Asymptotic behavior of solutions of a degenerate Fisher–KPP equation with free boundaries

Ningkui Sun

Department of Mathematics, Tongji University, Shanghai 200092, China

A R T I C L E   I N F O

Article history:
Received 21 December 2014
Received in revised form 23 January 2015
Accepted 31 January 2015

Keywords:
Reaction–diffusion equation
Free boundary problem
Asymptotic behavior
Sharp threshold

A B S T R A C T

The aim of this paper is to study the asymptotic behavior of solutions of a degenerate Fisher–KPP equation $u_t = u_{xx} + u^p(1 - u)$ ($p > 0$) in the domain $\{t, x\} \subset \mathbb{R}^2 : t > 0, x \in [g(t), h(t)]$, where $g(t)$ and $h(t)$ are two free boundaries. For $p > 1$ we obtain a trichotomy result: spreading $[g(t), h(t)] \to \mathbb{R}$ and $u(t, \cdot) \to 1$ locally uniformly in $\mathbb{R}$, vanishing $|h(t) - g(t)| < \infty$ and $u(t, \cdot) \to 0$ uniformly in $[g(t), h(t)]$, and virtual vanishing $[g(t), h(t)] \to \mathbb{R}$ and $u(t, \cdot) \to 0$ uniformly in $[g(t), h(t)]$. For $0 < p < 1$ we deduce that spreading can only happen, that is, 1 is the global attractor for all positive solutions. When spreading happens, we prove that the asymptotic spreading speed is continuous and strictly decreasing in $p$.

1. Introduction

In this paper, we consider the following free boundary problem

$$
\begin{align*}
&u_t = u_{xx} + u^p(1 - u), &g(t) < x < h(t), &t > 0, \\
&u(t, g(t)) = u(t, h(t)) = 0, &t > 0, \\
&g'(t) = -\mu u_x(t, g(t)), &t > 0, \\
&h'(t) = -\mu u_x(t, h(t)), &t > 0, \\
&-g(0) = h(0) = h_0, &u(0, x) = u_0(x), &-h_0 \leq x \leq h_0,
\end{align*}
$$

where $p > 0$, $x = g(t)$ and $x = h(t)$ are the moving boundaries, $\mu$ is a positive constant. The initial function $u_0$ belongs to $X(h_0)$ for some $h_0 > 0$, where

$$
X(h_0) := \{\phi \in C^2([-h_0, h_0]) : \phi(-h_0) = \phi(h_0) = 0, \phi'(-h_0) > 0, \phi'(h_0) < 0, \phi(x) > 0 \text{ in } (-h_0, h_0)\}.
$$

Problem (1.1) with $p = 1$ was recently studied in [1,2], etc. They used this model to describe the spreading of some new or invasive species, with $u(t, x)$ denoting the population density of a species over a one dimensional space, and the free boundaries $x = g(t)$ and $x = h(t)$ representing the spreading fronts, which are determined by the population gradient at the fronts and a coefficient $\mu$ in the Stefan condition together, namely, $g'(t) = -\mu u_x(t, g(t)), h'(t) = -\mu u_x(t, h(t))$ (for more background of such free boundary conditions, see [1,3,4] for example). In [5], the author studied problem (1.1) with the nonlinearity $1/(1 - u)$ and deduced the solution will go to 1 as $t$ goes to a finite time (this is called a quenching phenomenon), since 1 is a singular point of $1/(1 - u)$, which is different from this paper.
In [1,2], the authors obtained the following results for problem (1.1) with \( p = 1 \).

(A) for any initial data, problem (1.1) had a unique time-global solution \( (u, g, h) \) with \( u \in C^{(1+\alpha)/2,1+\alpha}((0, \infty) \times [g(t), h(t)]) \) and \( g, h \in C^{(1+\alpha)/2}(0, \infty)\) for any \( \alpha \in (0, 1) \), and \( 0 \leq u \leq 1 + \|u_0\|_\infty \) in \( (0, \infty) \times [g(t), h(t)] \);

(B) as \( t \to \infty \), either spreading happened in the sense that \( g_{\infty} := \lim_{t \to \infty} g(t) = -\infty \), \( h_{\infty} := \lim_{t \to \infty} h(t) = \infty \), and \( \lim_{t \to \infty} u(t, x) = 1 \) locally uniformly in \( \mathbb{R} \); or vanishing happened in the sense that \( h_{\infty} - g_{\infty} \leq \pi \) and \( \lim_{t \to \infty} \max_{g(t) \leq x \leq h(t)} u(t, x) = 0 \).

When \( p > 1 \), the equation in (1.1) arises in the study of isothermal autocatalytic reaction, see [6–9], and is studied by some authors, see [10–12] and references therein. It is well known that for \( p > 1 \), there is a unique \( c_0(p) > 0 \) such that for any \( c \geq c_0(p) \), the following equation \( u_t = u_{\alpha} + u^p(1 - u) \) \((t > 0, x \in \mathbb{R})\) has a traveling wave solution \( u(x) = U(x - ct) \) satisfying \( U'(z) < 0 \) for \( z \in \mathbb{R} \), \( U(-\infty) = 1 \) and \( U(\infty) = 0 \); while no such solution exists for \( c < c_0(p) \).

The number \( c_0(p) \) is called the minimal speed of the traveling waves.

In the first part of this paper, we consider problem (1.1) with \( p > 1 \). A similar discussion as in [1,2] shows that the result (A) remains hold in this case. However, the dichotomy result (B) is different. In particular, the vanishing case is replaced by two subcases: (1) \( u \to 0 \) while \( (g(t), h(t)) \) remains bounded, which is still called the vanishing phenomenon; (2) \( u \to 0 \) and \( (g(t), h(t)) \to \mathbb{R} \), which will be called the virtual vanishing phenomenon. The virtual vanishing phenomenon never occur for the classical logistic equation (i.e. the case where \( p = 1 \)). Furthermore, there is no critical size for initial occupying domain \([-h_0, h_0] \); for any \( h_0 > 0 \), vanishing will happen as long as \( \|u_0\|_\infty \) is small. Our first main result is stated as follows.

