Multi-Unit Auctions: Beyond Roberts

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Abstract

We exhibit incentive compatible multi-unit auctions that are not affine maximizers (i.e. are not of the VCG family) and yet approximate the social welfare to within a factor of $1 + \epsilon$. For the case of two-item two-bidder auctions we show that this family of auctions, termed triage mechanisms, are the only scalable ones that give an approximation factor better than 2. “Scalable” means that the allocation does not depend on the units in which the valuations are measured. We deduce from this that any scalable computationally-efficient incentive-compatible auction for $m$ items and $n \geq 2$ bidders cannot approximate the social welfare to within a factor better than 2. This is in contrast to arbitrarily good approximations that can be reached under computational constraints alone, and in contrast the optimal social welfare that can be obtained under incentive constraints alone.

1 Introduction

1.1 Background

The field of Algorithmic Mechanism Design [25] designs mechanisms for achieving various computational goals, under the assumption of rational selfishness of the involved parties. The notions used are taken from the economic field of Mechanism Design, and a basic notion is that of incentive-compatibility, or, equivalently, truthfulness – where rational players are motivated to act truthfully. For background and survey see part II of [26]. This paper will consider only the simplest and most robust notion of incentive compatibility, that of dominant strategies in quasi-linear settings with independent private values. The typical question in the field asks for a computationally-efficient incentive compatible mechanism that implements a certain type of outcome, usually the approximate optimization of some target “social” goal. There are two variants of this challenge, the first considers situations where incentive compatibility itself is hard to achieve and the computational efficiency is just an additional burden, with the prime example being approximate minimization of the makespan in scheduling problems [25]. The second variant focuses on cases where each of the two constraints of incentive compatibility and computational efficiency can be achieved separately, and the challenge is to get them simultaneously, with the prime example being approximate welfare maximization in various types of combinatorial auctions [22].

While there has been much work and some progress on these types of challenges, with particular emphasis on the problems mentioned above of combinatorial auctions (e.g., [20, 18, 3, 14, 21, 15, 12, 6]) and scheduling (e.g., [7, 19, 2]), the basic challenge remains mostly unanswered. As noted in [20], the main issue turns out to be the richness of the domain of player’s valuations, i.e. of their private information. On one extreme are single-dimensional domains where the private information of

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each participant is captured by a scalar (or domains very close to it, e.g., [22]). For these types of problems, incentive-compatible mechanisms are well characterized by a certain monotonicity condition and, in most cases, the challenge of reconciling incentives with computational efficiency has been met [22] [15] [9] [8]. On the other extreme are problems which are “fully dimensional” (or close to fully dimensional, e.g., [27] [16]) where there is no structure on valuations, in which case a key theorem of Roberts [28] characterizes incentive compatible mechanisms as “affine maximizers” “on a sub-range” – simple generalizations of the VCG mechanism. While such affine maximizers on a sub-range are not completely powerless in polynomial time, in most cases this characterization implies impossibility of good computationally efficient incentive compatible mechanisms. Most interesting problems, including those mentioned above, lie in an intermediate range where the valuation spaces are neither single dimensional not fully dimensional, a range for which very little is known. The main problem seems to be the lack of a good characterization of incentive compatibility in these intermediate ranges. In particular, the key unknown is whether any useful incentive compatible non-VCG mechanisms exist in the intermediate range.

1.2 Multi-unit auctions

As mentioned, the paradigmatic problems for the reconciliation of computational constraints with incentive constraints are the various subclasses of combinatorial auctions. In this paper we consider the simplest variant which exhibits this tension: multi-unit auctions. In this problem there are \( m \) identical items for sale among \( n \) bidders, where each bidder \( i \) has a valuation function \( v_i : \{0, \ldots, m\} \rightarrow \mathbb{R} \), where \( v_i(k) \) denotes player \( i \)'s value for receiving \( k \) items. The valuations \( v_i \) are assumed to be monotone non-decreasing (free disposal) with \( v_i(0) = 0 \) (normalization). Key and implicit here is that there are no externalities: the value of bidder \( i \) depends only on what he gets rather than also on the allocation to the others. The optimization goal is to find an allocation of items to the bidders, where bidder \( i \) gets \( s_i \) items, with \( \sum_i s_i \leq m \), that maximizes social welfare \( \sum_i v_i(s_i) \).

The problem becomes computationally challenging when the number of items \( m \) is “large”, i.e. when the running time of the mechanism is not allowed to be polynomial in \( m \) but rather just in \( \log m \). There are two variant models in this case, the first assumes that the valuation functions are given as “black boxes” that the algorithm may query, and the second assumes that the valuation functions are given in some succinct bidding language. Finding the optimal allocation is essentially a knapsack problem and is computationally hard in both models: in the black-box model it requires exponentially many queries, and in the succinct representation model, it is NP-hard. Just like Knapsack, the optimal social welfare can be approximated arbitrarily well (in both models) and has an FPTAS: approximation ratio of \( 1 + \epsilon \) obtained in time that is polynomial in \( n \), \( \log m \), and \( \epsilon^{-1} \). This FPTAS does not imply any incentive compatible approximation though, and the question boils down to what degree of approximation can be obtained in an incentive compatible way in polynomial time.

Already in Vickrey’s seminal paper [30] multi-unit auctions were considered, restricted to the case of downward sloping valuations, i.e. \( v_i(k+1) - v_i(k) \leq v_i(k) - v_i(k-1) \) for all \( 0 < k < m \). For this case the optimal allocation can be found efficiently, as an “equilibrium price” exists, which can be found by binary search (together with the optimal allocation it implies), and attaching the Vickrey payments – the point of his paper – gives incentive compatibility. For general valuations the exact optimum is computationally hard to achieve, so further work considered approximations. The single-dimensional “single minded” case was shown to have an incentive compatible FPTAS [5], improving an earlier 2-approximation [24]. The general case was studied in [13] where an incentive-compatible

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1 The “weak monotonicity” [20] [14] [29] characterization is from the point of view of a single player and is not specific enough to be useful in this regard.

2 With a single positive exception for certain multi-unit combinatorial auctions [9].

3 The usual query assumed is a “value query”, asking for \( v_i(k) \) for some \( k \), but most lower bounds hold for any queries, as they apply in the communication complexity model.
2-approximation was obtained using a maximal-in-range VCG mechanism (with better approximations for some restricted bidding languages). It was also shown there that no computationally-efficient maximum in range VCG mechanism can achieve a better approximation ratio, and the main open problem left was whether other non-VCG incentive compatible mechanisms exist that may get a better approximation ratio. Previous results showed that incentive compatible mechanisms between two players that always allocate all items cannot do so. It should be emphasized that the question regards deterministic mechanisms, as a randomized FPTAS was obtained.

1.3 Our Results

Our first result is a family of non-VCG mechanisms for multi-unit auctions that can achieve a $1 + \epsilon$ approximation for any $\epsilon > 0$, for two players.

**Theorem:** For every $\epsilon > 0$ there exists an incentive compatible mechanism for multi-unit auctions of $m$ items between two players with approximation ratio $1 + \epsilon$ which is not an affine maximizer.

We call these mechanisms *Triage mechanisms* as they split the valuation domain into three sub-domains, depending on the ratios $v(1)/v(m)$ and $v(m-1)/v(m)$. Their payment structure mimics VCG prices with two exceptions: in the “low sub-domain”, the price for a single item is decreased to a specific fraction of the value of all items, and in the “high sub-domain”, the payments of all non-empty bundles are increased, by the same amount, in a specific linear way. This family of mechanisms is parameterized by three parameters (specifying a weight and the “high thresholds” for both players), with all other parameters uniquely determined by them. We also exhibit two other families of finitely approximating incentive compatible mechanisms, but their approximation factor is worse.

Our next, and main, result shows that these Triage mechanisms are the only scalable incentive compatible mechanisms with a good approximation ratio for the case of $m = 2$ items and $n = 2$ players. Scalability means that the auction’s allocation rule does not depend on the “units” in which the valuations are given: multiplying all valuations by the same positive constant does not change the allocation. The Triage mechanisms and the other mechanisms mentioned above are all scalable.

**Theorem:** A scalable incentive compatible mechanism that achieves a $c$-approximation, for $c < 2$, of the optimal social welfare in a multi-unit auction of two items among two bidders, must be a Triage mechanism for some choice of parameters.

This is the first characterization of incentive compatibility in an auction domain or, more generally, in a domain with no externalities. Triage mechanisms are affine maximizers on the “middle sub-domain” and we show that this extends to auctions of an arbitrary number of items among two players.

**Theorem:** A scalable incentive compatible mechanism that achieves a $c$-approximation, for $c < 2$, of the optimal social welfare in a multi-unit auction of $m > 2$ items among $n = 2$ bidders, must be identical to an affine maximizer with VCG payments, on the sub-domain where $v_i(1) = 0$ and $v_i(m-1) = v_i(m-2)$ for both players $i$.

Interestingly, this theorem is not proved by direct characterization, but rather uses the 2-items...
characterization as a black box. The theorem immediately implies computational hardness, a first of a kind result for an auction domain:

**Theorem:** Fix a model of computation where finding the exact social-welfare maximizing allocation of $m$ items between two players is computationally hard, even with valuations restricted to $v_i(1) = 0$ and $v_i(m-1) = v_i(m-2)$. Then, getting a scalable incentive compatible $c$-approximation, for $c < 2$, of the social welfare in a multi-unit auction of $m$ items among any $n \geq 2$ players, is also computationally hard.

This implies an exponential lower bound on communication in the black-box model \cite{23} and implies NP-hardness in the succinct representation model, with, e.g., the bidding language allowing valuations to be specified by boolean circuits \cite{20}.

The main open problem is to get rid of the scalability assumption which we believe is not really necessary for all our theorems. We note that our reduction to the two item case from an arbitrary number of items does not require scalability, so the hurdle is really just in the characterization of the two-item two-bidder multi-unit auctions. The fixed small size would perhaps suggest a direct attack, perhaps even a computer-assisted one, but obviously we were not able to do so.

**Organization**

In Section 2 the setting and basic definitions are given. Section 3 discusses the new mechanisms for multi-unit auctions: the triage mechanism, and two additional ones. Section 4 shows that all truthful and scalable mechanisms for multi-unit auctions with 2 items and 2 bidders are triage mechanisms. Finally, Section 5 provides a characterization of mechanisms for any number of items.

## 2 Preliminaries

### 2.1 The Setting

In a multi-unit auction there is a set of $m$ identical items, and a set $N = \{1, 2, \ldots, n\}$ of bidders. Each bidder $i$ has a valuation function $v_i : [m] \to \mathbb{R}^+$, which is normalized ($v_i(0) = 0$) and non-decreasing. Denote by $V$ the set of all normalized non-decreasing valuations. An allocation of the items $\vec{s} = (s_1, \ldots, s_n)$ is a vector of non-negative integers such that $\sum s_i \leq m$. Denote the set of allocations by $S$. The goal is to find an allocation that maximizes the welfare: $\sum v_i(s_i)$.

In most of this paper we concentrate in the case where $n = 2$. For convenience, we name the bidders Alice and Bob. We usually denote Alice’s valuation by $v$, and Bob’s by $u$.

### 2.2 Truthfulness

The reader is referred to \cite{24} for the (standard) proofs missing in this subsection. An $n$-bidder mechanism for multi-unit auctions is a pair $(A, p)$ where $A : V^n \to S$ and $p = (p^{(1)}, \ldots, p^{(n)})$, where for each $i$, $p^{(i)} : V^n \to \mathbb{R}$.

**Definition 2.1** Let $(A, p)$ be a mechanism. $(A, p)$ is truthful if for all $i$, all $v_i, v'_i$ and all $v_{-i}$ we have that $v_i(A(v_i, v_{-i})) - p^{(i)}(v_i, v_{-i}) \geq v_i(A(v'_i, v_{-i})) - p^{(i)}(v'_i, v_{-i})$.

