Continuous Review Inventory Model with Fuzzy Stochastic Demand and Variable Lead Time

Nita H. Shah, Department of Mathematics, Gujarat University, Ahmedabad, Gujarat, India
Hardik N. Soni, Chimanhbai Patel Post Graduate Institute of Computer Applications, Ahmedabad, Gujarat, India

ABSTRACT

The present study considers a continuous review inventory system for the inventory model involving fuzzy random demand, variable lead-time with backorders and lost sales. The authors first use the triangular fuzzy number count upon lead-time to construct a lead-time demand. Using credibility criterion, the expected shortages are calculated. Without loss of generality, the authors have assumed that all the observed values of the fuzzy random variable, representing the demand are triangular fuzzy numbers. Consequently, the value of total expected cost in the fuzzy sense is derived using the expected value criterion or credibility criterion. For the proposed model, the authors provide a solution to find the optimal lead-time and the optimal order quantity along with the reorder point such that the total expected cost in the fuzzy sense has a minimum value. Numerical study is also provided to illustrate the results of proposed model.

Keywords: Continuous Review, Fuzzy Expected Value, Fuzzy Random Variable, Inventory, Lead-Time, Triangular Fuzzy Number Count

1. INTRODUCTION

In era of technology, every transaction in business can be reviewed on continuous basis for the purpose of monitoring and control. Being an important entity of business, inventory also requires continuous review for timely replenishment. In this context continuous review inventory system is an appropriate mathematical model to handle such problem. In traditional continuous review inventory systems, the lead-time demand deals with the probability theory and the annual average demand and the associated costs are presented by a crisp value (Hadley & Whitin, 1963; Singal et al., 1994). In addition, the issue of lead-time reduction has received a great deal of attention in the field of production/inventory management.

DOI: 10.4018/ijaie.2012070102
We begin by reviewing recent studies on how to control lead-time. Liao and Shyu (1991) stated that lead-time is negotiable and can be decomposed into several components, each having a different piece-wise linear crash cost function for lead-time reduction. Ben-Daya and Raouf (1994) extended Liao and Shyu’s (1991) work to consider both lead-time and order quantity as decision variables. Moon and Gallego (1994) assumed unfavorable lead-time demand distribution and solved both the continuous review and periodic review models with a mixture of backorders and lost sales using the minmax distribution free approach. Ouyang et al. (1996) generalized Ben-Daya and Raouf’s (1994) assumption that shortages were allowed and constructed variable lead-time from a mixed inventory model with backorders and lost sales. Moon and Choi (1998), Hariga and Ben-Daya (1999) and Lan et al. (1999) pointed out three improvements for Ouyang et al. (1996). Moon and Choi (1998) considered the reorder point as a new decision variable. Lan et al. (1999) constructed a simplified solution procedure. Hariga and Ben-Daya (1999) developed five stochastic inventory models with complete and partial information about the lead-time demand distribution such that the reorder point is a decision variable. Chu et al. (1999) improved the solution procedure of Ben-Daya and Raouf (1994) by using the Newton–Raphson method with an appropriate starting point. Ouyang and Chuang (2000) considered a lotsize-reorder point inventory model with fuzzy demand. Chang et al. (2004) presented a lead-time reduction model based on continuous review inventory systems in which the uncertainty of demand during lead-time is dealt with a probabilistic fuzzy set and annual average demand by a fuzzy number only. Chang et al. (2006) presented the same work considering lead-time demand as fuzzy random variable instead of probabilistic fuzzy set. Vijayan and Kumaran (2008) investigated the mixed \((Q, r)\) and \((R, T)\) inventory models including trapezoidal fuzzy costs. Thangam and Uthyakumar (2009) formulated a realistic supply chain model with imprecise demand, lead time and inventory costs. Recently, Shah and Soni (2011) developed continuous review inventory model for fuzzy price dependent inventory models with variable lead-time and present value.

There is no question that uncertainty plays a role in almost all scenarios dealing with inventory management. There are many reasons for variability and uncertainty in inventory systems. The conventional approaches for treating uncertainty in inventory management rely on probability theory. However, the probability based approaches may not be ample to reflect all uncertainties that may arise in a real world inventory system. On this view, several researchers used fuzzy set theory in modeling of inventory systems (cf. Chen et al., 1996; Chang & Yao, 1998; Yao & Lee, 1996, 1999; Yao & Chiang, 2003).