**Theorem 1.1.** Let \( p > 1 \), \( h_0 > 0 \), \( \phi \in \mathcal{X}(h_0) \) and \( (u, g, h) \) be a solution of problem (1.1) with initial data \( u_0 = \sigma \phi \). Then there exist \( \alpha_*, \alpha^* \in (0, \infty) \) with \( \alpha_* \leq \alpha^* \) such that

(i) Spreading happens when \( \sigma > \alpha^* \) in the sense that \( (g_{\infty}, h_{\infty}) = \mathbb{R} \) and

\[
\lim_{t \to \infty} u(t, x) = 1 \text{ locally uniformly in } \mathbb{R};
\]

(ii) Vanishing happens when \( \sigma < \alpha_* \) in the sense that \( (g_{\infty}, h_{\infty}) \) is a finite interval and

\[
\lim_{t \to \infty} \max_{g(t) \leq x \leq h(t)} u(t, x) = 0;
\]

(iii) Virtual vanishing happens when \( \sigma \in [\alpha_*, \alpha^*] \) in the sense that \( (g_{\infty}, h_{\infty}) = \mathbb{R} \) and

\[
\lim_{t \to \infty} \max_{g(t) \leq x \leq h(t)} u(t, x) = 0.
\]

In the second part of this paper, we consider problem (1.1) with \( 0 < p < 1 \). As far as we know, there is no much study on the equation in (1.1) with \( 0 < p < 1 \). One reason might be there is no classical traveling wave for such an equation. However, we will show that, equipped with a Stefan boundary condition as in (1.1), there exists a semi-wave with finite traveling speed, which is quite different from the Cauchy problem. The existence of solutions of problem (1.1) for \( 0 < p < 1 \) is proved by the Schauder fixed point theorem in the Appendix, while the contraction mapping theorem does not work in this case. We obtain a “hair-trigger effect”, this is, spreading always happen for any nontrivial initial function \( u_0(\alpha) \), regardless of its initial size and occupying domain \([-h_0, h_0] \), just like the case where \( p = 1 \) for the Cauchy problem cf. [13]. Our second main result is stated as follows.

**Theorem 1.2.** Let \( 0 < p < 1 \), \( h_0 > 0 \), \( u_0 \in \mathcal{X}(h_0) \) and \((u, g, h)\) be a solution of problem (1.1). Then spreading happens: \((g_{\infty}, h_{\infty}) = \mathbb{R}\) and

\[
\lim_{t \to \infty} u(t, x) = 1 \text{ locally uniformly in } \mathbb{R}.
\]

When spreading happens, it is interesting to study the asymptotic spreading speed. As in [1,2], the asymptotic spreading speed is determined by the following problem

\[
\begin{aligned}
\phi''_c - c\phi''_c + \phi'_c(1 - \phi_c) &= 0 \quad \text{for } z \in (0, \infty), \\
\phi_c(0) &= 0, \quad \mu\phi'_c(0) = \sigma, \quad \phi_c(\infty) = 1, \quad \phi_c(z) > 0 \quad \text{for } z > 0,
\end{aligned}
\]

(1.3)

where \( p = 1 \). More precisely, problem (1.3) with \( p = 1 \) has a unique solution pair \((\alpha^*, \phi_{\alpha^*})\) with \( \mu\phi''_{\alpha^*}(0) = \alpha^* > 0 \), and

\[
\lim_{t \to \infty} \frac{h(t)}{t} = \lim_{t \to \infty} \frac{g(t)}{t} = \alpha^*.
\]

Similarly, we can show that problem (1.3) for all \( p > 0 \) has a unique solution pair \((c, \phi) = (c^*(p, \mu), \phi_{c^*})\) with \( \mu\phi''_{c^*}(0) = c^*(p, \mu) > 0 \), where \( \phi_{c^*} \) is called a semi-wave with speed \( c^*(p, \mu) \) (see Lemma 3.2). By a similar argument as in [14], we have the following result.
**Proposition 1.3.** Suppose that \( p > 0 \) and spreading happens. Let \((u, g, h)\) and \((c^*(p, \mu), \varphi_c^*)\) be a solution of problem (1.1) and (1.3), respectively. Then

\[
\lim_{t \to \infty} h'(t) = -\lim_{t \to \infty} g'(t) = c^*(p, \mu).
\]

The last theorem of this paper is about the dependence of \( c^*(p, \mu) \) on \( p \) and \( \mu \).

**Theorem 1.4.** Let \( c^*(p, \mu) \) be the asymptotic spreading speed given by Proposition 1.3. Then

(i) for any given \( p > 0 \), \( c^*(p, \mu) \) is continuous and strictly increasing in \( \mu > 0 \);

(ii) for any given \( \mu > 0 \), \( c^*(p, \mu) \) is continuous and strictly decreasing in \( p > 0 \) and

\[
\lim_{p \to \infty} c^*(p, \mu) = 0.
\]

**Remark 1.5.** By a similar argument as the case where \( p = 1 \), we can deduce that \( \lim_{\mu \to 0} c^*(p, \mu) = 0 \) and \( \lim_{\mu \to \infty} c^*(p, \mu) = c_0(p) \), where \( c_0(p) \) is infinite when \( 0 < p < 1 \) (see Lemma 3.2), while it is the minimal speed of the traveling waves of \( u_t = u_x + u^p(1 - u) \) when \( p \geq 1 \).

The rest of this paper is organized as follows. Section 2 covers the proof of the convergence results for \( p > 1 \) and \( 0 < p < 1 \). Theorem 1.4 is proved in Section 3. The Appendix is devoted to the proof of the existence of solutions of (1.1) for \( 0 < p < 1 \) by using the Schauder fixed point theorem.

2. Convergence results

Throughout this section, we denote \( f(u) := u^p(1 - u) \) for convenience and study the convergence results for \( p > 1 \) and \( 0 < p < 1 \).

2.1. The comparison principle

In this subsection we give the comparison principle for all \( p > 0 \) which will be used later in this paper. In particular, the standard comparison principle for the case \( p > 1 \) is the same as that for the case \( p = 1 \) (see [1, Lemma 5.7]), since \( f \) is \( C^1 \) for \( p \geq 1 \). Thus, for brevity, we omit the details here. When \( 0 < p < 1 \), \( f \) is not \( C^1 \), the standard comparison principle does not hold for the case that some part of two initial datums coincide for lack of the uniqueness, however, it holds for two totally separated initial datums, see the below lemma. In fact, the below strong version of comparison principle is sufficient for our paper in the case \( 0 < p < 1 \).

**Lemma 2.1.** Suppose that \( 0 < p < 1 \), \( T \in (0, \infty) \), \( \overline{g}, \overline{h} \in C^1([0, T]) \), \( \overline{u} \in C(\overline{D}_T) \cap C^{1,2}(D_T) \) with \( D_T = \{(t, x) \in \mathbb{R}^2 : 0 < t \leq T, \overline{g}(t) < x < \overline{h}(t)\} \), and

\[
\begin{cases}
\overline{u}_t \geq \overline{u}_{xx} + f(\overline{u}), & 0 < t \leq T, \overline{g}(t) < x < \overline{h}(t), \\
\overline{u} = 0, & 0 < t \leq T, x = \overline{g}(t), \\
\overline{u} = 0, & 0 < t \leq T, x = \overline{h}(t).
\end{cases}
\]

If \( g(0) > \overline{g}(0) \), \( h(0) < \overline{h}(0) \) and \( u(0, x) < \overline{u}(0, x) \) in \([g(0), h(0)]\), and \((u, g, h)\) is a solution of (1.1), then \( g(t) > \overline{g}(t), h(t) < \overline{h}(t) \) in \([0, T]\) and

\[
u(t, x) < \overline{u}(t, x) \text{ for } t \in (0, T), x \in [g(t), h(t)].
\]

**Proof.** Note that the conclusion does not follow from the standard comparison principle directly, since \( f'(\cdot) \) is not a bounded function. Indeed, the monotonicity of \( f \) (that is, \( u^p(1 - u) \) is monotonically increasing in \( u \in [0, \frac{p}{p+1}] \)) plays a key role in our proof.