It is well known that an algorithm (to multi-unit auctions) is truthful if and only if each bidder is presented with a payment for each bundle $t$ that does not depend on bidder $i$’s valuation (i.e., $p^{(i)} : V^{n-1} \to \mathbb{R}$). Denote this payment by $p^{(i)}(v_{-i})$. Each bidder takes a bundle that maximizes his profit: $v_i(t) - p^{(i)}(v_{-i})$ (this is called the “taxation principle” – we will sometimes say that these payments are *induced* by $v_{-i}$). We note that we may assume without loss of generality that for
t > t’, \( p_t^{(1)}(v) \geq p_{t'}^{(1)}(v) \) (“payment monotonicity”): otherwise, we have a mechanism with the same allocation rule by using \( p_t^{(1)}(v) = p_{t'}^{(1)}(v) \) and the appropriate tie-breaking between bundles \( t \) and \( t' \) when \( u(t) = u(t') \).

**Definition 2.2** A is an affine maximizer if there exist a set of allocations \( \mathcal{R} \), a constant \( \alpha_i \geq 0 \) for each \( i \in N \), and a constant \( \beta \in \mathbb{R} \) for each \( s \in S \) such that \( A(v_1, \ldots, v_n) \in \arg\max_{\vec{s} \in (s_1, \ldots, s_n) \in \mathcal{R}} (\Sigma_i (\alpha_i v_i(s_i)) + \beta_s) \). A is called welfare maximizer if \( \beta_s = 0 \) for each \( s \in \mathcal{R} \) and \( \mathcal{R} \) equals the set of all allocations.

The following proposition is standard:

**Proposition 2.3** Let \( A \) be an affine maximizer (in particular, welfare maximizer). There are payments \( p \) such that \( (A, p) \) is a truthful mechanism.

We now specify the payments of a welfare maximizer for the \( n = 2 \) case: there is a constant \( w > 0 \) such that for each \( t \), a valuation \( v \) of Alice and a valuation \( u \) of Bob, \( p_t^{(2)}(v) = w(v(m) - v(m - t)) \) and \( p_t^{(1)}(u) = (u(m) - u(m - t))/w \).

We sometimes use a table notation to denote a 2-items instance. We illustrate this notation below for the case of a welfare maximizer:

<table>
<thead>
<tr>
<th>Number of items</th>
<th>Alice’s value</th>
<th>Alice’s payment</th>
<th>Bob’s value</th>
<th>Bob’s payment</th>
</tr>
</thead>
<tbody>
<tr>
<td>One</td>
<td>( v(1) )</td>
<td>((u(2) - u(1))/w )</td>
<td>( u(1) )</td>
<td>( w(v(2) - v(1)) )</td>
</tr>
<tr>
<td>Both</td>
<td>( v(2) )</td>
<td>( u(2)/w )</td>
<td>( u(2) )</td>
<td>( w \cdot v(2) )</td>
</tr>
</tbody>
</table>

As discussed above, Bob’s payments depend only on Alice’s value and similarly Alice’s payments depend only on Bob’s value.

### 2.3 Scalability

In this paper we consider two definitions of scalability.

**Definition 2.4** An auction is allocation scalable if multiplying the valuations of all bidders by the same positive factor does not change the allocation.

**Definition 2.5** An auction is payment scalable if for each bidder \( i \), valuations of the other bidders \( v_{-i} \), and \( \alpha > 0 \), \( \alpha \cdot p^{(i)}(v_{-i}) = p^{(i)}(\alpha \cdot v_{-i}) \).

The next proposition shows that every allocation scalable mechanism is also payment scalable, and thus in this paper we use the term *scalable* to denote the less restrictive notion of scalability – payment scalability.

**Proposition 2.6** Let \( A \) be an allocation scalable mechanism. Then, \( A \) is also payment scalable.

**Proof:** We prove the lemma for the case of \( n = 2 \) but the proof easily extends to \( n > 2 \) bidders. Fix a valuation \( u \) of Bob. Let \( B_t(u) \) be the set of all valuations \( v \) that assign Alice \( t \) items in input \((v, u)\). Formally, \( B_t(u) = \{v | A_1(v, u) = t \} \). We say that \( t \) is in the range of \( u \) if \( B_t \neq \emptyset \).

We claim that \( t \) is in the range of \( u \) if and only if \( t \) is in the range of \( \alpha \cdot u \). To see that, consider \( t \) that is in the range of \( u \). We have that \( t \) is also in the range of \( \alpha \cdot u \) since \( A(v, u) = A(\alpha \cdot v, \alpha \cdot u) \). The 'only if' direction is symmetric.
We now show that for every \( t, t' \) in the range of \( u \) and \( \alpha > 0 \), \( \alpha (p_t^{(i)}(u) - p_t^{(i)}(u)) = p_t^{(i)}(\alpha \cdot u) - p_t^{(i)}(\alpha \cdot u) \) (for bundles not in the range we set the payment to be equal to the payment of the next bigger bundle that is in the range, and use the appropriate tie-breaking rule). Fix some \( t \) in the range of \( u \) such that there exists \( v \in B_t(u) \) that is on the border of \( B_t(u) \) (in the usual topological sense). If there is no such point, Alice is always assigned \( t \) and we are done by letting \( p_t^{(i)}(\alpha \cdot u) = 0 \) for every \( \alpha \neq 0 \). Assume otherwise. There exists at least one \( t' \neq t \) which is in every \( \epsilon \)-neighborhood of \( v \) and in \( B_{t'} \), since \( v \) is on the border. Thus we have \( p_t^{(i)}(u) - p_{t'}^{(i)}(u) = v(t) - v(t') \). From scalability, we have that \( \alpha \cdot v \) is on the border of \( B_t(u) \) with \( t' \) playing the same role. We have that \( p_t^{(i)}(\alpha \cdot u) - p_{\alpha \cdot v}^{(i)}(u) = \alpha (p_t^{(i)}(u) - p_{t'}^{(i)}(u)) \).

We continue similarly. Fix \( t'' \neq t, t' \) where there exists \( v \in B_{t''}(u) \), and \( v \) is on the border of \( B_t \cup B_{t'} \).

Thus, in every \( \epsilon \)-neighborhood of \( v \) there exists a valuation \( v' \) for which \( v' \in B_t \) or \( v' \in B_{t''} \). Without loss of generality assume that \( v' \in B_t \) (otherwise, switch the roles of \( t \) and \( t' \)). By using scalability similarly to the arguments above, we get that \( v(t) - v(t'') = p_t^{(i)}(u) - p_{v'}^{(i)}(u) \), and consequently we also have \( \alpha (v(t) - v(t'')) = p_t^{(i)}(\alpha \cdot u) - p_{\alpha \cdot v'}^{(i)}(\alpha \cdot u) = \alpha (p_t^{(i)}(u) - p_{v'}^{(i)}(u)) \). The proof continues similarly until all bundles in the range are considered.

\[ \square \]

### 3 New Mechanisms for Multi-Unit Auctions

We now present three families of truthful mechanisms for multi-unit auctions that provide a bounded approximation ratio for multi-unit auctions with 2 bidders. The mechanisms in all three families are not affine maximizers. The first one, the Triage auction, provides an approximation ratio of \((1 + \epsilon)\) (for the correct choice of parameters), and we will prove in the next sections that Triage auctions are the only truthful and scalable mechanisms that provide an approximation ratio better than 2 if \( m = 2 \). The other mechanisms are for any number of items and provide an approximation of almost 2. To the best of our knowledge all of the previously known finitely-approximating mechanisms in the literature were (essentially) affine maximizers.

We provide the description of the mechanisms by specifying the payments functions for each of the bidders (each such function depends only on the other bidder). Truthfulness of the mechanisms is obvious and we are left only with proving feasibility and analyzing the approximation ratio.

#### 3.1 The Triage Auction

**Definition 3.1** The Triage auction is parameterized by three parameters, \( w, \theta_A, \theta_B \), for \( w > 0 \), \( 0 \leq \theta_A, \theta_B \leq 1 \), and \( \theta_A \geq 1 - \theta_B \). The payment functions are:

- \( p_m^{(2)}(v) = wv(m) \) if \( v(1) < \theta_A v(m) \), and \( p_m^{(2)}(v) = \frac{wv(1)}{\theta_A} \) otherwise.
- For \( 2 \leq k \leq m-1 \), \( p_k^{(2)}(v) = p_m^{(2)}(v) - w \cdot v(m-k) \).
- \( p_1^{(2)}(v) = p_m^{(2)}(v) - wv(m-1) \) if \( v(m-1) > (1 - \theta_B)v(m) \), and \( p_1^{(2)}(v) = p_m^{(2)}(v) - w(1 - \theta_B)v(m) \) otherwise.

and

- \( p_m^{(1)}(u) = w^{-1}u(m) \) if \( u(1) < \theta_B u(m) \), and \( p_m^{(1)}(u) = \frac{w^{-1}u(1)}{\theta_B} \) otherwise.
- For \( 2 \leq k \leq m-1 \), \( p_k^{(1)}(v) = p_m^{(1)}(u) - w^{-1} \cdot u(m-k) \).
- \( p_1^{(1)}(u) = p_m^{(1)}(u) - w^{-1}u(m-1) \) if \( u(m-1) > (1 - \theta_A)u(m) \), and \( p_1^{(1)}(v) = p_m^{(1)}(u) - w^{-1}(1 - \theta_A)u(m) \) otherwise.
For simplicity, in the analysis we assume the following tie-breaking rule: Alice takes the smallest bundle that maximize her profit, and Bob takes the largest bundle among all bundles that maximize his profit. We define the optimal allocation to be the allocation \((k, m-k)\) with the minimal \(k\) among all allocations that maximize \(v(k) + u(m-k)\). We use the following claim in the feasibility and truthfulness proofs:

**Claim 3.2** For triage auctions with \(w = 1\), if the optimal allocation is \((k, m-k)\) then Alice is allocated either \(k\) items, one item, or no items. Similarly, Bob is allocated either \(m-k\) items, one item, or no items.

**Proof:** To prove the lemma we will show that the equation \(v(k) - p_k^{(2)}(u) \geq v(t) - p_t^{(2)}(u)\) holds for \(t \neq 0, 1\). The equation implies that Alice prefers \(k\) items over \(t\) items. For each \(t \neq 1, 0\) we have that \(p_k^{(2)}(u) - p_t^{(2)}(u) \geq u(m-k) - u(m-t)\), thus the equation holds for \(t \neq 0, 1\) since we are given that \(v(k) + u(m-k) \geq v(t) + u(m-t)\) \(\square\).

**Lemma 3.3** The \((w, \theta_A, \theta_B)\)-Triage auction is feasible.

**Proof:** We first note that without loss of generality we may assume that \(w = 1\) (since multiplying one bidder’s prices by \(w\) and dividing the other’s by the same \(w\) maintains feasibility).

By Claim 3.2 if the optimal allocation is \((k, m-k)\) and \(k \neq 0, m\) then the mechanism is feasible, since Alice is allocated at most \(k\) items and Bob is allocated at most \(m-k\) items. We now assume that the optimal allocation is \((m, 0)\) (this implies in particular that \(v(m) \geq u(m)\)) and prove that the mechanism outputs a feasible solution. We have to show that If Bob does take one item then Alice does not take \(m\) items. Bob takes one item only if \(u(1) > p_k^{(2)}(v)\) which either means that \(u(1) > v(m) - v(m-1)\) or \(u(1) > v(m)\theta_B\), depending on the ratio between \(v(m-1)\) and \(v(m)\). The first case cannot happen since it implies that \(u(1) + v(m-1) > v(m)\), which is false since the optimal solution allocates \(m\) items to Alice. In the second case, since \(u(1) \geq \theta_B u(m)\) (because \(v(m) \geq u(m)\)), Alice will not take \(m\) items since for Alice to do so we must have that \(v(m) \geq \frac{u(1)}{\theta_B}\) \(\square\).