With the development of fuzzy set theory, the fuzzy/fuzzy stochastic approach has also been employed extensively for the characterizing the parameters of inventory problems. Along the direction of continuous review system several researchers have performed the investigation using different approaches in the fuzzy framework. For instance, Gen et al. (1997) proposed a new method for the continuous review inventory model where triangular fuzzy numbers represented input data. Kao and Hsu (2002) have considered a lotsize-reorder point inventory model with fuzzy demand. Chang et al. (2004) presented a lead-time reduction model based on continuous review inventory systems in which the uncertainty of demand during lead-time is dealt with a probabilistic fuzzy set. Vijayan and Kumaran (2008) investigated the mixed \((Q, r)\) and \((R, T)\) inventory models including trapezoidal fuzzy costs. Thangam and Uthyakumar (2009) formulated a realistic supply chain model with imprecise demand, lead time and inventory costs. Recently, Shah and Soni (2011) developed continuous review inventory model for fuzzy price dependent
demand and imprecise inventory costs. From literature survey, little work has been devoted in the continuous review inventory problems in which fuzziness and randomness appear simultaneously into an optimization setting. Dutta et al. (2007) have developed continuous review inventory model with constant lead-time in a mixed environment by incorporating fuzzy random variable as the customer demand. They treated lead-time demand as triangular fuzzy number and computed expected shortage using interval valued possibilistic mean. They took lead-time as constant but didn’t control it. Moreover, in many real-life inventory systems, it is more reasonable to assume that only a fraction of the demand during stock-out period is needed to be backordered, and remaining fraction is lost.

Based on above arguments, the present study considers a continuous review inventory system for the inventory model involving fuzzy random demand, variable lead-time with backorders and lost sales. The triangular fuzzy number count upon lead-time is used to construct a lead-time demand. Since the annual demand is a fuzzy random variable, the associated cost function is also a fuzzy random variable. Consequently, the value of total expected cost in the fuzzy sense is derived using the expected value criterion or credibility criterion. For the proposed model, we provide a solution procedure to find the optimal lead-time and the optimal order quantity along with the reorder point such that the total expected annual cost in the fuzzy sense has a minimum value.

The rest of this paper is organized as follows. In section 2, we recall some preliminary knowledge. In section 3 notations and assumptions are given which are used to develop the proposed model. Section 4 proposes a fuzzy expected value model. Mathematical analysis is carried out in section 5. Numerical example is provided in section 6 to demonstrate the effectiveness of the solution methodology. Section 7 furnishes conclusion and further extension.

2. PRELIMINARY

2.1. Possibility Measure, Necessity Measure and Credibility Measure

Let Θ be a nonempty set, P (Θ) the power set of Θ and Pos a possibility measure. Then the triplet (Θ, P(Θ), Pos) is called a possibility space. In this paper, the fuzzy variable is defined as follows:

**Definition 1**: A fuzzy variable is defined as a function from a possibility space (Θ, P(Θ), Pos) to the set of real numbers (Nahmias, 1978; Liu & Liu, 2002).

Let ξ be a fuzzy variable defined on the possibility space (Θ, P(Θ), Pos). Then its membership function µ is derived from the possibility measure through:

\[ µ(x) = \text{Pos} \{ \theta \in \Theta | \xi(\theta) = x \}, \quad x \in \mathbb{R} \]

**Definition 2**: Let ξ be a fuzzy variable with membership function µ. Then for any Borel set B of real numbers (Liu & Liu, 2002):

\[
\text{Pos} \{ \xi \in B \} = \sup_{x \in B} \mu(x) \quad (1)
\]

\[
\text{Nec} \{ \xi \in B \} = 1 - \text{Pos} \{ \xi \in B^c \} = 1 - \sup_{x \in B^c} \mu(x) \quad (2)
\]

\[
\text{Cr} \{ \xi \in B \} = \frac{1}{2} \left( \text{Pos} \{ \xi \in B \} + \text{Nec} \{ \xi \in B \} \right) \quad (3)
\]

Liu and Iwamura (1998) came to the equation shown in Box 1.

Also, * is any of the relation >, <, =, ≤, ≥:
Triangle fuzzy number (TFN): A TFN $\xi$ is specified by three parameters $a_1, a_2, a_3$, where $a_1 < a_2 < a_3$ and is characterized by possibility distribution $\mu_\xi$, given by:

$$\mu_\xi(x) = \begin{cases} 
\frac{x-a_1}{a_2-a_1} & \text{for } a_1 \leq x \leq a_2 \\
\frac{x-a_2}{a_3-a_2} & \text{for } a_2 \leq x \leq a_3 \\
0 & \text{otherwise}
\end{cases}$$

If $\xi$ be a TFN and $t$ be any crisp number. Then according to Liu and Iwamura [16] we define possibility measure and necessity measure as follows:

$$\text{Pos}(\xi \leq t) = \begin{cases} 
0 & \text{for } t < a_1 \\
\frac{t-a_1}{a_2-a_1} & \text{for } a_1 \leq t \leq a_2 \\
1 & \text{otherwise}
\end{cases}$$