Now, let us use an indirect argument and suppose that there exists \( t^* \in (0, T) \) such that

\[
\begin{align*}
u(t, x) & < \overline{u}(t, x) \text{ for } t \in [0, t^*), x \in [g(t), h(t)], \\
g(t) & > \overline{g}(t), h(t) < \overline{h}(t) \text{ for } t \in [0, t^*),
\end{align*}
\]

and at least one of the following results holds:

\[
\begin{cases}
g(t^*) = \overline{g}(t^*), \text{ or } h(t^*) = \overline{h}(t^*), \\
or \ u(t^*, x^*) = \overline{u}(t^*, x^*) \text{ for some } x^* \in (g(t^*), h(t^*)).
\end{cases}
\]

(2.1)
Denote $M := \max\{|(t, x)|(t, x) \in \overline{D}_t^+\} + 1$. For any $0 \leq \xi \leq \eta \leq M$, we claim that
\[
f(\eta) - f(\xi) \geq -B(\eta - \xi),
\]
where $B := -\min\{f'(s)\}_{s \in \overline{p}_{p+1}, s \leq s \leq M}$. In fact, when $0 \leq \xi \leq \eta \leq \frac{p}{p+1}$, we have $f(\eta) - f(\xi) \geq 0$ since $f$ is monotonically increasing in $u \in [0, \frac{p}{p+1}]$, thus (2.2) holds in this case. When $0 \leq \xi \leq \frac{p}{p+1} < \eta \leq M$ and $f(\eta) - f(\xi) \geq 0$, then $f(\eta) - f(\xi) \geq 0$, which implies (2.2). When $0 \leq \xi \leq \frac{p}{p+1} < \eta \leq M$ and $f(\eta) < f(\xi)$, there exists $\theta \in [\frac{p}{p+1}, \eta]$ such that $f(\xi) = f(\theta)$, thus
\[
f(\eta) - f(\xi) = f(\eta) - f(\theta) \geq -B(\eta - \theta) \geq -B(\eta - \xi).
\]
When $\frac{p}{p+1} \leq \xi \leq \eta \leq M$, we also have (2.2), since
\[
f(\eta) - f(\xi) = f'(\theta)(\eta - \xi) \quad \text{for some } \theta \in [p/(p+1), M].
\]
Therefore (2.2) holds for any $0 \leq \xi \leq \eta \leq M$. Since $0 \leq u(t, x) - \bar{u}(t, x) \leq M$ in $[0, t^*] \times [g(t), h(t)]$, thus we have that for $(t, x) \in [0, t^*] \times [g(t), h(t)]$,
\[
(\bar{u} - u)_t - (\Pi - u)_{x x} \geq f(\Pi) - f(u) \geq -B(\Pi - u).
\]
The standard comparison principle derives that $\Pi > u$ for $(t, x) \in [0, t^*] \times [g(t), h(t)]$, which is a contradiction to the assumption. Thus this lemma follows. \hfill \Box

**Remark 2.2.** The function $\Pi$, or the triple $(\Pi, g, h)$, in Lemma 2.1 is often called a supersolution of (1.1). A subsolution can be defined analogously by reversing all the inequalities.

2.2. The case $p > 1$

In this subsection we assume that $p > 1$ and prove Theorem 1.1. For convenience, we write the solution of (1.1) with initial data $u_0 = \sigma \phi$ for some $\phi \in \mathcal{X}(h_0)$ as $(u_\sigma(t, x), g^\sigma(t), h^\sigma(t))$. Set $g^\sigma_\infty := \lim_{t \to \infty} g^\sigma(t), h^\sigma_\infty := \lim_{t \to \infty} h^\sigma(t)$, and define
\[
\Sigma_1 := \{\sigma > 0 : \sup_{x \in [g^\sigma(t), h^\sigma(t)]} u_\sigma(t, \cdot) \rightarrow 0 \text{ as } t \to \infty \text{ and } h^\sigma_\infty - g^\sigma_\infty < \infty\},
\]
\[
\Sigma_2 := \{\sigma > 0 : u_\sigma(t, \cdot) \rightarrow 1 \text{ locally uniformly in } \mathbb{R} \text{ and } h^\sigma_\infty, -g^\sigma_\infty = \infty\}.
\]
When $\sigma \in \Sigma_1$ (resp. $\Sigma_2$) we say vanishing (resp. spreading) happens for $u_\sigma$.

**Lemma 2.3.** $\Sigma_1$ is a non-empty and open set, and $\Sigma_2$ is open.

**Proof.** For some $\phi \in \mathcal{X}(h_0)$, consider the following problem
\[
\begin{aligned}
u_t &= u_{x x} + \tilde{f}(u), & g(t) < x < h(t), & t > 0, \\
u(t, g(t)) &= 0, & g'(t) &= -\mu u(t, g(t)), & t > 0, \\
u(t, h(t)) &= 0, & h'(t) &= -\mu u(t, h(t)), & t > 0, \\
-g(0) = h(0) &= h_0, & \tilde{u}(0, x) &= \sigma \phi(x), & -h_0 \leq x \leq h_0.
\end{aligned}
\]
for various different $\tilde{f}(u)$ and $\tilde{u}(0, x)$.

In order to prove that $\Sigma_1$ is not empty for any $h_0 > 0$, we consider problem (2.3) with $\tilde{f}(u)$ replaced by $f_\gamma(u) := u^\gamma(1 - u) + \gamma u(1 - u)$, where $\gamma > 0$ is small such that $h_0 < \pi/(4\sqrt{\gamma})$. Since $f_\gamma(u)$ is a monostable nonlinearity with $f_\gamma'(0) = \gamma > 0$ then by [2, Theorem 5.2], we have that there exists $\tilde{\sigma}^* > 0$ such that vanishing happens for the solution of (2.3) with $\sigma \leq \tilde{\sigma}^*$. It then follows from the standard comparison principle that $(0, \tilde{\sigma}^*) \subset \Sigma_1$.