**Lemma 3.4** The \((1, \theta_A, \theta_B)\)-Triage auction provides an approximation ratio of \(\max(\frac{1}{\theta_A}, \frac{1}{\theta_B})\), for \(\theta_A > \frac{1}{1+\theta_B}\) and \(\theta_B > \frac{1}{1+\theta_A}\).

**Proof:** Let the optimal solution be \((k, m-k)\). If Alice is allocated \(k\) items and Bob \(m-k\) items then the approximation ratio is 1. By Claim 3.2 the only other cases to consider are when one of the bidders is allocated one or no items (whereas in the optimal solution his allocation is different).

Suppose that Alice is allocated no items. For that to happen we must have that \(v(k) < p_k^{(1)}(u)\) and therefore \(p_k^{(1)}(u) > u(m) - u(m-k)\) (otherwise Alice is assigned \(k\) items), which happens only if \(u(1) > \theta_B u(m)\). Alice is allocated no items and so \(v(k) < \frac{u(1)}{\theta_B} - u(m-k)\) which is equivalent to \(v(k) + u(m-k) < \frac{u(1)}{\theta_B}\). In other words, we showed that the optimal solution has value less than \(\frac{u(1)}{\theta_B}\), which also means that:

\[
v(m) \leq \frac{u(1)}{\theta_B} \tag{1}
\]

Since we have an upper bound on the value of the optimal solution, to prove that the approximation ratio is no worse than \(\frac{1}{\theta_B}\) it suffices to show that Bob is allocated at least one item. Suppose that Bob is not allocated one item: \(u(1) < p_k^{(2)}(v) \leq \theta_B v(m)\) (since from the conditions on \(\theta_A\) and \(\theta_B\): \(\frac{v(n)}{\theta_A} - v(m) < \theta_B v(m)\)). Rearranging, we have that \(\frac{u(1)}{\theta_B} < v(m)\), a contradiction to (1).
Suppose now that Alice is allocated one item. If in the optimal solution her allocation is different, then we have that \( v(1) \geq \theta_A u(m) \), and also that \( v(1) - p^{(1)}(u) > v(k) - p^{(1)}(u) \) which, by the definition of the payment functions, implies that \( v(1) + (1 - \theta_A)u(m) > v(k) + u(m - k) \). The approximation ratio is no worse than \( \frac{v(k) + u(m - k)}{v(1)} \leq \frac{v(1) + (1 - \theta_A)u(m)}{v(1)} \leq 1 + \frac{(1 - \theta_A)u(m)}{\theta_A u(m)} \leq \frac{1}{\theta_A} \).

\[\square\]

### 3.2 Shifted Welfare Maximizer

**Definition 3.5** An \( \alpha \)-shifted welfare maximizer is the following two bidders auction: Bob’s payment for \( t \) items, \( t \neq 0, m \) is \((1 + \alpha)v(m) - v(m - t)\), for \( t = 0 \) items is 0, and for \( t = m \) items is \( v(m) \). Alice’s payment for \( t \) items, \( t \neq 0, m \) is \((1 + \alpha)u(m) - u(m - t)\), for \( t = 0 \) items is 0, and for \( t = m \) items is \( u(m) \).

**Lemma 3.6** For any \( \alpha > 0 \) the \( \alpha \)-shifted welfare maximizer is feasible.

**Proof:** We assume without loss of generality that \( v(m) \geq u(m) \). Observe that this implies that Bob is not allocated \( m \) items. Suppose that Alice is allocated \( k \) items. We will now show that Bob is allocated at most \( m - k \) items, hence the mechanism is feasible. If Alice is allocated \( k \) items then we have that for every \( t \neq 0, m \):

\[
v(k) - ((1 + \alpha)u(m) - u(m - k)) \geq v(t) - ((1 + \alpha)u(m) - u(m - t))
\]

which implies that

\[
v(k) + u(m - k) \geq v(t) + u(m - t)
\]

Thus we have that Bob (weakly) prefers \( m - k \) items over \( m - t \) items, for every \( t \neq 0, m \):

\[
u(m - k) - ((1 + \alpha)v(m) - v(k)) \geq u(t) - ((1 + \alpha)v(m) - v(t))
\]

Notice that this concludes the proof for this case: we have already argued that Bob is not assigned \( m \) items, and if Bob is assigned 0 items feasibility still holds.

We now handle the case where Alice is allocated \( m \) items. We have that for every \( t \neq 0, m \):

\[
v(m) - u(m) \geq v(t) - ((1 + \alpha)u(m) - u(m - t))
\]

which implies that

\[
v(m) \geq v(t) + u(m - t) - \alpha u(m)
\]

which in turn implies that

\[(1 + \alpha)v(m) \geq v(t) + u(m - t)\]

Therefore, the profit of Bob from taking \( m - t \neq m, 0 \) items is at most 0:

\[u(m - t) - ((1 + \alpha)v(m) - v(t)) \leq 0\]

\[\square\]

**Lemma 3.7** For any \( 1 \geq \alpha > 0 \) the \( \alpha \)-shifted welfare maximizer provides an approximation ratio of \( 1 + \frac{1}{\alpha} \).
Proof: Again, we assume without loss of generality that \( v(m) \geq u(m) \). Observe that whenever Alice is allocated \( m \) items then we have a 2-approximation. Thus we concentrate on the case where Alice is assigned \( k < m \) items. Observe that \( k > 0 \), since taking \( m \) items has a positive profit for Alice. Furthermore, the payment for \( k \) items is at least \( \alpha u(m) \) thus the welfare of the allocation the the algorithm outputs is at least \( v(k) \geq \alpha u(m) \). To finish the proof we claim that the optimal solution is \( (k, m - k) \): the bundle of \( k \) items is a profit-maximizing bundle of Alice and thus for every \( t \neq 0, m \):

\[
v(k) - ((1 + \alpha)u(m) - u(m - k)) \geq v(t) - ((1 + \alpha)u(m) - u(m - t))
\]

which implies that

\[
v(k) + u(m - k)) \geq v(t) + u(m - t))
\]

We now show that we also have that \( v(k) + u(m - k) \geq v(m) + \alpha u(m) \):

\[
v(k) - ((1 + \alpha)u(m) - u(m - k)) \geq v(m) - u(m)
\]

which implies that

\[
v(k) + u(m - k) \geq v(m) + \alpha u(m)
\]

From the last inequality we also get that \( v(k) \geq \alpha u(m) \) since \( u(m - k) \leq v(k) \). Using this fact, we conclude that the approximation ratio that the algorithm provides is

\[
\frac{v(k) + u(m - k)}{v(k)} = 1 + \frac{u(m - k)}{v(k)} \leq \frac{u(m)}{\alpha u(m)} = 1 + \frac{1}{\alpha}.
\]

### 3.3 Fractions Auction

**Definition 3.8** Given constants \( 0 \leq \alpha_1 \leq \ldots \leq \alpha_{m-1} \leq 1 \), the \((\alpha_{m-1}, \ldots, \alpha_1)\)-fractions auction is the following: Bob’s payment for \( m \) items is \( v(m) \), for \( t \neq m, 0 \) items is \( \alpha_t \cdot v(m) \), for \( t = 0 \) items it is 0. Alice’s payment for \( t > 0 \) items is \( \max\{u(m), \frac{u(m-1)}{\alpha_{m-1}}, \ldots, \frac{u(m-t)}{\alpha_{m-t}}\} \). Alice’s payment for \( t = 0 \) items is 0.

**Lemma 3.9** The \((\alpha_{m-1}, \ldots, \alpha_1)\)-fractions auction is feasible.

**Proof:** Suppose that Bob is allocated \( k \) items by the mechanism. We will show that the left bidder is assigned at most \( m - k \) items, hence the mechanism is feasible. If Bob is assigned \( k \) items then:

\[
u(k) \geq \alpha_k v(m)
\]

Now observe that for every \( t > m - k \) the price of Alice is \( \max\{u(m), \frac{u(m-1)}{\alpha_{m-1}}, \ldots, \frac{u(t)}{\alpha_k}\} \geq \frac{u(k)}{\alpha_k} \). For Alice to take \( m - k \) items (and not 0 items) we must therefore have that \( v(m) \geq v(m - k) > \frac{u(k)}{\alpha_k} \). However, this is not true since we know that \( u(k) \geq \alpha_k v(m) \).

**Lemma 3.10** The \((\alpha_{m-1}, \ldots, \alpha_1)\)-fractions auction provides an approximation ratio of \( \frac{2}{\alpha_1} \).

**Proof:** Suppose that Bob is allocated \( k > 0 \) items. Since the bundle of \( k \) items is profitable we have that \( u(k) \geq \alpha_k v(m) \), and this is a lower bound to the welfare of the allocation. The welfare of the optimal allocation is at most \( v(m) + u(m) \). Observe that since Bob is allocated \( k \) and not \( m \) items we have that \( u(k) - \alpha_k v(m) > u(m) - v(m) \) and therefore \( u(m) - u(k) \leq (1 - \alpha_k) v(m) \). The approximation ratio of the algorithm in this case is at least

\[
\frac{v(m) + u(m)}{u(k)} \leq \frac{v(m) + u(k) + (1 - \alpha_k) v(m)}{u(k)} \leq \frac{u(k) + v(m) + (1 - \alpha_k) v(m)}{u(k)} \leq 1 + \frac{v(m) + (1 - \alpha_k) v(m)}{\alpha_k v(m)}
\]
The other case is when Bob is allocated no items at all. Let \( k \) be the number of items that Alice is assigned. Observe that \( k > 0 \): since Bob is not allocated any items, we have that for all \( t > 0 \), \( u(t) \leq \alpha_1 \cdot v(m) \) and \( u(t) \leq v(t) \). Thus Alice has a non-negative profit from taking the bundle of \( m \) items, therefore we may assume that without loss of generality the mechanism assigns Alice \( k > 0 \) items. In particular we have that \( v(k) \geq \max\{u(m), \frac{u(m-k)}{\alpha_{m-k}}\} \). As in the previous case, we also have that, since Alice prefers \( k \) items over \( m \) items, \( v(m) - \max\{u(m), \frac{u(m-1)}{\alpha_{m-1}}, \ldots, \frac{u(1)}{\alpha_1}\} \leq v(k) - \max\{u(m), \frac{u(m-1)}{\alpha_{m-1}}, \ldots, \frac{u(1)}{\alpha_1}\} \). Thus, \( v(m) - v(k) \leq \frac{u(m)}{\alpha_1} - u(m) \).

Again, an upper bound on the value of the optimal allocation is \( v(m) + u(m) \). The approximation ratio of the algorithm is at least

\[
\frac{u(m) + v(m)}{v(k)} = \frac{v(m)}{v(k)} + \frac{u(m)}{v(k)} \leq \frac{v(k) + (\frac{u(m)}{\alpha_1} - u(m))}{v(k)} + \frac{u(m)}{v(k)} \leq 1 + \frac{u(m)}{\alpha_1 v(k)} \leq 1 + \frac{1}{\alpha_1}
\]

where the last inequality is since \( u(m) \leq v(k) \).

### 4 Characterization of Scalable 2-Item Auctions

This section is devoted to proving the following characterization result:

**Theorem 4.1 (characterization)** The only feasible, scalable and truthful auctions with an approximation ratio strictly better than 2 for two identical goods and two bidders are triage auctions for some \((w, \theta_A, \theta_B)\).

We now provide a brief road map to the proof of the theorem. Very differently from Roberts’ theorem proof, our proof analyzes the payment functions of the bidders (rather then the social choice functions) and shows that the payment functions are identical to some triage auction’s payment functions. Recall that in the two item case, the payment functions of a triage auction are defined using three different regions that correspond to the ratio between the value for two items and the value for one item: high, mid, and low. The proof of the theorem is divided to subsections that roughly correspond to these regions.

