,\beta} x_n),
\]
\[
\tag{71}

\]
where \( \{\alpha_n\} \) is a sequence in \( (0, 1) \) satisfying \( \alpha_n \to 0 \) and \( \sum_{n=0}^{\infty} \alpha_n = \infty \). Then the sequences \( \{x_n\}, \{y_n\}, \) and \( \{z_n\} \) generated to be (42) converge to \( x^*, y^*, \) and \( z^* \), respectively, where \( (x^*, y^*, z^*) \) is the unique element in \( \Omega_1 \times \Omega_2 \times \Omega_3 \) such that the following three inequalities are satisfied:
\[
\langle x^* - f_1 (y^*), x - x^* \rangle \geq 0, \quad \forall x \in \Omega_1,
\]
\[
\langle y^* - f_2 (z^*), y - y^* \rangle \geq 0, \quad \forall y \in \Omega_1,
\]
\[
\langle z^* - f_3 (x^*), z - z^* \rangle \geq 0, \quad \forall z \in \Omega_1.
\] (72)

**Corollary 14.** Let \( A_1, M_1, \Omega_1, K_1, \) and \( K_{1,\beta} \) satisfy conditions (C1) and (C2), and let \( F : H \to H \) be \( \mu \)-Lipschitzian and \( r \)-strongly monotone mapping. Let \( \{x_n\}, \{y_n\}, \) and \( \{z_n\} \) be three sequences defined by
\[
x_0, y_0, z_0 \in H,
\]
x_{n+1} = (1 - \alpha_n) K_{1,\beta} x_n + \alpha_n f_1 (K_{1,\beta} y_n),
y_{n+1} = (1 - \alpha_n) K_{1,\beta} y_n + \alpha_n f_2 (K_{1,\beta} z_n),
z_{n+1} = (1 - \alpha_n) K_{1,\beta} z_n + \alpha_n f_3 (K_{1,\beta} x_n),
\]
\[
\tag{73}

\]
where \( f_1 := I - \rho F, f_2 := I - \eta F, f_3 := I - \xi F \) with \( \rho, \eta, \xi \in (0, 2r/\mu^2) \), and \( \{\alpha_n\} \) is a sequence in \( (0, 1) \) satisfying \( \alpha_n \to 0 \) and \( \sum_{n=0}^{\infty} \alpha_n = \infty \). Then the sequences \( \{x_n\}, \{y_n\}, \) and \( \{z_n\} \) converge to \( x^*, y^*, \) and \( z^* \), respectively, where \( (x^*, y^*, z^*) \) is the unique element in \( \Omega_1 \times \Omega_2 \times \Omega_3 \) such that the following three inequalities are satisfied:
\[
\langle \rho F (y^*) + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in \Omega_1,
\]
\[
\langle \eta F (z^*) + y^* - z^*, y - y^* \rangle \geq 0, \quad \forall y \in \Omega_1,
\]
\[
\langle \xi F (x^*) + z^* - x^*, z - z^* \rangle \geq 0, \quad \forall z \in \Omega_1.
\] (74)

Setting \( A_1 = A_2 = A_3, f_1 = f_2 = f_3, \) and \( x_0 = y_0 = z_0 \) in Theorem II, we obtain the following corollary.
Corollary 15. Let $A_1, M_1, \Omega_1, K_1$, and $K_{1, \beta}$ satisfy conditions (C1) and (C2), and let $f : H \rightarrow H$ be contractions with a contractive constant $h \in (0, 1)$. Let $\{x_n\}$ be the sequences defined by
\[
x_0 \in H, \quad x_{n+1} = (1 - \alpha_n) K_{1, \beta} x_n + \alpha_n f (K_{1, \beta} x_n),
\]
where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying $\alpha_n \rightarrow 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the sequences $\{x_n\}$ converge to $x^* \in \Omega_1$ such that the following three inequalities are satisfied:
\[
\langle x^* - f_1 (x^*), x - x^* \rangle \geq 0, \quad \forall x \in \Omega_1.
\]

Corollary 16. Let $A_1, M_1, \Omega_1, K_1$, and $K_{1, \beta}$ satisfy conditions (C1) and (C2), and let $F : H \rightarrow H$ be $\mu$-Lipschitzian and $\strongly$-monotone mapping. Let $\{x_n\}$ be the sequences defined by
\[
x_0 \in H, \quad x_{n+1} = (1 - \alpha_n) K_{1, \beta} x_n + \alpha_n (I - \rho F) (K_{1, \beta} x_n),
\]
where $\rho \in (0, 2\mu^2)$ and $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying $\alpha_n \rightarrow 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the sequences $\{x_n\}$ converge to $x^* \in \Omega_1$ such that the following three inequalities are satisfied:
\[
\langle F (x^*), x - x^* \rangle \geq 0, \quad \forall x \in \Omega_1.
\]

Conflict of Interests

The authors declare that there is no conflict of interests regarding to the publication of this paper.

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References


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