Let us prove $\Sigma_1$ is an open set. Fix $\sigma_1 \in \Sigma_1$, definition of $\Sigma_1$ implies that for any small $\delta > 0$, there exists $T > 0$ large such that the solution $(u_{\sigma_1}, g_{\sigma_1}^+, h_{\sigma_1}^+)$ of (1.1) with $u_0 = \sigma_1 \phi$ satisfies $\|u_{\sigma_1}(T, \cdot)\|_\infty \leq \delta$. Moreover, we can find a constant $K > 0$ such that $h_{\sigma_1}^+(T) - g_{\sigma_1}^+(T) < K$. By the continuous dependence of the solution of (1.1) on its initial values, we choose $\varepsilon > 0$ sufficiently small, then the solution $(\tilde{u}, \tilde{g}^+, \tilde{h}^+)$ of (1.1) with $u_0 = (\sigma_1 + \varepsilon) \phi$ satisfies
\[
\|\tilde{u}(T, \cdot)\|_\infty < 2\delta \quad \text{and} \quad \|(\tilde{h}^+(T) - \tilde{g}^+(T))\| < K + 1.
\]
Considering problem (2.3) with $\tilde{u}(0, x)$ and $f_\gamma(u)$ replaced by $\tilde{u}(T, x)$ and $u^\varepsilon(1 - u) + \varepsilon u(1 - u)$, respectively, where $l = \frac{\pi^2}{2(k+1)^2}$, it is easy to check that there exists $\delta_1 > 0$ small such that when $\|\tilde{u}(0, x)\|_\infty \leq \delta_1$, vanishing happens for the solution of (2.3) in this case. Since $\delta$ is arbitrary, choosing $\delta = \delta_1/2$, then one can obtain that vanishing happens for $(\tilde{u}, \tilde{g}^+, \tilde{h}^+)$, which derives that $\sigma_1 + \varepsilon \in \Sigma_1$. On the other hand, the standard comparison principle yields that $\sigma \in \Sigma_1$ for any $\sigma < \sigma_1$. Therefore, $\Sigma_1$ is an open set.
If $\Sigma_2$ is empty, then it is open. It remains to show that $\Sigma_2$ is open when $\Sigma_2$ is not empty. Fix a $\sigma_2 \in \Sigma_2$, definition of $\Sigma_2$ yields that for the solution $v(x)$ of
\begin{equation}
\begin{cases}
v_{xx} + v^p (1 - v) = 0,
\quad -Z < x < Z, \\
v(Z) = v(-Z) = 0,
\end{cases}
\end{equation}
with $Z < \infty$, there exists $T_1 > 0$ large such that the solution $(u_{c_2}, g^{c_2}, h^{c_2})$ of (1.1) with $u_0 = \sigma_2 \phi$ satisfies $u_{c_2}(T_1, x) > v(x)$ for $x \in [-Z, Z] \subset (g^{c_2}(T_1), h^{c_2}(T_1))$. By the continuous dependence of the solution of (1.1) on its initial values, we have that there exists $\epsilon > 0$ small such that the solution $(u, g^*, h^*)$ of (1.1) with $u_0 = (\sigma_2 - \epsilon) \phi$ satisfies $u(1, x) > v(x)$ for $x \in [-Z, Z] \subset (g^*(T_1), h^*(T_1))$. Therefore, $\Sigma_2$ is an open set in this case.

Remark 2.4. Indeed $\Sigma_2$ can be empty when $p$ is large and $h_0$ is small. Considering problem (2.3) with $\tilde{f}$ replaced by $f$, and using [2, Proposition 5.4], we can deduce that there exists $h_0 < \pi/(2\sqrt{\bar{f}})$ such that for any $\sigma > 0$ and $h_0 < h_0$, vanishing happens for the solution of (2.3) provided $p > 1$. The standard comparison principle implies that $\Sigma_2 = \emptyset$ in this case. On the other hand, $\Sigma_2$ cannot be empty when $h_0$ is large. In fact, using [2, Proposition 5.12], we obtain that there exists $L > 0$ such that for $\phi \in \mathcal{X}(h_0)$ with $h_0 > L$ and for any large $\sigma > 0$, spreading happens for the solution of (2.3) with a combustion type of nonlinearity $f$. It follows from the standard comparison principle that spreading happens for the solution of (1.1) with the same initial data, which implies that $\Sigma_2 \neq \emptyset$.

2.3. The case $0 < p < 1$

Throughout this subsection we assume that $0 < p < 1$ and prove Theorem 1.2. Since $f'(u) \to \infty$ as $u \to 0$, then we can use this property of $f$ to construct a subsolution to show the asymptotic behavior of problem (1.1). As to the existence of solutions in this case, we refer to Theorem A.1 in the Appendix.

Proof of Theorem 1.2. Let $(u, g, h)$ be a solution of problem (1.1) with $0 < p < 1$. By definition of $f$, one can deduce that there exists a sufficiently small constant $\epsilon_0 < p/8$ such that for any $0 < \epsilon \leq \epsilon_0$, the equation $f(u) = f(\epsilon)$ has roots $\pm 1 - a(\epsilon)$ with $a(\epsilon) > 0$ and $a(\epsilon) \to 0$ as $\epsilon \to 0$. By simple calculation, we find that $f''(u) < 0$ for all $u > 0$. For any $\epsilon \in (0, \epsilon_0)$, define $f_\epsilon(u) := f(u + \epsilon) - f(\epsilon)$. It is easy to check that
\begin{equation}
\begin{cases}
f_\epsilon \in C^2([0, \infty)), \\
f_\epsilon'(0) = f'(\epsilon) > 0, \\
f_\epsilon(0) = f_\epsilon(1 - a(\epsilon) - \epsilon) = 0, \\
f_\epsilon(u) > 0 \text{ for } u \in (0, 1 - a(\epsilon) - \epsilon).
\end{cases}
\end{equation}

Let $(u_\epsilon, g_\epsilon, h_\epsilon)$ be a solution of
\begin{equation}
\begin{cases}
u_{xx} + v^p (1 - v) = 0, \\
u(t, g(t)) = 0, \\
g(t) = -\mu u_\epsilon(t, g(t)), \\
t > 0, \\
u(t, h(t)) = 0, \\
h(t) = -\mu u_\epsilon(t, h(t)), \\
0 < h_\epsilon < h_0, \\
-\mu u_\epsilon(x) = u_\epsilon(x), \\
-\mu u_\epsilon(x) = u_\epsilon(x).
\end{cases}
\end{equation}

where $0 < h_0 < h_0, u_\epsilon(x) \in \mathcal{X}(h_0)$ and $u_\epsilon(x) < u_\epsilon(x)$ in $[-h_0, h_0]$. By the results of [2], one can obtain that $|h'(t)|, |g'(t)| \leq D$, where $D$ is a positive constant independent of $t$.