Subsection 4.1 gives an alternative definition of the triage auction that is easier to work with, for the special case where \( m = 2 \). In subsection 4.2 we give characterize the payment function for two items. The results of subsection 4.1 hold for any scalable mechanism with a bounded approximation ratio, not just ones with an approximation ratio better than 2. The next subsections are devoted to characterizing the payment functions for one item. Subsection 4.3 defines and “separates” the high-range from the mid and low ranges: it shows that, roughly speaking, the payment for one item induced by a valuation that is in the mid-range or in the low-range is also in the mid range or in the low range. Subsection 4.4 proves some basic properties, like continuity, of the payment function in the low and mid range. The central part of the proof is Subsection 4.5 which shows that the payment functions in the mid range are equivalent to the payment functions of weighted VCG. Finally, we conclude the proof with Subsection 4.6 that handles the value of the transition points between the high and mid range, and with Subsection 4.7 that gives the payment function for the high range.
<table>
<thead>
<tr>
<th>Number of items</th>
<th>Alice’s value</th>
<th>Alice’s payment</th>
<th>Bob’s value</th>
<th>Bob’s payment</th>
</tr>
</thead>
<tbody>
<tr>
<td>One</td>
<td>(rv)</td>
<td>(g(s) \cdot q(s) \cdot u)</td>
<td>(su)</td>
<td>(f(r) \cdot p(r) \cdot v)</td>
</tr>
<tr>
<td>Both</td>
<td>(v)</td>
<td>(q(s) \cdot u)</td>
<td>(u)</td>
<td>(p(r) \cdot v)</td>
</tr>
</tbody>
</table>

**Proposition 4.3** For any \(p, q : [0, 1] \rightarrow \mathbb{R}^+\) and \(f, g : [0, 1] \rightarrow [0, 1]\), the scalable mechanism based on them is scalable and truthful (but may be infeasible and allocate more than 2 items). Any truthful scalable mechanism (even a non-feasible one as long as it allocates at most two items to any bidder) is equivalent to one based on some functions.

**Proof:** The first direction is trivial. Truthfulness implies the existence of payments \(p_1^2 (v(1), v(2))\) and \(p_2^2 (v(1), v(2))\) for Bob that depend only on \(v(1), v(2)\). So now define \(p(r) = p_2^2 (r, 1)\) and \(f(r) = p_1^2 (r, 1)/p_2^2 (r, 1)\). Our mechanism on input \((v(1), v(2))\) by definition gives the two-item payment \(v(2) \cdot p(v(1)/v(2)) = v(2) \cdot p_2^2(v(1)/v(2), 1) = p_2^2(v(1), v(2))\), where the last equality follows from the scalability of \(p_2^2\). Similarly the payment given for one items is \(v(2) \cdot p(v(1)/v(2)) \cdot f(v(1)/v(2)) = v(2) \cdot p_2^2(v(1)/v(2), 1) \cdot p_1^2(v(1)/v(2), 1)/p_2^2(v(1)/v(2), 1) = v(2) \cdot p_1^2(v(1), v(2))\), where the last equality follows from the scalability of \(p_1^2\).

We now give an alternative (equivalent) definition of the triage auction, for the \(m = 2\) case.

**Definition 4.4** The \((w, \theta_A, \theta_B)\)-triage auction for \(w > 0\) and \(0 \leq \theta_A, \theta_B \leq 1\), \(\theta_A \geq 1 - \theta_B\), is the scalable mechanism based on:

- For \(r \leq 1 - \theta_B\): \(f(r) = \theta_B\), and \(p(r) = w\).
- For \(1 - \theta_B \leq r \leq \theta_A\): \(f(r) = 1 - r\), and \(p(r) = w\).
- For \(r \geq \theta_A\): \(f(r) = 1 - \theta_A\), and \(p(r) = wr/\theta_A\).

and

- For \(s \leq 1 - \theta_A\): \(g(s) = \theta_A\), and \(q(s) = w^{-1}\).
- For \(1 - \theta_A \leq s \leq \theta_B\): \(g(s) = 1 - s\), and \(q(s) = w^{-1}\).
- For \(s \geq \theta_B\): \(g(s) = 1 - \theta_B\), and \(q(s) = w^{-1}s/\theta_B\).

### 4.2 Characterizing the Payment for Two Items

The results in this section hold for any feasible scalable and truthful mechanism with a bounded approximation ratio. We will usually only prove the theorem for \(p\) where the fact for \(q\) is symmetric.

**Lemma 4.5** The function \(p\) is monotone non-decreasing.

**Proof:** Assume towards contradiction that for some \(r' > r\) we have \(p(r') < u < u' < p(r)\). Since \(u > p(r')\), on inputs \((r', 1)\) and \((0, u)\) Bob must win both items, so the first cannot win anything. Notice that \((r, 1)\) wins nothing against \((0, u'(1+\epsilon))\) (by using the scalability of the payments, \(u'(1+\epsilon)\) induces bigger payments than \(u'\), and Alice did not win any items with the bigger valuation \((r, 1)\)), but also Bob does not win both items since \(u'(1+\epsilon) < p(r)\) for small enough \(\epsilon > 0\), so the total welfare achieved is 0 contradicting finite approximation.
Lemma 4.6 (weighting) \( p(0) \cdot q(0) = 1 \).

**Proof:** Consider the following input:

<table>
<thead>
<tr>
<th>Number of items</th>
<th>Alice’s value</th>
<th>Alice’s payment</th>
<th>Bob’s value</th>
<th>Bob’s payment</th>
</tr>
</thead>
<tbody>
<tr>
<td>One</td>
<td>0</td>
<td>?</td>
<td>0</td>
<td>?</td>
</tr>
<tr>
<td>Both</td>
<td>( v )</td>
<td>( uq(0) )</td>
<td>( u )</td>
<td>( vp(0) )</td>
</tr>
</tbody>
</table>

The only allocations that will give the required approximation ratio on inputs of the form \((0, v)\) and \((0, u)\) are those that give both items to one of the bidders. If \( u < vp(0) \) then Bob does not win two items; whereas if \( u > vp(0) \) then he wins both items, and dually for Alice. So we get a contradiction to feasibility if \( u > vp(0) \) and \( v > uq(0) \), i.e., if \( p(0)q(0) < 1 \). On the other hand, if \( u < vp(0) \) and \( v < uq(0) \), i.e., if \( p(0)q(0) > 1 \), then we get a total welfare of 0, contradicting finite approximation ratio.

Lemma 4.7 (low range) If \( r < g(0) \) then \( p(r) = p(0) \).

**Proof:** Assume that \( p(r) \neq p(0) \) then, using monotonicity, let \( p(r) > u' > u > p(0) \). On input \((0, 1)\) and \((0, u)\) Bob gets both items (since \( u > p(0) \)) and so Alice must get none. On inputs \((r, 1)\) and \((0, u')\) Alice gets at most 1 item (since, by the scalability of the payments, the payment induces by Bob for two items has increased), but since \( u' < p(r) \) Bob does not get two items, and so for finite approximation, Alice must get an item, so \( r \geq u'g(0)q(0) \geq p(0)q(0)g(0) = g(0) \).

Lemma 4.8 \( p(r) \geq r/(g(0)q(0)) \).

**Proof:** Let \( u > p(r) \), then on input \((r, 1)\) and \((0, u)\) Bob gets both items. Alice’s payment for a single item is \( uq(0)q(0) \) which for feasibility must be at least \( r \). Since this holds for all \( u > p(r) \) we get that \( r \leq p(r)g(0)q(0) \) as required.

Lemma 4.9 (high range) If \( r > g(0) \) then \( p(r) = r/(g(0)q(0)) \).

**Proof:** We will prove the contra-positive which by the previous lemma assumes \( p(r) > u > r/(g(0)q(0)) \). Consider the following input:

<table>
<thead>
<tr>
<th>Number of items</th>
<th>Alice’s value</th>
<th>Alice’s payment</th>
<th>Bob’s value</th>
<th>Bob’s payment</th>
</tr>
</thead>
<tbody>
<tr>
<td>One</td>
<td>( r )</td>
<td>( uq(0)g(0) )</td>
<td>0</td>
<td>?</td>
</tr>
<tr>
<td>Both</td>
<td>1</td>
<td>( uq(0) )</td>
<td>( u )</td>
<td>( p(r) )</td>
</tr>
</tbody>
</table>

In this case Bob can not win both items so he gets a value of 0. By the choice of \( u \), Alice’s payment for two items is \( uq(0) > r/g(0) \) is greater than 1, thus she cannot win two items. Thus for finite approximation she must win one item and thus \( r \geq uq(0)g(0) \) and since this is true for every \( u < p(r) \), we have \( r \geq p(r)g(0)q(0) \), contradiction.

At this point we have completed the required characterization of \( p \) and \( q \).

**Definition 4.10** Let \( w = p(0) \), \( \theta_A = g(0) \), and \( \theta_B = f(0) \).

**Lemma 4.11 (summary of subsection)** For \( r \leq \theta_A \) we have that \( p(r) = w \) and for \( r \geq \theta_A \) we have \( p(r) = wr/\theta_A \). Similarly, for \( s \leq \theta_B \) we have that \( q(s) = w^{-1} \) and for \( s \geq \theta_B \) we have \( q(s) = w^{-1}s/\theta_B \).
Proof: The low range lemma states the required fact for $r < \theta_A$. The high range lemma states the required fact for $r \geq \theta_A$, taking into account the inverse lemma, $p(0)q(0) = 1$, the same holds for $q$, replacing $w$ with $w^{-1}$, again relying on $p(0)q(0) = 1$. For $r = \theta_A$ we observe that $p(r) = w$ since $p$ is a monotone function and approaches $w$ above and below $w$.

4.3 Separating the high range

In this section we show that for $r \leq \theta_A$ we have that $f(r) \leq \theta_B$. Similarly it follows that for $s \leq \theta_B$ we have that $g(s) \leq \theta_A$. Note that by the previous section $r \leq \theta_A$ if and only if $p(r) > w$, and this last condition is what drives this section.

Lemma 4.12 (one-side inverse) If $r \leq \theta_A$ then for any $\delta > 0$, $wg(f(r) - \delta)q(f(r) - \delta) \geq r$.

Proof: Assume to the contrary $wg(f(r) - \delta)q(f(r) - \delta) < r$ and consider the following input:

<table>
<thead>
<tr>
<th>Number of items</th>
<th>Alice’s value</th>
<th>Alice’s payment</th>
<th>Bob’s value</th>
<th>Bob’s payment</th>
</tr>
</thead>
<tbody>
<tr>
<td>One</td>
<td>$r$</td>
<td>$wg(f(r) - \delta)q(f(r) - \delta)(1 + \epsilon) &lt; r$</td>
<td>$(f(r) - \delta)w(1 + \epsilon)$</td>
<td>$w(f(r) - \delta)$</td>
</tr>
<tr>
<td>Both</td>
<td>1</td>
<td>$q(1)$</td>
<td>$w(1 + \epsilon)$</td>
<td>$w$</td>
</tr>
</tbody>
</table>

Bob takes two items. However, when $\epsilon$ is small enough, Alice gets positive utility from one item so she will take (at least) a single item, contradicting feasibility.

At this point we separate into two cases, according to whether $r > wf(r)$. We start with the easy case:

Lemma 4.13 (case I) If $r \leq \theta_A$ and $r \leq wf(r)$ then $f(r) \leq \theta_B$.