For TFN $\xi$ and $t$ be any crisp number. Then credibility distribution of TFN is given by:

$$\Phi(t) = \begin{cases} 
0 & \text{for } t < a_1 \\
\frac{t-a_1}{2(a_2-a_1)} & \text{for } a_1 \leq t \leq a_2 \\
\frac{t+a_3-2a_2}{2(a_3-a_2)} & \text{for } a_2 \leq t \leq a_3 \\
1 & \text{otherwise}
\end{cases}$$

Definition 4: Let $\xi$ be a fuzzy variable then the expected value $E[\xi]$ is defined as (Liu and Liu, 2002):

$$E[\xi] = \int_{-\infty}^{\infty} Cr\{\xi \geq r\} \, dr - \int_{-\infty}^{0} Cr\{\xi \leq r\} \, dr$$

provided that at least one of the two integral is finite:

Proposition 1: Let $\xi$ is a fuzzy variable with credibility distribution $\Phi$ (Liu, 2007). If:

$$\lim_{t \to -\infty} \Phi(t) = 0, \quad \lim_{t \to \infty} \Phi(t) = 1$$

and the Lebesgue-Stieltjes integral $\int_{-\infty}^{\infty} t \, d\Phi(t)$ is finite, then we have:

$$E[\xi] = \int_{-\infty}^{\infty} t \, d\Phi(t)$$
Hence, using (7) and (8) the TFN
\[ \xi = (a_1, a_2, a_3) \]
has an expected value:
\[ E[\xi] = \frac{a_1 + 2a_2 + a_3}{4} \tag{9} \]

### 2.2. Fuzzy Random Variable (FRV) and Its Expectation

Let \((\Omega, A, P)\) be the probability space, and \(\hat{X}\) be a random variable on \((\Omega, A, P)\) with probability density function \(f(x)\). Fuzzy random variable \(\hat{X}\) is a mapping from \(\mathbb{R}\) to a family of fuzzy numbers (Kwakernaak, 1978). Intuitively speaking, fuzzy random variables are the random variables whose values are fuzzy numbers.

In this paper assuming that demand is fuzzy random in nature, denoting it by \(\hat{D}\), having mean \(\hat{d}\) with membership function:

\[
\mu_d(x) = \begin{cases} 
\frac{x - d + \Delta_1}{\Delta_1} & d - \Delta_1 \leq x \leq d \\
\frac{d + \Delta_2 - x}{\Delta_2} & d \leq x \leq d + \Delta_2 \\
0 & \text{otherwise}
\end{cases}
\]

where \(d\) is the observation of \(\hat{D}\) satisfying the condition \(d > \max\{\Delta_1, \Delta_2\}\) and \(\Delta_1\) and \(\Delta_2\) are determined by decision maker (DM):

**Example 1:** Consider the two types of data, first type is in stochastic and second type is in fuzzy stochastic environment (see Tables 1 and 2).

Here \(\hat{d}_i\) is the value around \(d_i\) with probability \(p_i\), where \(\{d_i - \Delta_1, d_i, d_i + \Delta_2\}\) (for TFN) and \(\sum_{i=1}^{n} p_i = 1\). Then the mean of two types of data are respectively:
\[ \sum_{i=1}^{n} d_i p_i = d \text{ (say)} \]
and:
\[ \sum_{i=1}^{n} \hat{d}_i \otimes p_i = \left( \sum_{i=1}^{n} d_i p_i - \Delta_1, \sum_{i=1}^{n} d_i p_i, \sum_{i=1}^{n} d_i p_i + \Delta_2 \right) = (d - \Delta_1, d, d + \Delta_2) = d \]

In next section we present the notations and assumptions used in developing the model.

### 3. NOTATION AND ASSUMPTIONS

We use the following notation and assumptions to develop inventory models with crashing component lead-time, backorders and lost sales:

- \(A\): Fixed ordering cost per order;
- \(\hat{D}\): Demand rate in units per year which is fuzzy random in nature;
- \(\hat{d}\): Expected demand rate in units per year which is fuzzy in nature;
- \(h\): Inventory holding cost per item per year which is fuzzy in nature;
- \(L\): Lead-time (decision variable) that has \(n\) mutually independent components. The i-th component has a minimum duration \(a_i\) and normal duration \(b_i\) with a crashing cost \(c_i\) per unit time under the assumption \(c_1 \leq c_2 \leq \ldots \leq c_n\). The components of \(L\) are crashed one at a time starting from the component of least \(c_i\) and so forth. Hence, the range for \(L\) is from \(\sum_{j=1}^{n} a_j\) to \(\sum_{j=1}^{n} b_j\);
• $L_r$: Length of lead-time with components 1; 2;...; $r$ crashed to their minimum durations. We define:

$$L_n = \sum_{j=1}^{n} a_j$$

and:

$$L_r = L_n + \sum_{k=r+1}^{n} (b_k - a_k)$$

for:

$$r = 0, 1, ..., n - 1$$

Since $b_r > a_r$, it follows that $L_{r-1} > L_r$ for $r = 1, 2, ..., n$.