Since $\lim_{t \to \infty} f'(0) = \infty$, no matter how small the initial data $h_0$ is, there must exist $\epsilon^* \leq \epsilon_0$ sufficiently small such that for any $0 < \epsilon \leq \epsilon^*$ we have $h_0 \geq \pi/(2\sqrt{\bar{f}'(0)})$, then the solution $(u_\epsilon, g_\epsilon, h_\epsilon)$ of (2.6) spreads in the sense that
\begin{equation}
\begin{aligned}
\lim_{t \to \infty} u_\epsilon(t, \cdot) &= 1 - a(\epsilon) - \epsilon \text{ locally uniformly in } \mathbb{R}, \\
\lim_{t \to \infty} g_\epsilon(t) &= \lim_{t \to \infty} h_\epsilon(t) = \infty.
\end{aligned}
\end{equation}
Since \( f(u) \leq 1 - u \) for \( u \geq 1 \), we can consider the following problem

\[
\begin{aligned}
q'(t) &= 1 - q, \quad t > 0, \\
q(0) &= \|u_0\|_\infty + 1,
\end{aligned}
\]

(2.9)

and obtain that \( u(t, x) \leq q(t) = 1 + \|u_0\|_\infty e^{-t} \). Combining with (2.7), (2.8), (2.9) and Lemma 2.1, we deduce that \( u(t, x) \) (and so any solution) converges to 1 locally uniformly in \( \mathbb{R} \), while \( h(t) \) and \( -g(t) \) goes to infinity as \( t \to \infty \). Then Theorem 1.2 follows. \( \square \)

3. The asymptotic spreading speed

In this section we suppose that spreading happens and study the asymptotic spreading speed based on the phase plane analysis. Denote \( f_j(u) := u^p(1 - u) \) for all \( p > 0 \) in order to show the dependence on exponent \( p \). Let \( c_0(p) \) be \( \infty \) for \( 0 < p < 1 \) (see Lemma 3.2), and the minimal speed of traveling waves of the Cauchy problem for \( p \geq 1 \), respectively. Firstly, we have the following lemma, which shows the properties of \( c_0(p) \) for \( p \geq 1 \).

Lemma 3.1 ([9, Theorem 1] and [10, Theorem 1]). \( c_0(p) \) is a continuous and strictly decreasing function for \( p \geq 1 \). Moreover, \( c_0(p) \to 0 \) as \( p \to \infty \).

Let us consider the problem

\[
\begin{aligned}
\varphi'' - c\varphi' + f_p(\varphi) &= 0 \quad \text{for } z \in (0, \infty), \\
\varphi(0) &= 0, \quad \varphi(\infty) = 1, \quad \varphi(z) > 0 \quad \text{for } z \in (0, \infty).
\end{aligned}
\]

(3.1)

Denote \( \frac{d\varphi}{dz} \) by \( v' \), then the equation in (3.1) can be rewritten as following

\[
v' = w, \quad w' = cw - f_p(\varphi).
\]

(3.2)

A solution \( (v(z), w(z)) \) of (3.2) traces out a trajectory in the phase plane. If \( w(z) = v'(z) > 0 \) for all \( z > 0 \), then we denote the trajectory by a function \( w = W_p^c(v), v \in [0, 1] \), which satisfies

\[
\frac{dW_p^c}{dv} \equiv (W_p^c)' = c - \frac{f_p(\varphi)}{W_p^c} \quad \text{for } v \in (0, 1).
\]

(3.3)

Using the phase plane analysis, it is easy to check that when \( p \geq 1, W_p^c(0) > 0 \) for \( 0 < c < c_0(p) \) and \( W_p^c(0) = 0 \) for \( c \geq c_0(p) \); while, when \( 0 < p < 1, W_p^c(0) > 0 \) for \( 0 < c < \infty \) and \( W_p^c(0) \to W_p^\infty \) as \( c \to \infty \) with \( W_p^\infty \geq 0 \) (see Lemma 3.2 and its proof).

Next, we prove the existence and uniqueness of semi-waves and study the properties of \( W_p^c(0) \) with \( c > 0 \) for \( 0 < p < 1 \), which are important for this part and different from the case \( p = 1 \). However, the existence and uniqueness of semi-waves for \( p > 1 \) can be proved by the same argument as the case \( p = 1 \) in [1,2], so we omit the details here.

Lemma 3.2. Suppose that \( 0 < p < 1 \), then \( c_0(p) = +\infty \). For any \( c > 0 \), the following problem

\[
\begin{aligned}
\varphi''_z - c\varphi'_z + \varphi'_z(1 - \varphi_z) &= 0 \quad \text{for } z \in (0, \infty), \\
\varphi_z(0) &= 0, \quad \varphi_z(\infty) = 1, \quad \varphi_z(z) > 0 \quad \text{for } z > 0,
\end{aligned}
\]

(3.4)

admits a unique solution \( \varphi_z(z) \) for \( z \in [0, \infty) \). Moreover, for any given \( \mu > 0, (1.3) \) has a unique solution pair \( (c, \varphi_z) = (c^*(p, \mu), \varphi_z) \) with \( c^*(p, \mu) > 0 \).

Proof. Let \( \{f_n(u)\} \) be a class of increasing classical Fisher–KPP functions, which satisfies

\[
\begin{aligned}
f_n(0) \in C^1([0, \infty)), \quad f_n(u) \to f(u) \quad \text{as } n \to \infty, \quad f_n'(0) = n, \\
f_n(u) \leq f_n'(0)u, \quad f_n(u) \leq f(u) \quad \text{for } u \in (0, \infty).
\end{aligned}
\]

(3.5)

Consider the Cauchy problem:

\[
u_t = u\nu + f(u) \quad (x \in \mathbb{R}, \ t > 0), \quad u(0, x) = u_0(x) \quad (x \in \mathbb{R}),
\]

(3.5)

and denote by \( u(t, x) \) the solution of (3.5) with initial data \( u_0(x) = 2\delta_{1-2.2}(x) \). For any \( n \to 0 \), let \( u_n(t, x) \) be the solution of (3.5) with \( f(u) = f_n(u) \) and initial data \( u_0(0, x) = 1_{[-1,1]}(x) \). Then by the comparison principle, we can deduce that \( u(t, x) \geq u_n(t, x) \) for \( t \geq 0, x \in \mathbb{R} \). Since \( c_{n,0} \geq 2\sqrt{n} \) (where \( c_{n,0} \) is denoted by the minimal traveling wave speed for \( f_n(u) \)), and \( n \) is arbitrary large, then it follows that \( c_0(p) = +\infty \), in other words, there is no classical traveling wave for \( 0 < p < 1 \). Using the phase plane analysis, one can deduce \( W_p^c(0) > 0 \) for any \( c > 0 \), which implies that problem (3.4) admits a unique solution \( \varphi_z(z) \) for any \( c > 0 \).