Proof: Assume by way of contradiction that $f(r) > \theta_B$ and so $q(f(r)) > w^{-1}$, and for $\epsilon$ small enough consider the input:

<table>
<thead>
<tr>
<th>Number of items</th>
<th>Alice’s value</th>
<th>Alice’s payment</th>
<th>Bob’s value</th>
<th>Bob’s payment</th>
</tr>
</thead>
<tbody>
<tr>
<td>One</td>
<td>$r$</td>
<td>?</td>
<td>$f(r)(1 - \epsilon)w$</td>
<td>$w$</td>
</tr>
<tr>
<td>Both</td>
<td>1</td>
<td>$w(1 - \epsilon)q(f(r)) &gt; 1$</td>
<td>$f(r)w$</td>
<td>$(1 - \epsilon)w$</td>
</tr>
</tbody>
</table>

Notice that Bob gets negative utility from taking an item or two items and thus takes nothing. Alice gets negative utility from taking two items so can take at most a single item. The total welfare is thus at most $r$, whereas the social optimum is at least $r + wf(r)$. Since $r \leq wf(r)$ this is a contradiction to better than 1/2-approximation.

For the second case we first need to prepare a lemma.

Lemma 4.14 For $r > \theta_A$ we have that $r > f(r)p(r)$.

Proof: Consider the input:

<table>
<thead>
<tr>
<th>Number of items</th>
<th>Alice’s value</th>
<th>Alice’s payment</th>
<th>Bob’s value</th>
<th>Bob’s payment</th>
</tr>
</thead>
<tbody>
<tr>
<td>One</td>
<td>$r$</td>
<td>?</td>
<td>$f(r)p(r)(1 - \epsilon)$</td>
<td>$f(r)p(r)$</td>
</tr>
<tr>
<td>Both</td>
<td>1</td>
<td>$q(f(r))p(r)(1 - \epsilon)$</td>
<td>$p(r)(1 - \epsilon)$</td>
<td>$p(r)$</td>
</tr>
</tbody>
</table>
Bob has negative utility for either one item or two items. Since \( p(r) > w \) and \( q(f(r)) \geq w^{-1} \), Alice has negative utility for two items, as long as \( \epsilon \) is small enough. Thus the total welfare is at most \( r \), whereas the social optimum is at least \( r + f(r)p(r)(1 - \epsilon) \), so for better than 1/2-approximation we must have \( r > f(r)p(r) \).

**Corollary 4.15** If \( f(r) > \theta_B \) then \( f(r) > g(f(r))q(f(r)) \).

**Proof:** This is the previous lemma with the roles of the players switched and with \( s = f(r) \).

**Lemma 4.16 (case II)** If \( r \leq \theta_A \) and \( r > w f(r) \) then \( f(r) \leq \theta_B \).

**Proof:** Assume towards contradiction that there exists \( \delta > 0 \) such that \( f(r) - \delta > \theta_B \). Combining the one-sided inverse lemma and the previous corollary we have that \( f(r) - \delta > g(f(r) - \delta)q(f(r) - \delta) \geq r/w \); Contradiction.

Which concludes this subsection:

**Lemma 4.17** For \( r \leq \theta_A \) we have \( f(r) \leq \theta_B \). For \( s \leq \theta_B \) we have \( q(s) \leq \theta_B \).

**Proof:** The Case I and Case II lemmas cover all possibilities for \( f \); for \( g \) the situation is symmetric.

### 4.4 The low and mid range

In this subsection we continue dealing with the range \( r < \theta_A \) and show that it splits into two sub-ranges, the low-range \( r \leq l_A \), for which \( f(r) = \theta_B \), and the mid-range, \( l_A < r < \theta_A \), for which \( l_B < f(r) < \theta_B \).

**Lemma 4.18 (monotone consistency)** For \( r \leq \theta_A \) and \( s \leq \theta_B \) we have that \( s \leq f(r) \) if and only if \( r \leq g(s) \).

**Proof:** We will prove the only if direction, and the other direction is symmetric. Assume towards contradiction that \( r > g(s) \) and for small enough \( \epsilon > 0 \) consider the input:

<table>
<thead>
<tr>
<th>Number of items</th>
<th>Alice’s value</th>
<th>Alice’s payment</th>
<th>Bob’s value</th>
<th>Bob’s payment</th>
</tr>
</thead>
<tbody>
<tr>
<td>One</td>
<td>( r )</td>
<td>( g(s)(1 + \epsilon) &lt; r )</td>
<td>( s(1 + \epsilon)w )</td>
<td>( wf(r) )</td>
</tr>
<tr>
<td>Both</td>
<td>1</td>
<td>?</td>
<td>( (1 + \epsilon)w )</td>
<td>( w )</td>
</tr>
</tbody>
</table>

The utility of Bob from one item is at most \( s \) fraction of his utility for two items which is positive, so Bob wins two items. Alice wins at least one item, contradiction to feasibility.

**Corollary 4.19 (one-sided inverse)** If \( r \leq \theta_A \) then \( g(f(r)) \geq r \).

**Proof:** Use the previous lemma with \( s = f(r) \).

**Corollary 4.20 (monotonicity)** \( f \) is monotone non-increasing on \( r \leq \theta_A \).

**Proof:** Assume that \( f(r) < f(r') \), for some \( r, r' < \theta_A \). Take \( s = f(r') \). First, since \( s > f(r) \) we apply the contra-positive of the previous to get \( r > g(s) \). Second, since \( s \leq f(r') \) we apply the previous lemma to get \( r' \leq g(s) \). Putting these two inequalities together gives \( r > r' \).

We start by showing that \( f \) and \( g \) cannot decrease too quickly, and satisfy a Lipschitz condition.
Lemma 4.21 (Lipschitz condition) If $0 \leq r' < r < \theta_A$ then $f(r') - f(r) \leq (r - r') \cdot f(r)/(1 - r)$ and if $0 \leq s' < s < \theta_B$ then $g(s') - g(s) \leq (s - s') \cdot g(s)/(1 - s)$.

Proof: As usual we will prove for $f$, and the case for $g$ is similar. The required outcome is equivalent to $f(r')/f(r) \leq (1 - r')/(1 - r)$, so assume towards contradiction that $f(r')/f(r) > \alpha > (1 - r')/(1 - r)$, and consider the following input:

<table>
<thead>
<tr>
<th>Number of items</th>
<th>Alice’s value</th>
<th>Alice’s payment</th>
<th>Bob’s value</th>
<th>Bob’s payment</th>
</tr>
</thead>
<tbody>
<tr>
<td>One</td>
<td>$r\alpha$</td>
<td>$g(f(r'))$</td>
<td>$w$</td>
<td>$\alpha w r &lt; w f(r')$</td>
</tr>
<tr>
<td>Both</td>
<td>$\alpha$</td>
<td>1</td>
<td>$w$</td>
<td>?</td>
</tr>
</tbody>
</table>

Bob gets positive utility for one item so he takes (at least) an item. Since $\alpha > (1 - r')/(1 - r) > 1$, taking two items is profitable for Alice and she will take both items as long that is preferable to taking one item, i.e., if $\alpha - 1 > \alpha r - g(f(r'))$. But our assumption that $\alpha > (1 - r')/(1 - r)$ is equivalent to $\alpha - 1 > \alpha r - r'$, and that implies the previous inequality since, by the one-sided inverse corollary, $g(f(r')) \geq r'$. Thus all together three items are allocated. Contradiction.$\square$

Corollary 4.22 (continuity) The function $f$ is continuous on $[0, \theta_A)$ and $g$ is continuous on $[0, \theta_B)$.

Proof: The Lipschitz condition implies continuity.$\square$

4.5 The mid range

This is the central part of the proof in which we show that in the middle range $f$ and $g$ are linear, and in fact the payments are identical to those given by weighted VCG. We first wish to separate the region for which $f(r) = \theta_B$ from that which $f(r) < \theta_B$. Monotonicity implies that the first is a prefix, so let us define:

Definition 4.23 Let $l_A = \sup \{r \leq \theta_A \mid f(r) = \theta_B\}$ and $l_B = \sup \{s \leq \theta_B \mid g(s) = \theta_A\}$.

Corollary 4.24 (inverses) For $l_A < r < \theta_A$ we have $g(f(r)) = r$.

Proof: The monotone consistency lemma implies that for $s \leq f(r)$ we have $g(s) \geq r$ whereas its contra-positive states that for $s > f(r)$ we have $g(s) < r$. So continuity implies that for $s = f(r)$ we have $g(s) = r$. $\square$

Corollary 4.25 (bijective) The function $f(r)$ is bijective from the interval $(l_A, \theta_A)$ to the interval $(l_B, \theta_B)$ and the function $g(s)$ is bijective from the interval $(l_B, \theta_B)$ to the interval $(l_A, \theta_A)$.

The following lemma shows that the Lipschitz bound above is actually tight and defined a constant derivative for $f$, at least for small enough $r$.

Lemma 4.26 (differences) Let the approximation ratio of the mechanism be at least $(1 + \delta)/2$. For every $l_A < r' < r < \theta_A$ and $r \leq (1 + \delta)w f(r')(1 - r)/(1 - r')$ we have $f(r') - f(r) = (r - r') \cdot f(r)/(1 - r)$.

Proof: The required outcome is equivalent to $f(r')/f(r) = (1 - r')/(1 - r)$, so assume towards contradiction that $f(r')/f(r) < \alpha < (1 - r')/(1 - r)$ (since the other direction was shown to be impossible), and consider the following input:
Bob will not take both items since \( \alpha > f(r')/f(r) > 1 \), and will not take a single item since \( \alpha f(r) > f(r') \). Alice will not take both items if taking one item is preferable, i.e., if \( \alpha - 1 < \alpha r - g(f(r')) = \alpha r - r' \) which is equivalent to our assumption that \( \alpha < (1 - r')/(1 - r) \). Thus the total utility obtained is \( r \alpha \) whereas the optimum is at least \( r \alpha + w f(r') \) in contradiction to \( 1/2 + \delta \)-approximation since \( \alpha < (1 - r')/(1 - r) \) implies \( r \alpha < (1 + \delta) w f(r') \).

\[ \square \]

**Lemma 4.27** For \( l_A < r < \theta_A \) with \( r < (1 + \delta) w f(r) \) we have that \( f(r) = c \cdot (1 - r) \) for some constant \( c > 0 \).

**Proof:** Since \( f \) is continuous, we have that there exists \( \epsilon > 0 \) so that for all \( r - \epsilon < r'' < r' < r \) we have that \( r < (1 + \delta) w f(r') (1 - r)/(1 - r') \) as well as \( r < (1 + \delta) w f(r'') (1 - r)/(1 - r') \) and \( r' < (1 + \delta) w f(r'') (1 - r)/(1 - r'') \). We thus apply the previous lemma three times: for \( r' < r \), for \( r'' < r \), and for \( r'' < r', \) and get \( (r - r') f(r)/(1 - r) = (r - r') f(r)/(1 - r) + (r' - r'') f(r')/(1 - r') \), which implies that \( f(r)/(1 - r) = f(r')/(1 - r') \). Thus we have the required result in an open interval around \( r \), and since this is true for every \( r \) and the intervals are overlapping, it must be the same constant \( c \) everywhere.

\[ \square \]

**Corollary 4.28** For \( l_A < r < \theta_A \) with \( r (1 + \delta) > w f(r) \) we have that \( f(r) = 1 - c' r \) for some constant \( c' > 0 \).

**Proof:** The symmetric version of the previous lemma for \( g \) states that for \( s < (1 + \delta) w^{-1} g(s) \) we have \( g(s) = c'(1 - s) \). Use \( s = f(r) \) and the fact that \( g(f(r)) = r \) to obtain that for \( f(r) < (1 + \delta) w^{-1} r \) we have \( r = g(f(r)) = c'(1 - f(r)) \), which implies the corollary.

\[ \square \]

**Corollary 4.29** For every \( l_A < r < \theta_A \) we have that \( f(r) = 1 - r \).

**Proof:** Since \( f \) is continuous, the range \( r (1 + \delta) > w f(r) \) overlaps the range \( r < (1 + \delta) w f(r) \). Thus for every \( r \) in this interval we have that \( c(1 - r) = f(r) = 1 - c' r \). This implies \( c = c' = 1 \).