• $Q$: Order quantity (decision variable);
• $Q_r$: Optimal order quantity when lead-time is $L_r$;

• $C(L)$: Lead-time crashing cost per cycle for a given $L \in [L_r, L_{r-1}]$ is given by:

$$C(L) = c_r (L_{r-1} - L) + \sum_{k=1}^{r-1} c_k (b_k - a_k)$$

• $R$: Reorder point (decision variable) given by $R = \text{expected demand during lead-time} + \text{safety stock}$. Inventory is continuously reviewed. Replenishments are made whenever the inventory level falls to the reorder point $R$;

• $d_L$: Random demand during lead-time;
• $B_r$: Number of stock-out per cycle;
• $\beta$: Fraction of the demand during the stock-out period that will be backordered ($0 \leq \beta \leq 1$);

$$\pi - \pi = \pi + (1 - \beta) \pi_0$$

with $\pi$ the fixed penalty cost per unit short and $\pi_0$ the marginal profit per unit;

• $x^+$: Maximum value of $x$ and 0, i.e. $x^+ = \max\{x, 0\}$. 

### Table 1. Stochastic data type

<table>
<thead>
<tr>
<th>Demand</th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>...</th>
<th>$d_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>$p_1$</td>
<td>$p_2$</td>
<td>...</td>
<td>$p_n$</td>
</tr>
</tbody>
</table>

### Table 2. Fuzzy stochastic environment data type

<table>
<thead>
<tr>
<th>Demand</th>
<th>$\tilde{d}_1$</th>
<th>$\tilde{d}_2$</th>
<th>...</th>
<th>$\tilde{d}_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>$p_1$</td>
<td>$p_2$</td>
<td>...</td>
<td>$p_n$</td>
</tr>
</tbody>
</table>
• $E(\bullet)$: Expected value of $(\bullet)$.

4. MATHEMATICAL FORMULATION

By the above assumptions and considering that only a fraction $\beta$ of the demand during the stock-out period can be backordered, Ouyang et al. (1996), established the total annual cost as shown in Box 2.

Here the demand is taken as random variable. But in actual market, it is very difficult to determine a precise value of demand. In this situation, management needs to collect the demand information from experts. When the experts’ opinion are imprecise, like demand is about some fixed quantity and that fixed quantity is randomly chosen, then the demand can be vaguely expressed. Therefore, total annual demand is treated as fuzzy random variable (FRV). When the parameter $D$ becomes FRV, the cost function in (10) is also a FRV and can be written as shown in Box 3.

As the lead-time is variable and the total annual demand is characterized by fuzzy randomly, the value of lead-time demand (LTD) may have variation depending upon the length of lead-time in this uncertain environment. Generally, the demand estimation during the lead-time period is based on the DM’s imprecise apperception from his intrinsic understanding and hence it can be imprecisely expressed. Therefore, associate with the variable lead-time $L$ we consider fuzzy LTD $\tilde{d}_L$ as TFN which is defined as by letting $d_L = p_2L$:

$$
\mu_{d_L}(x) = \begin{cases} 
\frac{x - p_1L}{(p_2 - p_1)L} & \text{for } p_1L \leq x \leq p_2L \\
\frac{x - p_1L}{(p_2 - p_3)L} & \text{for } p_2L \leq x \leq p_3L \\
0 & \text{otherwise}
\end{cases}
$$

Box 2.

$$
\begin{align*}
C(Q, R, L) &= \text{setup cost} + \text{holding cost} + \text{stock-out cost} + \text{lead-time crashing cost} \\
&= A\frac{D}{Q} + h\left[\frac{Q}{2} + R - E[d_L] + \left(1 - \beta\right)E[B_R]\right] + \frac{D}{Q}\left[\pi E[B_R]\right] + \frac{D}{Q}C(L) \\
&= \frac{D}{Q}\left[A + \pi E[B_R] + C(L)\right] + h\left[\frac{Q}{2} + R - E[d_L] + \left(1 - \beta\right)E[B_R]\right] \\
&= A\frac{D}{Q} + \pi E[(d_L - R)^+] + C(L) + h\left[\frac{Q}{2} + r - E[d_L] + \left(1 - \beta\right)E[(d_L - R)^+]\right] \\
&= \tilde{C}(Q, R, L)
\end{align*}
$$

Box 3.

$$
\tilde{C}(Q, R, L) = \frac{D}{Q}\left[A + \pi E[(d_L - R)^+] + C(L)\right] + h\left[\frac{Q}{2} + r - E[d_L] + \left(1 - \beta\right)E[(d_L - R)^+]\right]
$$
where $0 < p_1 < p_2 < p_3$ are confirmed by decision maker are reflect a kind of fuzzy apprehension from his intrinsic understanding.