In the following, we will show that problem (1.3) admits a unique solution for any given \( \mu > 0 \). Since \( W_p^c(0) \) is positive and decreasing in \( c > 0 \), therefore \( \lim_{c \to \infty} W_p^c(0) \) exists, denote it by \( W_p^\infty(0) \), which satisfies \( 0 \leq W_p^\infty(0) < W_p^0(0) \). Let us
consider the continuous function \( \eta(c) = W_p'(0) - c/\mu \) for \( c \in [0, \infty) \). By the above discussion we know that \( \eta(c) \) is strictly decreasing in \( c \). Moreover, \( \eta(0) = W_p^0(0) > 0 \) and \( \lim_{c \to \infty} \eta(c) = -\infty \), thus there exists a unique \( c^*(p, \mu) \in (0, \infty) \) such that \( \eta(c^*(p, \mu)) = 0 \). Later, by a similar argument as in [2], we can deduce that \( (c^*(p, \mu), \phi_{c^*}) \) with \( c^*(p, \mu) > 0 \) is the unique solution pair of (1.3).

The proof is complete. \( \square \)

Finally, based on the previous results, we can show the proof of Theorem 1.4

**Proof of Theorem 1.4.** First, let us prove two claims.

**Claim 1.** If \( p_1 > p_2 > 0 \) and \( \mu > 0 \) is fixed, then \( W_{p_1}^c(v) < W_{p_2}^c(v) \) for \( v \in [0, 1) \), \( c \in [0, c_0(p_2)) \), and \( c^*(p_1, \mu) < c^*(p_2, \mu) \).

Without confusion, we denote \( W_i(v) := W_{p_i}^c(v) \) for \( i = 1, 2 \) for convenience in this part. When \( p_1 > p_2 > 0 \), the direct calculation yields that \( W_i'(1) = W_i^c(1) \). The phase plane analysis implies that there exists \( \delta > 0 \) small such that \( W_i'(v) < 0 \) for \( v \in (1 - \delta, 1) \) and \( i = 1, 2 \). Since

\[
W_2'(v) < c - f_{p_1}(v)/W_2(v) \quad \text{and} \quad f_{p_1}(v) < f_{p_2}(v) \quad \text{for} \quad v \in (0, 1),
\]

we find that as \( v \) decreases from \( v = 1 \), the curve \( w = W_1(v) \) is separated from the curve \( w = W_2(v) \) and the curve \( w = W_1(v) \) remains below the curve \( w = W_2(v) \) for \( v \in (1 - \delta, 1) \) and \( c \in [0, c_0(p)) \). From (3.6), we can deduce that \( W_1(v) < W_2(v) \) for \( v \in (0, 1) \) and \( W_1(0) \leq W_2(0) \). Let us prove that \( W_1(0) < W_2(0) \) for \( c \in [0, c_0(p)) \). Otherwise, there exists \( \bar{c} \in [0, c_0(p)) \) such that \( W_1(0) = W_2(0) \), where \( W_i(v) := W_{p_i}^c(v) \) for \( i = 1, 2 \). Set \( T(v) := W_1(v) - W_2(v) \), then the function satisfies

\[
T'(v) > d(v)T(v) \quad \text{for} \quad v \in (0, 1), \quad \text{and} \quad T(0) = 0,
\]

where \( d(v) := \frac{f_{p_1}(v)}{W_1(v)W_2(v)} \). This implies that \( T(v) > 0 \) for \( v \in (0, 1) \), a contradiction.

In a similar way as in [2], we obtain that \( (c^*(p, \mu), c^{(p, \mu)}_0) \) is the unique intersection point of the decreasing curve \( y = W_p^c(0) \) with the increasing line \( y = \frac{c}{\mu} \) in the cy-plane and \( W_p^c(v) \) is decreasing in \( p \) for \( v \in [0, 1) \), \( c \in [0, c_0(p_2)) \). Thus one can deduce \( c^*(p_1, \mu) < c^*(p_2, \mu) \) for \( p_1 > p_2 \), which proves Claim 1.

**Claim 2.** \( c^*(p, \mu) \) is continuous in \( p > 0 \).

For any small \( \varepsilon > 0 \), set \( f_{p+\varepsilon}(u) := u^{p+\varepsilon}(1 - u) \). Multiplying (3.3) by \( 2W_p^c \) and integrating on \([0, 1]\) we conclude that

\[
(W_p^c(0))^2 + 2c\int_0^1 W_p^c(v)dv = 2\int_0^1 f_p(s)ds. \tag{3.7}
\]

Similarly, we have that

\[
(W_{p+\varepsilon}^c(0))^2 + 2c\int_0^1 W_{p+\varepsilon}^c(v)dv = 2\int_0^1 f_{p+\varepsilon}(s)ds. \tag{3.8}
\]

From Claim 1, we deduce that \( 0 \leq W_{p+\varepsilon}^c(v) < W_p^c(v) \) for \( v \in [0, 1) \) and \( c \in [0, c^*(p, \mu)] \). Note that \( W_{p+\varepsilon}^c(0) \geq 0 \) for \( c \in [0, c^*(p, \mu)] \), thus

\[
W_p^c(0) + W_{p+\varepsilon}^c(0) \geq W_{p+\varepsilon}^c(0) \geq W_p^{c^*(p, \mu)}(0) > 0 \quad \text{for} \quad c \in [0, c^*(p, \mu)]. \tag{3.9}
\]

Thanks to (3.7)–(3.9), we can deduce \( W_p^c(0) - W_{p+\varepsilon}^c(0) \leq k(\varepsilon) \varepsilon \) for \( c \in [0, c^*(p, \mu)] \), where \( k(\varepsilon) = \frac{1}{W_p^{c^*(p, \mu)}(0)} \cdot \frac{4}{(p+1)^2(p+2)} \).

In the cy-plane, \( (c^*(p, \mu), c^{(p, \mu)}_0) \) (resp. \( (c^*(p + \varepsilon, \mu), c^{(p+\varepsilon, \mu)}_0) \)) is the unique intersection point of \( y = W_p^c(0) \) (resp. \( y = W_{p+\varepsilon}^c(0) \)) with \( y = c/\mu \). Thus

\[
0 < c^*(p, \mu) - c^*(p + \varepsilon, \mu) < \mu k(\varepsilon) \varepsilon.
\]

Therefore \( c^*(p, \mu) \) is right continuous in \( p \) for \( p > 0 \). In a similar way as above we deduce \( c^*(p, \mu) \) is left continuous in \( p \) for \( p > 0 \). Therefore \( c^*(p, \mu) \) is continuous in \( p \). This proves Claim 2.