Which leads us to the conclusion of this subsection:

**Lemma 4.30** For every \( 1 - \theta_B < r < \theta_A \) we have that \( f(r) = 1 - r \). For every \( \theta_A < s < \theta_B \) we have that \( g(s) = 1 - s \). For every \( r < 1 - \theta_B \) we have that \( f(r) = \theta_B \). For every \( s < 1 - \theta_A \) we have that \( g(s) = \theta_A \).

**Proof:** As \( g \) is monotone decreasing, continuous and onto, we must have \( \lim_{s \to \theta_B} g(s) = l_A \). The previous corollary allows directly evaluating the limit to be \( \theta_B \) and the \( l_A = 1 - \theta_B \), which gives the desired result as the statements of the previous corollary as well as the definition of \( l_A \) and \( l_B \).

\[ \square \]
4.6 The Value of \( f(\theta_A) \) and \( g(\theta_B) \)

Lemma 4.31 \( f(\theta_A) = 1 - \theta_A \) and \( g(\theta_B) = 1 - \theta_B \).

We prove the lemma for \( f(\theta_A) \) but it symmetrically holds for \( g(\theta_B) \). The proof consists of the following two claims:

Claim 4.32 \( f(\theta_A) \geq 1 - \theta_A \).

Proof: Suppose towards a contradiction that \( f(\theta_A) < 1 - \theta_A \). Consider the following instance, where \( \delta > 0 \):

<table>
<thead>
<tr>
<th>Number of items</th>
<th>Alice’s value</th>
<th>Alice’s payment</th>
<th>Bob’s value</th>
<th>Bob’s payment</th>
</tr>
</thead>
<tbody>
<tr>
<td>One</td>
<td>( \theta_A )</td>
<td>( (1 - \delta)\theta_A )</td>
<td>( (1 - \delta)(w_f(\theta_A) + 1 - \theta_A) )</td>
<td>( w_f(\theta_A) )</td>
</tr>
<tr>
<td>Both</td>
<td>1</td>
<td>( (1 - \delta) )</td>
<td>( (1 - \delta)w )</td>
<td>( w )</td>
</tr>
</tbody>
</table>

Alice is allocated 2 items (her most profitable bundle). If \( \delta > 0 \) is such that \( (1 - \delta)(w_f(\theta_A) + 1 - \theta_A) > w_f(\theta_A) \) then Bob is allocated at least one item. This is a contradiction to the feasibility of the mechanism.

Claim 4.33 \( f(\theta_A) \leq 1 - \theta_A \).

Proof: Suppose towards a contradiction that \( f(\theta_A) > 1 - \theta_A \). Choose \( t \) to be such that \( f(\theta_A) > t > 1 - \theta_A \). Consider the following instance:

<table>
<thead>
<tr>
<th>Number of items</th>
<th>Alice’s value</th>
<th>Alice’s payment</th>
<th>Bob’s value</th>
<th>Bob’s payment</th>
</tr>
</thead>
<tbody>
<tr>
<td>One</td>
<td>( \theta_A )</td>
<td>( (1 + \delta)g(t) )</td>
<td>( (1 + \delta)wt )</td>
<td>( w_f(\theta_A) )</td>
</tr>
<tr>
<td>Both</td>
<td>1</td>
<td>( (1 + \delta) )</td>
<td>( (1 + \delta)w )</td>
<td>( w )</td>
</tr>
</tbody>
</table>

If \( \delta > 0 \), Bob is allocated 2 items. Also notice that \( g(t) < \theta_A \), since \( t \) is in the mid range. Thus if \( \delta \) is small enough taking one item is profitable for Alice. Hence the mechanism is allocating at least 3 items, in contradiction to the feasibility of the mechanism.

4.7 The high range

In this section we characterize the high range. We prove that for every \( r > \theta_A \), \( f(r) = \frac{r}{\theta_A} - r \), and for every \( r > \theta_B \), \( g(r) = \frac{r}{\theta_B} - r \).

Claim 4.34 \( \theta_A > w_f(\theta_A) \). Similarly, \( \theta_B > g(\theta_A)/w \).

Proof: We prove only the first part. We use the fact established in the previous section that \( f(\theta_A) = 1 - \theta_B \). The proof is similar to the proof of Lemma 4.14. Suppose towards a contradiction that \( \theta_A \leq w_f(\theta_A) \). Consider the following instance, for \( \gamma > 0 \):

<table>
<thead>
<tr>
<th>Number of items</th>
<th>Alice’s value</th>
<th>Alice’s payment</th>
<th>Bob’s value</th>
<th>Bob’s payment</th>
</tr>
</thead>
<tbody>
<tr>
<td>One</td>
<td>( \theta_A + \epsilon )</td>
<td>?</td>
<td>( w_f(r) - \gamma )</td>
<td>( w_f(r) )</td>
</tr>
<tr>
<td>Both</td>
<td>1</td>
<td>&gt; 1</td>
<td>( w(1 + \gamma) )</td>
<td>( \frac{w_f}{\theta_A} )</td>
</tr>
</tbody>
</table>
If $\gamma$ is small enough then Bob is not allocated any items. Alice is allocated at most one item. Thus, when $\gamma, \epsilon$ approach 0 the approximation ratio approaches 2: the solution that allocates one item to each of the bidders has value of $\theta_A + \epsilon + wf(r) - \gamma$ whereas the algorithm returns a solution with value at most $\theta_A \leq wf(\theta_A)$.

**Claim 4.35** Let Bob’s valuation be $u = (w + (\theta_A - w(1 - \theta_A)), \theta_A)$. Denote Alice’s payment for one item by $p$. Then, $p = r$. Similarly, let Alice’s valuation be $v = (\frac{1}{w} + (\theta_B - (1 - \theta_B)/w), \theta_B)$. Bob’s payment for one item is $\theta_B$.

**Proof:** We prove only the first part; The second part is very similar. Consider the following instance:

<table>
<thead>
<tr>
<th>Number of items</th>
<th>Alice’s value</th>
<th>Alice’s payment</th>
<th>Bob’s value</th>
<th>Bob’s payment</th>
</tr>
</thead>
<tbody>
<tr>
<td>One</td>
<td>$\theta_A$</td>
<td>$(1+\delta)p$</td>
<td>$(1+\delta)\theta_A$</td>
<td>$w(1-\theta_A)$</td>
</tr>
<tr>
<td>Both</td>
<td>1</td>
<td>$&gt;1$</td>
<td>$(1+\delta)(w+(\theta_A-w(1-\theta_A)))$</td>
<td>$w$</td>
</tr>
</tbody>
</table>

When $\delta = 0$, the profit of Bob from either one item or two items is equal and non-negative (by Claim 4.33). Thus, when $\delta > 0$ Bob is allocated two items and Alice is allocated no items. This implies that $p \geq (1 + \delta)\theta_A$. When $\delta < 0$ the no bidder is allocated two items. If $\delta < 0$ is small enough, to preserve the approximation ratio, each bidder, and in particular Alice, must be allocated one item (since each bidder contributes about a half of the value of the solution that allocates one item to each bidder). In this case we therefore have that $p \leq (1 + \delta)\theta_A$. Taking $\delta$ to 0 from above and below we get that $p = \theta_A$, as needed.

**Lemma 4.36** Let $r > \theta_A$. Then $f(r) = \frac{r}{\theta_A^2} - r$. Similarly, for every $r > \theta_B$, $g(r) = \frac{r}{\theta_B^2} - r$.

**Proof:** We prove only the first statement. Let $1 \geq t \geq \theta_A$. Consider the following instance (Alice’s payment for one item is by Claim 4.35):

<table>
<thead>
<tr>
<th>Number of items</th>
<th>Alice’s value</th>
<th>Alice’s payment</th>
<th>Bob’s value</th>
<th>Bob’s payment</th>
</tr>
</thead>
<tbody>
<tr>
<td>One</td>
<td>$\theta_A$</td>
<td>$(1+\delta)\theta_A$</td>
<td>$(1+\delta)\theta_A$</td>
<td>$wf(f(\frac{\theta_A}{t}))$</td>
</tr>
<tr>
<td>Both</td>
<td>$t$</td>
<td>$&gt;1$</td>
<td>$(1+\delta)(w+(\theta_A-w(1-\theta_A)))$</td>
<td>$w$</td>
</tr>
</tbody>
</table>

We will show that when $\delta = 0$ for every $1 \geq t \geq \theta_A$, Bob’s payments are identical: for one item the payment is always $w(1 - \theta_A)$ and for 2 items it is $w$. This implies, using scalability, that $tf(\frac{\theta_A}{t}) = f(\theta_A) = 1 - \theta_A$. From the last equation we can calculate $f(r)$ for every $1 \geq r \geq \theta_A$: using $t$ such that $r = \frac{\theta_A}{t}$, we have that $\frac{\theta_A}{t}f(r) = 1 - \theta_A$, and therefore $f(r) = \frac{r}{\theta_A^2} - r$, as needed.

We start by showing that Bob’s payment for two items is identical for all $1 \geq t \geq \theta_A$. Using scalability and the formula for the payment for two items, the payment is $w \cdot t \cdot \frac{r}{\theta_A^2} = w$.

We now show that the Bob’s payment for one item is the same for all such $t$. Suppose for contradiction that for some $t$, $tf(\frac{\theta_A}{t}) > 1 - \theta_A$. Observe that for small enough $\delta < 0$ the Bob is allocated two items, whereas Alice is allocated one item – a contradiction to the feasibility of the mechanism. Now suppose that for some $t$, $tf(\frac{\theta_A}{t}) < 1 - \theta_A$. In this case fix some small enough $\delta > 0$ and observe that Bob is allocated one item, whereas Alice is allocated no items at all. In this case the value of the solution that allocates one item to each bidder is $\theta_A + (1 + \delta)\theta_A$, but the value of the solution the algorithm obtains is only $(1 + \delta)\theta_A$. The algorithm provides an approximation ratio that approaches 2 as $\delta$ approaches 0. A contradiction.
5 Characterizing Mechanisms for any Number of Items

In the previous sections we showed that every scalable mechanism for two items that provides an approximation ratio better than 2 is a triage auction. This section gives an almost complete characterization for truthful and scalable mechanisms for any number of items that have an approximation ratio better than 2. In particular this section’s characterization implies that truthful and scalable mechanisms for multi-unit auctions cannot guarantee an approximation ratio better than 2 in polynomial-time.

The 2-items characterization is used as a black box to prove the characterization of mechanisms for three or more items. Importantly, the scalability assumption is not used in this section. In other words, proving that triage auctions are the only truthful mechanisms (scalable or not) that provide an approximation ratio better than 2 in multi-unit auctions with only two items, would immediately imply our characterization result any number of items, and in particular would imply an unconditional lower bound on the power of all polynomial time truthful mechanisms.

5.1 Induced Mechanisms: Definition and Basic Properties

Definition 5.1 Let \( l_1, l_2, h_1, h_2 \) be such that \( 1 \leq l_1 < h_1 \leq m \) and \( 1 \leq l_1 < h_1 \leq m \). The \((l_1, h_1)\)-extension of a valuation \( v \), denoted \( v^{l_1, h_1} \), is defined as follows: for every \( k < l_1 \), \( v^{l_1, h_1}(k) = 0 \). For every \( h_1 > k \geq l_1 \), \( v^{l_1, h_1}(k) = v(1) \). For every \( h_1 \geq h_1 \), \( v^{l_1, h_1}(k) = v(2) \).