The expected value of this fuzzy lead-time demand i.e. $E \left[ \tilde{d}_L \right]$ can be obtained using (9). Using (7) the credibility distribution of lead-time demand is given by the equation shown in Box 4.

Now the continuous review inventory model in mixed fuzzy and stochastic environment can be rewritten as shown in Box 5 where $E \left[ \tilde{d}_L - R \right]^+$ is the expected shortage during each cycle.

In order to obtain the specific expression of $E \left[ \left( \tilde{d}_L - R \right)^+ \right]$, we must get the function form of $E \left[ \left( \tilde{d}_L - R \right)^+ \right]$. Note that if $R \geq E \left[ \tilde{d}_L \right]$ then the non-negative criterion of safety stock is always satisfied. Consequently, the lower bound of reorder point $R$ is $E \left[ \tilde{d}_L \right]$. When the actual demand ($d$ say) during lead-time in each cycle is greater than $R$, there is an obvious shortage of amount $d - R$. In order to determine the expected amount of shortage in each cycle two cases will arise according to the position of $R$ in $[p_1L, p_3L]$ subject to the condition $R \geq E \left[ \tilde{d}_L \right]$. 

**Case 1:** Let $R \in [p_1L, p_2L]$ (see Figure 1)

Using (8) of Proposition 1, we come to the equation shown in Box 6:

**Case 2:** Let $R \in [p_2L, p_3L]$ (see Figure 2).

Using (8) of Proposition 1:

**Box 4.**

$$
\Phi_1(t) = \begin{cases} 
0 & \text{for } t < p_1L \\
\frac{t - p_1L}{2(p_2 - p_1)L} & \text{for } p_1L \leq t \leq p_2L \\
\frac{t + p_3L - 2p_2L}{2(p_3 - p_2)L} & \text{for } p_2L \leq t \leq p_3L \\
1 & \text{otherwise}
\end{cases}
$$

**Box 5.**

$$
\bar{C}(Q, R, L) = \frac{D}{Q} \left( A + \pi E \left[ (\tilde{d}_L - R)^+ \right] + C(L) \right) + h \left[ \frac{Q}{2} + R - E[\tilde{d}_L] + (1 - \beta)E[\left( \tilde{d}_L - R \right)^+] \right]
$$

(13)
Figure 1. When \( R \in [p_1L, p_2L] \)

Box 6.

\[
E\left[\left(\tilde{d}_L - R\right)^+\right] = \int_R^{p_2L} (t - R) d\Phi_1(t) = \int_R^{p_1L} (t - R) d\Phi_1(t) + \int_{p_1L}^{p_2L} (t - R) d\Phi_1(t)
\]

\[
= \frac{2p_2L^2 - p_2L\left((p_1 - p_3)L + 4R\right) + 2Rp_1L + R^2 - p_1p_3L^2}{4(p_2 - p_1)L}
\] (14)

Figure 2. When \( R \in [p_2L, p_3L] \)
Now, corresponding to the crisp expected value of a positive random variable, we recall the expectation of a fuzzy random variable is a unique fuzzy number (see Example 1). Let \( \tilde{d} = (d - \Delta_1, d, d + \Delta_2) \) be the expected value of \( \tilde{D} \). Consequently, the total annual cost \( \tilde{C}(Q, R, L) \) becomes a fuzzy quantity. Using linearity of operator \( E \), the expected total annual cost is given by the equation shown in Box 7 where using (9) \( E[d] = d + \frac{\Delta_2 - \Delta_1}{4} \) and \( E[\tilde{d}_L - R]^+ \) given by either (14) or (15) according to the position of \( R \) in \( [E[\tilde{d}_L], p_3L] \).

**5. MATHEMATICAL ANALYSIS**

As the purpose is to determine the optimal order quantity, reorder point and lead-time such that the expected total annual cost given by (16) is minimum, the single objective crisp problem can be given by the equation shown in Box 8:

**Case 1:** For \( R \in [p_1L, p_2L] \) we have the equation shown in Box 9.

It is to be noted that for fixed \( Q \) and \( R \), the minimum value of \( E[\tilde{C}(Q, R, L)] \) will occur at the end points of the interval \( [L_r, L_{r-1}] \).