Later, based on Claims 1 and 2 and Lemma 3.1, we can deduce the following results:

(i) for any given \( p > 0 \), \( c^*(p, \mu) \) is strictly increasing and continuous in \( \mu > 0 \);
(ii) for any given \( \mu > 0 \), \( c^*(p, \mu) \) is strictly decreasing and continuous in \( p > 0 \) and

\[
0 \leq \lim_{p \to \infty} c^*(p, \mu) \leq \lim_{p \to \infty} c_0(p) = 0.
\]

This completes the proof of Theorem 1.4 (see Fig. 1 for the asymptotic spreading speed). \( \square \)
with

\[ p \]

mapping theorems do not work in this case, which is used to study the existence and uniqueness of solutions for the case

\[ p \geq 1 \]

(see [1]). The main result in this section is the following:

**Theorem A.1.** Suppose that \( 0 < p < 1 \). For any given \( u_0 \in w(h_0) \) and any \( \alpha \in (0, 1) \), there is a \( T > 0 \) such that problem (1.1) admits a solution

\[
(u, g, h) \in C^{1+\alpha/2, 1+\alpha}((0, T) \times [g(t), h(t)]) \times C^{1+\alpha/2}((0, T)) \times C^{1+\alpha/2}((0, T)).
\]

**Proof.** Step 1. Use the same notations \( \xi(y) \) and \( \xi(y) \) and make the same transformation \((t, y) \to (t, x)\) as in [1], then \( u(t, x) \to w(t, y) \) and problem (1.1) becomes

\[
\begin{align*}
& w_t - A(g, h, y)w_{yy} + B(g, h, y)w_y = f(w), \quad y_0 < y < y_0, \quad t > 0, \\
& w(t, y_0) = w(t, y_0), \quad t > 0, \\
& g'(t) = -\mu u_y(t, y_0), \quad t > 0, \\
& h'(t) = -\mu u_y(t, y_0), \quad t > 0, \\
& g(0) = h(0), \quad y, y_0 = u_0(y), \quad y_0 \leq y \leq y_0,
\end{align*}
\]

(A.1)

with \( f(w) = f(u) \) and \( A(g, h, y) = [1 + \xi'(y)(h(t) - h_0) + \xi'(y)(g(t) + h_0)]^{-1} \).

\[
B(g, h, y) = \{\xi''(y)(h(t) - h_0) + \xi''(y)(g(t) + h_0)\}A(g, h, y)^2 - [\xi(y)h'(t) + \xi(y)g'(t)]A(g, h, y)^2.
\]

Denote \( h_1 = -\mu u'_0(h_0), h_2 = \mu u'_0(h_0) \), \( \Omega_T = [0, T] \times [-h_0, h_0], C^{1}(\Omega_T) \) is a Banach space with the norm

\[
\| (w, g, h) \|_X = \| w \|_{C^{1}(\Omega_T)} + \| g \|_{C^{1}(\Omega_T)} + \| h \|_{C^{1}(\Omega_T)}.
\]

Since \( D_T \) is a bounded and closed subspace of \( X \), then \( D_T \) is a Banach space.

Note that \( f(w) \in C^1([0, \infty)) \) for \( v \in (0, p) \). Applying standard \( L^q \) theory and the Sobolev embedding theorem, we can deduce that for any \( (w, g, h) \in D_T \), the following linear problem:

\[
\begin{align*}
& \tilde{w}_t - A(g, h, y)\tilde{w}_{yy} + B(g, h, y)\tilde{w}_y = f(w), \quad t > 0, \quad -h_0 < y < h_0, \\
& \tilde{w}(t, \pm h_0) = 0, \quad t > 0, \\
& \tilde{w}(0, y) = u_0(y), \quad -h_0 \leq y \leq h_0,
\end{align*}
\]

(A.2)
admits a unique \( \tilde{w}(t, y) \in C^{1+\alpha,1+\alpha}(\Omega_T) \), which satisfies
\[
\|\tilde{w}\|_{C^{1+\alpha,1+\alpha}(\Omega_T)} \leq C_1, \tag{A.3}
\]
where \( C_1 \) is a constant dependent on \( h_0, \alpha \) and \( \|u_0\|_{C^2([-h_0,h_0])} \).

Defining \( \tilde{h} \) and \( \tilde{g} \) by \( \tilde{h}(t) = h_0 - \int_0^t \mu \tilde{w}_y(\tau, h_0) \, d\tau \), \( \tilde{g}(t) = -h_0 - \int_0^t \mu \tilde{w}_y(\tau, -h_0) \, d\tau \), we deduce
\[
\tilde{h}'(t) = -\mu \tilde{w}_y(t, h_0), \quad \tilde{h}(0) = h_0, \quad \tilde{h}'(0) = -\mu \tilde{w}_y(t, h_0) = h_1,
\]
and thus \( \tilde{h}' \in C^\frac{\alpha}{2}([0, T]) \), which satisfies
\[
\|\tilde{h}'\|_{C^\frac{\alpha}{2}([0, T])} \leq \mu C_1 := C_2. \tag{A.4}
\]
Similarly, we have \( \tilde{g}' \in C^\frac{\alpha}{2}([0, T]) \), which satisfies
\[
\|\tilde{g}'\|_{C^\frac{\alpha}{2}([0, T])} \leq \mu C_1 := C_2. \tag{A.5}
\]

Step 2. We define an operator \( \mathcal{F} : D_T \to X \) by \( \mathcal{F}(w, g, h) = (\tilde{w}, \tilde{g}, \tilde{h}) \).
Clearly \( (w, g, h) \in D_T \) is a solution of (A.1) if and only if it is a fixed point of \( \mathcal{F} \). We will show that if \( T > 0 \) is small enough, then \( \mathcal{F} \) has a fixed point by using the Schauder fixed point theorem. Note that we have no uniqueness of solution of (1.1) with \( 0 < p < 1 \).

By (A.3)-(A.5), we deduce that
\[
\|\tilde{w} - u_0\|_{C(\Omega_T)} \leq C_1 T^{\frac{1+\alpha}{2}}, \quad \|\tilde{h}' - h_1\|_{C([0, T])} \leq C_2 T^{\frac{\alpha}{2}}, \quad \|\tilde{g}' + h_2\|_{C([0, T])} \leq C_2 T^{\frac{\alpha}{2}}.
\]
Therefore if we choose \( T \leq \min\{C_2^{-\frac{2}{\alpha}}, C_1^{-\frac{2}{1+\alpha}}\} \), then \( \mathcal{F} \) maps \( D_T \) into itself. In fact, \( \mathcal{F} \) maps \( D_T \) into \( S_T \), where \( S_T := \left( C^{\frac{1+\alpha}{2},1+\alpha}(\Omega_T) \times C^{1+\frac{\alpha}{2}}([0, T]) \times C^{1+\frac{\alpha}{2}}([0, T]) \right) \) for \( \alpha \in (0, 1) \), which yields that \( \mathcal{F} \) is compact.