Definition 5.2 (Induced Mechanism) Let \( A \) be a mechanism for multi-unit auctions with \( m \) items and 2 bidders. Let \( l_1, h_1, l_2, h_2 \) be integers with the following constraints: \( l_1 < h_1 \leq m \), \( l_2 < h_2 \leq m \), \( l_1 + l_2 \leq m \), \( l_1 + h_2 > m \) and \( l_2 + h_1 > m \). Define the induced mechanism \( A^{l_1, h_1, l_2, h_2} \) for 2 items as follows: given two valuations \( v \) and \( u \) run \( A \) with the \((l_1, h_1)\)-extended valuation \( v^{l_1, h_1} \) and the \((l_2, h_2)\)-extended valuation \( u^{l_2, h_2} \). Let \( (a_1, a_2) \) be the output allocation of \( A \) and \((p_1, p_2)\) be the payments the bidders are charged in \( A \). If \( a_1 \leq l_1 \) then let \( a'_1 = 0 \), if \( l_1 \leq a_1 < h_1 \) then let \( a'_1 = 1 \), otherwise let \( a'_1 = 2 \). If \( a_2 < l_2 \) then let \( a'_2 = 0 \), if \( l_2 \leq a_2 < h_2 \) then let \( a'_2 = 1 \), otherwise let \( a'_2 = 2 \). The output of \( A^{l_1, h_1, l_2, h_2} \) on \( v \) and \( u \) is \((a'_1, a'_2)\). The payment of Alice is \( p_1 \) and the payment of the Bob is \( p_2 \).

Proposition 5.3 Let \( A \) be a scalable mechanism for multi-unit auctions with \( m \) items and 2 bidders that provides an approximation ratio of \( \alpha \). Let \( A^{l_1, h_1, l_2, h_2} \) be an induced mechanism. \( A^{l_1, h_1, l_2, h_2} \) is feasible, truthful, scalable, and provides an approximation ratio of \( \alpha \).

Proof: We prove that the four properties hold. Recall that \((a_1, a_2)\) is the output of \( A \) and \((a'_1, a'_2)\) be the output of the induced mechanism.

- **Feasibility:** Since \( l_1 + h_2 > m \) if Bob is allocated two items in \( A^{l_1, h_1, l_2, h_2} \) then Alice is allocated no items. Similarly, \( l_2 + h_1 > m \) so if Alice is allocated two items then Bob is allocated no items. Thus \( A^{l_1, h_1, l_2, h_2} \) is feasible if there is a bidder that is allocated two items. Feasibility is obvious when each of the bidders is allocated at most one item in \( A^{l_1, h_1, l_2, h_2} \).

- **Truthfulness:** We prove for Alice with valuation \( v \). The proof for Bob with valuation \( u \) is similar. Observe that \( v^{l_1, h_1}(a_1) = v(a'_1) \) and that the payment of Alice is identical in \( A \) and in \( A^{l_1, h_1, l_2, h_2} \). Hence, the profit of Alice from taking \( t \) items in \( A \) is identical to her profit from taking \( t' \) items in the induced mechanism (\( t' = 0 \) if \( t < l_1 \), \( t' = 2 \) if \( t \geq h_1 \) and \( t' = 1 \) otherwise). Thus, since \( A \) is a truthful mechanism and Alice is allocated her most profitable bundle in \( A \), she is also allocated her most profitable bundle in the induced mechanism. To conclude this proof, observe that since \( A \) is truthful the payment of Alice depends only on Bob’s valuation.

- **Approximation Ratio:** Observe that for every allocation of 2 items \( s' = (s'_1, s'_2) \) there is an allocation for \( m \) items \( s = (s_1, s_2) \) such that \( v(s'_1) + u(s'_2) = v^{l_1, h_1}(s_1) + u^{l_2, h_2}(s_2) \), and vice versa.
In particular, the value of the optimal solution in the instance \((v, u)\) and in \((v_{1, h_1}^{1}, u_{1, h_2}^{1})\) is the same. Thus, if the allocation of \(m\) items \((a_{1}, a_{2})\) provides an approximation ratio of \(\alpha\) to the welfare, so does the allocation of 2 items \((a_{1}', a_{2}')\).

- **Scalability:** \(A\) is scalable and thus, for every \(\alpha > 0\), \((a_{1}, a_{2}) = A(v_{1, h_1}^{1}, u_{1, h_2}^{1}) = A(\alpha \cdot v_{1, h_1}^{1}, \alpha \cdot u_{1, h_2}^{1})\). Since the output of the induced mechanism, \((a_{1}', a_{2}')\), depends only on the output of \(A\), \((a_{1}, a_{2})\), we have that, for every \(\alpha > 0\), \((a_{1}', a_{2}') = A_{1, h_1}^{1, h_2, h_2}(v, u) = A_{1, h_1}^{1, h_2, h_2}(\alpha \cdot v, \alpha \cdot u)\).

In this section we denote the \(p^{(2)}\) function of \(A\) (the payments induced by Alice) by \(f\) and by \(f_{1, h_1}^{1, h_1} l_2, h_2\) the \(p^{(2)}\) function of the induced mechanism \((l_1, h_1, l_2, h_2)\). We denote by \(g\) the \(p^{(1)}\) function of \(A\) (the payments induced by Bob) and by \(g_{1, h_1}^{1, h_1} l_2, h_2\) the \(p^{(1)}\) function of the induced mechanism \((l_1, h_1, l_2, h_2)\).

**Corollary 5.4** Let \(v\) be a valuation for 2 items and let \(v_{1, h_1}^{1}\) be its \((l_1, h_1)\)-extension. Let \(l_2, h_2\) be such that \(A_{1, h_1}^{1, h_2, h_2}\) is an induced mechanism. \(f_1 f_{1, h_1}^{1, h_1} l_2, h_2(v) = f_{1, h_1}^{1, h_2, h_2}(v)\) and \(f_2 f_{1, h_1}^{1, h_1} l_2, h_2(v) = f_{2, h_1}^{1, h_2, h_2}(v)\).

Symmetrically, let \(u\) be a valuation for 2 items and let \(u_{1, h_2}^{1}\) be its \((l_2, h_2)\)-extension. Let \(l_1, h_1\) be such that \(A_{1, h_1}^{1, h_1, l_2, h_2}\) is an induced mechanism. \(g_1 g_{1, h_1}^{1, h_1} l_2, h_2(u) = g_{1, h_1}^{1, h_1, l_2, h_2}(u)\) and \(g_2 g_{1, h_1}^{1, h_1} l_2, h_2(u) = g_{2, h_1}^{1, h_1, l_2, h_2}(u)\).

### 5.2 Relations between Induced Mechanisms

Let \(A\) be a scalable and truthful mechanism for multi-unit auctions for \(m\) items that provides an approximation ratio better than 2. By our characterization and the discussion above we have that all the induced mechanisms of \(A\) are triage auctions. Denote the parameters of the triage mechanism \(A_{1, h_1}^{1, h_1, l_2, h_2}\) by \(\theta_A^{1, h_1} l_2, h_2, \theta_B^{1, h_1} l_2, h_2, u_{1, h_1}^{1, h_1, l_2, h_2}\).

**Claim 5.5** Let \(A\) be a truthful and scalable mechanism for multi unit auction with \(m\) items that provides an approximation ratio better than 2. Let \(A_{1, h_1}^{1, h_1, l_2, h_2}\) and \(A_{1, h_1}^{1, h_1, l_2, h_2}\) be two induced mechanisms of \(A\). Then, \(w_{1, h_1}^{1, h_1, l_2, h_2} = w_{1, h_1}^{1, h_1, l_2, h_2}, \theta_A^{1, h_1} l_2, h_2, \theta_B^{1, h_1} l_2, h_2\), and \(\theta_B^{1, h_1} l_2, h_2, \theta_B^{1, h_1} l_2, h_2\). Symmetrically, let \(A_{1, h_1}^{1, h_1, l_2, h_2}\) and \(A_{1, h_1}^{1, h_1, l_2, h_2}\) be two induced mechanisms of \(A\). Then, \(w_{1, h_1}^{1, h_1, l_2, h_2} = w_{1, h_1}^{1, h_1, l_2, h_2}, \theta_A^{1, h_1} l_2, h_2, \theta_B^{1, h_1} l_2, h_2\), and \(\theta_B^{1, h_1} l_2, h_2, \theta_B^{1, h_1} l_2, h_2\).

**Proof:** We prove only the first statement, the second one is proved using symmetric arguments. Consider the payment functions \(f_{1, h_1}^{1, h_1} l_2, h_2(v)\) and \(f_{1, h_1}^{1, h_1} l_2, h_2(v)\). These functions must be the same since by corollary 5.3, they equal \(f_{1, h_1}^{1, h_1} l_2, h_2(v)\), where \(v\) is the \((l_1, h_1)\)-extension of \(v\). Notice that the equality of these two payment functions implies equality of all the parameters that define the single-item payment:

\[
1 - \theta_B^{1, h_1} l_2, h_2 = 1 - \theta_A^{1, h_1} l_2, h_2\]

is the infimum of all \(r\) such that the derivative of \(f_{1, h_1}^{1, h_1} l_2, h_2((1, r, 0)) = f_{1, h_1}^{1, h_1} l_2, h_2((1, 0, r))\) is negative. \(\theta_A^{1, h_1} l_2, h_2 = \theta_B^{1, h_1} l_2, h_2\) is the infimum of all \(r \geq 1\) such that the derivative of \(f_{1, h_1}^{1, h_1} l_2, h_2((1, r, 0)) = f_{1, h_1}^{1, h_1} l_2, h_2((1, 0, r))\) is positive. Finally, to see that \(w_{1, h_1}^{1, h_1, l_2, h_2} = w_{1, h_1}^{1, h_1, l_2, h_2}\), notice that \(w_{1, h_1}^{1, h_1, l_2, h_2} \theta_B^{1, h_1} l_2, h_2 = f_{1, h_1}^{1, h_1} l_2, h_2((1, 0, 0)) = f_{1, h_1}^{1, h_1} l_2, h_2((1, 0, 0)) = w_{1, h_1}^{1, h_1, l_2, h_2} \theta_B^{1, h_1} l_2, h_2\). Since we already have that \(\theta_B^{1, h_1} l_2, h_2 = \theta_B^{1, h_1} l_2, h_2\), we conclude that \(w_{1, h_1}^{1, h_1, l_2, h_2} = w_{1, h_1}^{1, h_1, l_2, h_2}\).

**Claim 5.6** Let \(A\) be a truthful and scalable mechanism for multi unit auction with \(m\) items that provides an approximation ratio better than 2. Let \(A_{1, h_1}^{1, h_1, l_2, h_2}\) and \(A_{1, h_1}^{1, h_1, l_2, h_2}\) be two induced mechanisms of \(A\). Then, \(\theta_A^{1, h_1} l_2, h_2, \theta_B^{1, h_1} l_2, h_2\), and \(w_{1, h_1}^{1, h_1, l_2, h_2} = w_{1, h_1}^{1, h_1, l_2, h_2}\). Symmetrically, let \(A_{1, h_1}^{1, h_1, l_2, h_2}\) and \(A_{1, h_1}^{1, h_1, l_2, h_2}\) be two induced mechanisms of \(A\). Then, \(\theta_B^{1, h_1} l_2, h_2, \theta_B^{1, h_1} l_2, h_2\), and \(w_{1, h_1}^{1, h_1, l_2, h_2} = w_{1, h_1}^{1, h_1, l_2, h_2}\).
Proof: We prove only the first statement, the second one is symmetric. As in the proof of Claim 5.5 we have that the equality \( f_2^{l_1,h_1,l_2,h_2}(v) = f_2^{l_1,h_1,l_2,h_2}(v) \), for every valuation \( v \) for two items. This equality implies equality in all parameters that define the two-items payment. To see that, we use the face that \( f_2^{l_1,h_1,l_2,h_2}((1,0,0)) = f_2^{l_1,h_1,l_2,h_2}((1,0,0)) \) and that \( f_2^{l_1,h_1,l_2,h_2}((1,1,0)) = f_2^{l_1,h_1,l_2,h_2}((1,1,0)) \). From the first equality we get that \( w_0^{l_1,h_1,l_2,h_2} = w_0^{l_1,h_1,l_2,h_2} \). From the second equality we get that \( \frac{w_1^{l_1,h_1,l_2,h_2}}{\theta_A^{l_1,h_1,l_2,m}} = \frac{w_2^{l_1,h_1,l_2,h_2}}{\theta_A^{l_1,h_1,l_2,m}} \) and thus \( \theta_A^{l_1,h_1,l_2,h_2} = \theta_B^{l_1,h_1,l_2,h_2} \).  