Because, \( \frac{\partial^2 E[\tilde{C}(Q, R, L)]}{\partial L^2} > 0 \) for all \( L \in [L_r, L_{r-1}] \), i.e. \( E[\tilde{C}(Q, R, L)] \) is convex in \( L \in [L_r, L_{r-1}] \) for fixed \( Q \) and \( R \). Consequently, the minimum value of \( E[\tilde{C}(Q, R, L)] \) will occur at the point \( (Q, R, L_{r-1}) \) which satisfies:

\[
\begin{align*}
E[\tilde{d}_L - R]^+ &= \int_R^{p_3L} (t - R) d\Phi_1(t) \\
&= \left( \frac{p_3L - R}{4} \right) L \\
&= \left( \frac{p_3L - p_2L}{4} \right) L
\end{align*}
\]

**Box 7.**

\[
E[\tilde{C}(Q, R, L)] = \frac{E[\tilde{d}]}{Q} \left[ A + \pi E[\tilde{d}_L - R]^+] + C(L) \right] \\
+ h \left[ \frac{Q}{2} + R - E[\tilde{d}_L] + (1 - \beta) E[\tilde{d}_L - R]^+] \right] \\
\]

**Box 8.**

\[
\text{Minimize} \quad E[\tilde{C}(Q, R, L)] = \frac{E[\tilde{d}]}{Q} \left[ A + \pi E[\tilde{d}_L - R]^+] + C(L) \right] \\
+ h \left[ \frac{Q}{2} + R - E[\tilde{d}_L] + (1 - \beta) E[\tilde{d}_L - R]^+] \right]
\]
Box 9.

\[
E[\widetilde{C}(Q,R,L)] = \left[ A + \frac{-2p_2L^2 - p_2L\left(p_1 - p_3\right)L + 4R + 2Rp_1L + R^2 - p_1p_3L^2}{4\left(p_2 - p_1\right)L} \right] + C(L)
\]

\[
\frac{\partial}{\partial Q} E[\widetilde{C}(Q,R,L_{r-1})] = 0 \quad (18)
\]

\[
\frac{\partial}{\partial R} E[\widetilde{C}(Q,R,L_{r-1})] = 0 \quad (19)
\]

From the above two equations, we come to the equations shown in Box 10.

By equating the square of (21) with (20), \( R \) can be found using Maple or any numerical programming tools. The optimal \( Q \) is then solved by substituting \( R \) into (21).

Since the objective function \( E[\widetilde{C}(Q,R,L_{r-1})] \) is convex in \( Q \) and \( R \) (see Appendix-A.1), then the optimal \( Q \) and \( R \) derived from (20) and (21) respectively give the global minimum value of the objective function:

Case 2: For \( R \in [p_2L, p_3L] \) we have the equation shown in Box 11.

As argued above, for fixed \( Q \) and \( R \), the minimum value of \( E[\widetilde{C}(Q,R,L)] \) will occur at the end points of the interval \( [L_r, L_{r-1}] \). Hence the minimum value of \( E[\widetilde{C}(Q,R,L)] \) will occur at the point \( (Q,R,L_r) \) which satisfies:

Box 10.

\[
Q^2 = \frac{2E[\hat{d}]}{h} \left[ A + \pi \left( \frac{p_3L - R}{4} \right)^2 + \frac{p_3 - p_2}{4L} \right] + C(L) \quad (20)
\]

and:

\[
Q = \frac{-\pi E[\hat{d}]}{h} \left( \frac{R + (2p_2 - p_1)L}{(1 - \beta)R + (2p_2 - p_1)\beta - p_1L} \right) \quad (21)
\]
Box 11.

\[
E\left[\tilde{C}(Q, R, L)\right] = \frac{E[d]}{Q} \left[ A + \pi \left\{ \frac{(p_3 L - r)^2}{4(p_3 - p_2) L} \right\} + C(L) \right] + h \left( \frac{Q}{2} + R - E[\tilde{d}_L] + (1 - \beta) \left\{ \frac{(p_3 L - r)^2}{4(p_3 - p_2) L} \right\} \right)
\]

6. NUMERICAL EXAMPLE

Example 2: Let a retailer estimate the expected yearly demand of some product to be around 600 units, which is represented by the triangle fuzzy number \(\left(570, 600, 650\right)\), i.e. \(\Delta_1 = 30\) and \(\Delta_2 = 50\). Other related parameters are as follows:
- \(A = $300\) per order;
- \(h = $25\) per unit per year;
- \(\pi = $50\) per unit short;
- \(\pi_0 = $100\) per unit lost;
- \(\beta = 0.6\);
- \(p_1 = 7.8\);
- \(p_2 = 10\);
- \(p_3 = 12.5\), and the lead-time has four components with the data shown in Table 3.