Step 3. Let us prove that \( \mathcal{F} \) is continuous. Let \( (w_1, g_1, h_1) \in D_T \), \( (w_2, g_2, h_2) \in D_T \), and denote \( (\tilde{w}_1, \tilde{g}_1, \tilde{h}_1) = \mathcal{F}(w_1, g_1, h_1) \). It then follows from (A.3)-(A.5) that
\[
\|\tilde{w}_1\|_{C^{1+\alpha,1+\alpha}(\Omega_T)} \leq C_1, \quad \|\tilde{h}_1\|_{C^\frac{\alpha}{2}([0, T])} \leq C_2 \quad \text{and} \quad \|\tilde{g}_1\|_{C^\frac{\alpha}{2}([0, T])} \leq C_2.
\]
Denote \( d := \|(w_1, g_1, h_1) - (w_2, g_2, h_2)\|_X \) and suppose \( d \leq 1 \) for convenience.

Setting \( U(t, y) = \tilde{w}_1(t, y) - \tilde{w}_2(t, y) \), we obtain that \( U(t, y) \) satisfies
\[
\begin{aligned}
U_t - A(g_2, h_2, y)U_{yy} + B(g_2, h_2, y)U_y &= \tilde{f}(t, y), \quad -h_0 < y < h_0, \ t \in (0, T), \\
U(t, h_0) &= U(t, -h_0) = 0, \quad \ t \in (0, T), \\
U(0, y) &= 0, \quad \ t \in (0, T), \\
&\text{where } \tilde{f}(t, y) = (A_1 - A_2)\tilde{w}_{1,yy} - (B_1 - B_2)\tilde{w}_{1,y} + f(w_1) - f(w_2),
\end{aligned} \tag{A.6}
\]
with \( \tilde{f}(t, y) = (A_1 - A_2)\tilde{w}_{1,yy} - (B_1 - B_2)\tilde{w}_{1,y} + f(w_1) - f(w_2) \), where \( A_i := A(g_i, h_i, y) \) and \( B_i := B(g_i, h_i, y) \) for \( i = 1, 2 \). Since \( f \in C^\nu([0, \infty)) \) for \( \nu \in (0, p) \), then \( \|f(w_1) - f(w_2)\|_{C^\nu(\Omega_T)} \leq C_5 d^{\nu}, \) where \( q > 1 \) and \( C_5 \) is a constant dependent on \( h_0 \) and \( \|u_0\|_{C^2} \). Moreover, we deduce that
\[
\|(A_1 - A_2)\tilde{w}_{1,yy} - (B_1 - B_2)\tilde{w}_{1,y}\|_{C^\nu(\Omega_T)} \leq C_6 d,
\]
where \( C_6 \) is a constant dependent on \( C_1, C_2, h_0 \) and \( \|u_0\|_{C^2} \). Therefore, we obtain
\[
\|\tilde{f}(t, y)\|_{C^\nu(\Omega_T)} \leq C_7 d^{\nu},
\]
where \( C_7 \) is a constant dependent on \( C_1, C_2, h_0 \) and \( \|u_0\|_{C^2} \). (Note that the \( \nu \) arises from the sublinearity of \( u^p(1 - u) \) as \( 0 < p < 1 \); in the case \( p \geq 1 \), one can choose \( \nu = 1 \) and the contraction mapping theorem can be applied as in [1].)

Using the \( L^p \) estimates and the Sobolev embedding theorem, we can deduce that
\[
\|\tilde{w}_1 - \tilde{w}_2\|_{C^{1+\alpha,1+\alpha}(\Omega_T)} \leq \hat{C} d^{\nu}, \tag{A.7}
\]
for \( \alpha \in (0, 1) \) and \( \hat{C} \) depends on \( C_1, C_2, h_0 \) and \( \|u_0\|_{C^2} \). Taking the difference of the equations for \( \tilde{h}_1 \) and \( \tilde{h}_2 \) results in
\[
\|\tilde{h}_1 - \tilde{h}_2\|_{C([0, T])} \leq \mu \|\tilde{w}_{1,y} - \tilde{w}_{2,y}\|_{C^{\frac{\alpha}{2}}(\Omega_T)} \leq \mu \hat{C} d^{\nu}. \tag{A.8}
\]
By definitions, we can deduce that
\[
\|\tilde{h}_1 - \tilde{h}_2\|_{C([0, T])} \leq \mu T\|\tilde{w}_{1,y} - \tilde{w}_{2,y}\|_{C(\Omega_T)} \leq \mu T\hat{C} d^{\nu}. \tag{A.9}
\]
Similarly, we can obtain that
\[
\|\tilde{g}_1' - \tilde{g}_2'\|_{C([0, T])} \leq \mu \tilde{C} d^\nu, \quad \|\tilde{g}_1 - \tilde{g}_2\|_{C([0, T])} \leq \mu T \tilde{C} d^\nu.
\]
(A.10)
Combining (A.7)–(A.10), and assuming \(T \leq 1\), we have that
\[
\|\tilde{w}_1 - \tilde{w}_2\|_{C^1 + \alpha/2} + \|\tilde{g}_1 - \tilde{g}_2\|_{C^1([0, T])} + \|\tilde{h}_1 - \tilde{h}_2\|_{C^1([0, T])} \leq \tilde{C} d^\nu,
\]
with \(\tilde{C}\) depending on \(\tilde{C}\) and \(\mu\). For \(T = \min\{1, C\} - \frac{2}{\tilde{C} - 2}\alpha + \frac{h_0}{16(1 + \max(h_1, h_2))}\}, we obtain
\[
\|\tilde{w}_1, \tilde{g}_1, \tilde{h}_1\|_\infty - \|\tilde{w}_2, \tilde{g}_2, \tilde{h}_2\|_\infty \leq \tilde{C} d^\nu,
\]
then \(\mathcal{F}\) is a continuous mapping.

**Step 4.** Using the Schauder fixed point theorem, \(\mathcal{F}\) has a fixed point \((w, g, h) \in D_T\). Moreover, by the Schauder estimates, we have additional regularity for \((w, g, h)\) as a solution of (A.1), namely
\[
(w, g, h) \in C^{1 + \alpha/2, 2 + \alpha}((0, T) \times [-h_0, h_0]) \times C^{1 + \alpha/2}((0, T)) \times C^{1 + \alpha/2}((0, T)),
\]
then \((w, g, h)\) is a local classical solution of (A.1). In other words, we can deduce a local classical solution \((u, g, h)\) of (1.1) by \((w, g, h)\). This completes the proof. □

It is easy to check that the local solution obtained in *Theorem A.1* can be extended to all \(t > 0\).

**References**