Lemma 5.7 Let \( A \) be a truthful and scalable mechanism for multi unit auction with \( m \) items that provides an approximation ratio better than \( 2 \). Let \( A^{l_1,h_1,l_2,h_2} \) and \( A^{l_1',h_1',l_2',h_2'} \) be two induced mechanisms of \( A \). Then, \( w_0^{l_1,h_1,l_2,h_2} = w_0^{l_1',h_1',l_2',h_2'} \). 

Proof: We construct the following graph: each node represents an induced mechanism of \( A \) with parameters \((l_1,h_1,l_2,h_2)\). Construct the following edges: an edge between a node \((l_1,h_1,l_2,h_2)\) and a node \((l_1',h_1',l_2',h_2')\) exists if and only if exactly one of the following equalities does not hold: \( l_1 = l_1' \), \( h_1 = h_1' \), \( l_2 = l_2' \), \( h_2 = h_2' \). Notice that by Claims 5.5 and 5.6 if two nodes \((l_1,h_1,l_2,h_2)\) and \((l_1',h_1',l_2',h_2')\) are connected then \( w_0^{l_1,h_1,l_2,h_2} = w_0^{l_1',h_1',l_2',h_2'} \). Thus to prove the lemma it suffices to show that the graph is connected. We will show this by observing that there is a path from every node \((l_1,h_1,l_2,h_2)\) to \((1,1,1,1)\): starting from \((l_1,h_1,l_2,h_2)\), we first increase \( h_A \) and then \( h_B \) all the way up to \( m \) and then reduce \( l_A \) and \( l_B \) to 1 while changing one index at a time. 

Hence from now on we denote the \( w_0^{l_1,h_1,l_2,h_2} \) parameter of every induced mechanism \( A^{l_1,h_1,l_2,h_2} \) by \( w \).

5.3 Some Induced Mechanisms are Welfare Maximizers

Lemma 5.8 Let \( A \) be a truthful and scalable mechanism for multi unit auction with \( m \) items that provides an approximation ratio better than \( 2 \). Let \( A^{l_1,h_1,l_2,h_2} \) where \( l_1,l_2 \geq 2 \). Then, \( \theta_A^{l_1,m,l_2,m} = \theta_B^{l_1,m,l_2,m} = 1 \). 

Proof: The proof uses the following two claims.

Claim 5.9 Let \( m - 2 \geq 1 \). Then, \( \theta_A^{l_1,l_1+1,m-l_1,m-l_1+1} = 1 \). Similarly, let \( m - 2 \geq 2 \). Then, \( \theta_B^{l_1-l_2,m-l_2+1,l_2+1} = 1 \). 

Proof: We prove only the first statement. The second one is symmetric. Consider the two induced mechanisms \( A^{l_1,l_1+1,m-l_1,m-l_1+1} \) and \( A^{l_1-1,l_1,m-l_1+1,m-l_1+2} \). Let \( u' \) be the \((m-l_1+1,m-l_1+2)\)-extension of the valuation \((1,1,0)\). Notice that \( u' \) is also the the \((m-l_1+1,m-l_1+2)\)-extension of the valuation \((1,0,0)\). Thus we have that \( \frac{w}{\theta_A^{l_1-1,l_1,m-l_1+1,m-l_1+2}} = g^{l_1-1,l_1,m-l_1+1,m-l_1+2}(u') = g^{l_1,l_1+1,m-l_1,m-l_1+1}(u') = w_0^{l_1,l_1+1,m-l_1,m-l_1+1} \). Since the \( \theta_A \) and \( \theta_B \) parameters of all induced mechanisms take values between 0 and 1, we have that \( \theta_A^{l_1,l_1+1,m-l_1,m-l_1+1} = \theta_B^{l_1,l_1+1,m-l_1,m-l_1+1} = 1 \). 

Claim 5.10 Let \( m - 2 \geq 1 \). Then, \( \theta_A^{l_1,m,l_2,m} = 1 \). Similarly, let \( m - 2 \geq 2 \). Then, \( \theta_B^{l_1,m,l_2,m} = 1 \). 

Proof: We prove only the first statement. The second one is symmetric. By Claim 5.9 we have that \( \theta_A^{l_1,l_1+1,m-l_1,m-l_1+1} = 1 \). By applying claim 5.5 twice, first increasing the value of \( h_1 \) to \( m \) and then the value of \( h_2 \) to \( m \) we have that \( \theta_A^{l_1,m-l_1,m} = 1 \). By applying claim 5.6 while changing the third coordinate from \( m-l_1 \) to \( l_2 \), we conclude that \( \theta_A^{l_1,m,l_2,m} = 1 \). 

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The lemma follows by applying the last claim twice, once for Alice and once for Bob.

An immediate corollary of the lemma is the following:

**Definition 5.11** A valuation \( v \) is called \( l \)-simple if there exists some \( 0 < l < m \) such that for every \( k < l \), \( v(k) = 0 \), and for every \( l \leq k < m \) we have that \( v(k) = v(l) \).

**Corollary 5.12** Let \( l \geq 2 \). For every \( l \)-simple valuation \( v \) we have that \( f_m(v) = wv(m) \) and for all \( 1 < t < m - 1 \) such that \( l + t \leq m \) we have that \( f_t(v) = w(v(m) - v(m - t)) \). Similarly, for every \( l \)-simple valuation \( u \) we have that \( f_m(u) = u(m)/w \) and for all \( 1 < t < m - 1 \) such that \( l + t \leq m \) we have that \( g_t(u) = (u(m) - u(m - t))/w \).

**Proof:** For the first statement, observe that \( A^{l,m,t,m} \) is an induced mechanism for every \( t \) such that \( l + t \leq m \) and apply Lemma 5.3. The proof of the second statement is similar.

### 5.4 Concluding the Characterization

**Lemma 5.13** For each \( v \) where \( v(1) = 0 \), \( f_m(v) = wv(m) \). Symmetrically, for each \( u \) where \( u(1) = 0 \), \( g_m(u) = \frac{u(m)}{w} \).

**Proof:** We prove only the first statement. The second one is symmetric. The proof consists of the following two claims.

**Claim 5.14** For each \( v \), \( f_m(v) \geq wv(m) \).

**Proof:** Assume towards a contradiction that for some \( v \), we have that \( f_m(v) = w(v(m) - \epsilon) \), for some \( \epsilon > 0 \). Consider the instance \( (v, u) \) where \( u = (wv(m) - w\frac{\epsilon}{2}, 0, \ldots, 0) \). Bob’s profit for \( m \) items is positive whereas his profit for every \( k \neq m \) items is at most 0, since \( u(k) = 0 \). Thus Bob is allocated \( m \) items. On the other hand, since \( u \) is \( k \)-simple, \( g_m(u) = v(m) - \frac{\epsilon}{w} \), by Corollary 5.12. Thus the profit of Alice from taking \( m \) items is positive, hence Alice is not allocated the empty bundle. In conclusion, more than \( m \) items are allocated, a contradiction to the feasibility of the mechanism.

**Claim 5.15** For each \( v \) where \( v(1) = 0 \), \( f_m(v) \leq wv(m) \).

**Proof:** Assume towards a contradiction that for some \( v \), we have that \( f_m(v) = w(v(m) + \epsilon) \), for some \( \epsilon > 0 \). Consider the instance \( (v, u) \) where \( u = (wv(m) + \frac{\epsilon}{2}, 0, \ldots, 0) \). Notice that the profit of the Bob is negative for the bundle of all items and therefore his contribution to the welfare is 0. The contribution of Alice to the welfare is 0 too: \( u \) is (in particular) a 2-simple valuation. Thus, by Corollary 5.12, \( g_t(u) = (v(m) + \frac{\epsilon}{2}) > v(m) \) for every \( t \neq m, 1 \). In addition, the bundle of all items has a negative profit for Alice. Thus Alice is allocated at most one item, but \( v(1) = 0 \) (observe that by the monotonicity of the payments, \( g_{m-1}(u) \geq g_{m-2}(u) > v(m) \), thus Alice is not allocated \( m - 1 \) items). In conclusion, the mechanism outputs an allocation with a welfare of 0, a contradiction to the fact that the mechanism provides a bounded approximation ratio.

**Lemma 5.16** Let \( v \) be a valuation with \( v(1) = 0 \). For every \( k \neq 1, m - 1 \) we have that \( f_k(v) = w(v(m) - v(m - k)) \). Similarly, for every valuation \( u \) with \( u(1) = 0 \) we have, for every \( k \neq 1, m - 1 \) that \( g_k(u) = w^{-1}(u(m) - u(m - k)) \).
Proof: We prove only the first statement. The second one is symmetric. Notice that the lemma holds for $k = m$, by Lemma [5.13]. Next, towards a contradiction, assume that for some valuation $v$ and some $k$, $f_k(v) > w(v(m) - v(m - k))$ (we will consider the case where $f_k(v) > w(v(m) - v(m - k))$ later). Let $\epsilon = f_k(v) - w(v(m) - v(m - k))$. Let $u$ be the valuation where $u(m) = w(m) + \epsilon/4$, $u(l) = w(v(m) - v(m - k)) + \epsilon/2$, for $k \leq l < m$ and $u(l) = 0$ for every $l < k$. Consider the instance $(v, u)$. Bob is allocated $m$ items since this is its only profitable alternative. Observe that the payment induce by Bob for $m - k$ items is: $w^{-1}(u(m) - u(k)) = w^{-1}(w(m) + \epsilon/4 - w(v(m) - v(m - k)) - \epsilon/2) = v(m - k) - w^{-1}\epsilon/2$. Hence Alice’s profit from taking $m - k$ items is positive and the mechanism allocates more than $m$ items. A contradiction to the feasibility of the mechanism.

Now, towards a contradiction, assume that $f_k(v) < w(v(m) - v(m - k))$. Let $\epsilon = w(v(m) - v(m - k)) - f_k(v)$. Let $u$ be the valuation where $u(m) = w(v(m) - \epsilon/4)$, $u(l) = w(v(m) - v(m - k) - \epsilon/2)$, for $k \leq l < m$ and $u(l) = 0$ for every $l < k$. Consider the instance $(v, u)$. Bob is allocated at least $k$ items. Observe, however, that Alice is allocated more than $m - k$ items since by Corollary 5.12 her payment for every $t$ items, $1 < t < m - k$, is $v(m) - \epsilon/4 - (v(m) - v(m - k) - \epsilon/2) = v(m - k) + \epsilon/4$ and that her profit for the bundle of $m$ items is positive. A contradiction to the feasibility of the mechanism. \[\Box\]

From the lemma we get our final characterization result:

**Definition 5.17** A valuation $v$ is called degenerate if $v(1) = 0$ and $v(m-1) = v(m-2)$.

**Theorem 5.18 (Characterization of mechanisms for any number of items)** Let $A$ be a truthful and scalable mechanism for $m \geq 2$ items and two bidders that provides an approximation ratio better than 2. There exists a constant $\omega > 0$ such that on all inputs $(v, u)$ where $v$ and $u$ are degenerate, $A$ outputs a solution with value $\max_k(v(k) + wu(m-k))$.

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