Using data given in Table 3, we have:

Box 12.

\[
Q^2 = \frac{2E[\tilde{d}]}{h} \left[ A + \pi \left\{ \frac{(p_3 L - R)^2}{4(p_3 - p_2) L} \right\} + C(L) \right]
\]

\[
Q = \frac{-\pi E[\tilde{d}]}{h} \left\{ \frac{(p_3 L - R)}{(1-\beta) R + \left(1 + \beta\right) p_3 - 2p_2} \right\}
\]
\[ L_0 = 70 \text{ days (10 weeks)} \]
\[ L_1 = 70 - 14 = 56 \text{ days (8 weeks)} \]
\[ L_2 = 56 - 14 = 42 \text{ days (6 weeks)} \]
\[ L_3 = 42 - 7 = 35 \text{ days (5 weeks)} \]

and:

\[ L_4 = 35 - 7 = 28 \text{ days (4 weeks)} \]

Hence:

\[ L_4 = \min_{0 \leq r \leq 4} L_r = 4 \text{ weeks} \]
\[ L_0 = \min_{0 \leq r \leq 0} L_r = 10 \text{ weeks} \]

Note that \( E\left[ \tilde{d}_L \right] = 10.075 \) \( L \) therefore, range of \( R \) is \([10.075L, 12.5L]\) where \( MC = \frac{\min \{ E\left[ C\left(Q, R, L\right) \right] \}}{\text{Minimum}} \).

From Table 4 the optimal order quantity is 125.3053 units, optimal reorder point 73.3129, optimal lead-time 6 weeks and corresponding minimum total expected cost is $3454.68. Figure 3 depicts the total expected cost \( E\left[ C\left(Q, R, L\right) \right] \), which attains its minimum at \( Q = 125.3053 \) and \( R = 73.3129 \) for lead-time demand \( \tilde{d}_L = (46.8, 60.75) \).

Now, let us consider the cases for \( \beta = 0, 0.2, 0.6, 1 \) with various sets of \((p_1, p_2, p_3)\). By the procedure outlined above, we obtain the computed results as presented in Table 5. Note that the numerical results depend on the given values of problem parameters, which therefore, for other cases, it may get different results.

### Table 3. Lead-time data for different components

<table>
<thead>
<tr>
<th>Lead-Time Component (r)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal duration ( b_r ) (days)</td>
<td>20</td>
<td>20</td>
<td>16</td>
<td>14</td>
</tr>
<tr>
<td>Minimum duration ( a_r ) (days)</td>
<td>6</td>
<td>6</td>
<td>9</td>
<td>7</td>
</tr>
<tr>
<td>Unit crashing cost ( c_r ) ($/day)</td>
<td>0.01</td>
<td>0.4</td>
<td>1.8</td>
<td>6.0</td>
</tr>
</tbody>
</table>

### Table 4. Optimum result for different values of \( r \)

<table>
<thead>
<tr>
<th>Lead-Time Component (r)</th>
<th>( MC )</th>
<th>( Q )</th>
<th>( R )</th>
<th>( L )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3584.71</td>
<td>121.8444</td>
<td>122.2641</td>
<td>10</td>
</tr>
<tr>
<td>1</td>
<td>3473.38</td>
<td>121.6973</td>
<td>97.8138</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>3454.68</td>
<td>125.3053</td>
<td>73.3129</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>3640.36</td>
<td>134.9830</td>
<td>60.9881</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>4130.96</td>
<td>156.9193</td>
<td>48.5994</td>
<td>4</td>
</tr>
</tbody>
</table>
Figure 3. Total expected cost when \( L = 6 \)

![Figure 3. Total expected cost when \( L = 6 \)](image)

Table 5. The optimal solutions of proposed model

<table>
<thead>
<tr>
<th>((p_1, p_2, p_3))</th>
<th>((4.5, 6, 9))</th>
<th>((5, 7, 11))</th>
<th>((7.8, 10, 12.5))</th>
<th>((10, 12, 15.5))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\beta)</td>
<td>(\beta)</td>
<td>(\beta)</td>
<td>(\beta)</td>
<td>(\beta)</td>
</tr>
<tr>
<td>0</td>
<td>0.2</td>
<td>0.6</td>
<td>1</td>
<td>0.2</td>
</tr>
<tr>
<td>(Q)</td>
<td>125.0589</td>
<td>125.2542</td>
<td>124.9615</td>
<td>125.1564</td>
</tr>
<tr>
<td>(R)</td>
<td>52.8011</td>
<td>64.3990</td>
<td>74.0016</td>
<td>91.6002</td>
</tr>
<tr>
<td>(MC)</td>
<td>3490.75</td>
<td>3617.00</td>
<td>3463.24</td>
<td>3563.25</td>
</tr>
<tr>
<td>0.2</td>
<td>0.2</td>
<td>0.6</td>
<td>1</td>
<td>0.2</td>
</tr>
<tr>
<td>(Q)</td>
<td>125.1524</td>
<td>125.3795</td>
<td>125.0393</td>
<td>125.6408</td>
</tr>
<tr>
<td>(R)</td>
<td>52.6120</td>
<td>64.1461</td>
<td>73.8444</td>
<td>90.6318</td>
</tr>
<tr>
<td>(MC)</td>
<td>3488.40</td>
<td>3613.86</td>
<td>3461.29</td>
<td>3551.23</td>
</tr>
<tr>
<td>0.6</td>
<td>0.2</td>
<td>0.6</td>
<td>1</td>
<td>0.2</td>
</tr>
<tr>
<td>(Q)</td>
<td>125.4727</td>
<td>125.8095</td>
<td>125.3053</td>
<td>125.2034</td>
</tr>
<tr>
<td>(R)</td>
<td>51.9728</td>
<td>63.2900</td>
<td>73.3129</td>
<td>85.3129</td>
</tr>
<tr>
<td>(MC)</td>
<td>3480.46</td>
<td>3603.25</td>
<td>3454.68</td>
<td>3529.68</td>
</tr>
<tr>
<td>1</td>
<td>0.2</td>
<td>0.6</td>
<td>1</td>
<td>0.2</td>
</tr>
<tr>
<td>(Q)</td>
<td>126.3721</td>
<td>127.0230</td>
<td>126.0504</td>
<td>126.6963</td>
</tr>
<tr>
<td>(R)</td>
<td>50.2402</td>
<td>60.9611</td>
<td>71.8748</td>
<td>88.6023</td>
</tr>
<tr>
<td>(MC)</td>
<td>3459.06</td>
<td>3574.60</td>
<td>3436.88</td>
<td>3526.21</td>
</tr>
</tbody>
</table>

\[ MC = \text{Minimum } E \left[ C \left( Q, R, L \right) \right] \]
demand are triangular fuzzy numbers. So the expected total cost is triangular fuzzy number. Consequently, the value of total expected annual cost in the fuzzy sense is derived using the expected value criterion or credibility criterion. The optimal order quantity, the optimal reorder point, the optimal lead time and the optimal expected cost are obtained. Moreover, the effect of the fuzzy randomness of lead time demand and backorder rate on the optimum solution is analyzed.

The formulation technique can be applied to other inventory models such as vendor-buyer inventory model, two-warehouse inventory model, models with discount, etc.

REFERENCES


APPENDIX A

At point \( (Q, R, L_{r-1}) \):

\[
\frac{\partial^2 E[\tilde{C}(Q, R, L_{r-1})]}{\partial Q^2} = 2E[\tilde{d}] \left[ A + \pi \left( \frac{(p_2 L - R)^2}{4(p_2 - p_1)L} + p_2 L - 2R + p_3 L \right) + C(L) \right] > 0
\]

\[
\frac{\partial^2 E[\tilde{C}(Q, R, L_{r-1})]}{\partial R^2} = \frac{h(1 - \beta)Q + \pi E[\tilde{d}]}{2Q(p_2 - p_1)L} > 0
\]

\[
\left[ \frac{\partial^2 E[\tilde{C}(Q, R, L_{r-1})]}{\partial Q^2} \right] \left[ \frac{\partial^2 E[\tilde{C}(Q, R, L_{r-1})]}{\partial R^2} \right] = \frac{E[\tilde{d}]}{Q^4(p_2 - p_1)^2 L^2}
\]

\[
\left\{ \frac{\pi E^2}{4} \left[ E[\tilde{d}] \left( p_3 - p_1 \right) \left( p_1 - 2p_2 + p_3 \right) \pi + Q(1 - \beta)h(2p_2^2 + (p_3 - p_1)p_2 - p_3p_1) \right] 
\right. \\
+ L \left[ \frac{(1 - \beta)R}{2} \left( p_1 - 2p_2 \right) Q + \left( E[\tilde{d}]\pi + (1 - \beta)Qh \right) \left( p_2 - p_1 \right) \left( A + C(L) \right) \right] \\
\left. \frac{QR^2\pi h(1 - \beta)}{4} \right\} > 0
\]

APPENDIX B

At point \( (Q, R, L_r) \):

\[
\frac{\partial^2 E[\tilde{C}(Q, R, L_r)]}{\partial Q^2} = 2E[\tilde{d}] \left[ A + \pi \left( \frac{(p_3 L - R)^2}{4(p_3 - p_2)L} + C(L) \right) \right] > 0
\]

\[
\frac{\partial^2 E[\tilde{C}(Q, R, L_r)]}{\partial R^2} = \frac{h(1 - \beta)Q + \pi E[\tilde{d}]}{2Q(p_3 - p_2)L} > 0
\]

and:
\[
\begin{align*}
\frac{\partial^2 E[\tilde{C}(Q,R,L_r)]}{\partial Q^2} \frac{\partial^2 E[\tilde{C}(Q,R,L_r)]}{\partial Q \partial R} \frac{\partial^2 E[\tilde{C}(Q,R,L_r)]}{\partial R^2} &= \frac{E[d]}{Q^4 (p_3 - p_2)^2 L^2} \\
\left\{ Q \pi h (1 - \beta) \left( p_3 L - R \right)^2 \right. &+ \left. (p_3 - p_2) L \left( A + C(L) \right) \right\} \left\{ h Q (1 - \beta) + E[d] \right\} > 0
\end{align*}
\]