On the blow-up of a non-local parabolic problem

N.I. Kavallaris*, D.E. Tzanetis

Department of Mathematics, School of Applied Mathematical and Physical Sciences, National Technical University of Athens, Zografou Campus, 157 80 Athens, Greece

Received 7 October 2005; accepted 4 November 2005

Abstract

We investigate the conditions under which the solution of the initial-boundary value problem of the non-local equation

$$u_t = \Delta u + \lambda \frac{f(u)}{\left( \int_\Omega f(u) \, dx \right)^{p}}$$

where $\Omega$ is a bounded domain of $\mathbb{R}^N$ and $f(u)$ is a positive, increasing, convex function, performs blow-up.

© 2005 Elsevier Ltd. All rights reserved.

Keywords: Non-local parabolic problems; Blow-up

1. Introduction

Consider the problem

$$u_t = \Delta u + \frac{\lambda f(u)}{\left( \int_\Omega f(u) \, dx \right)^{p}}, \quad x \in \Omega, \quad t > 0,$$

$$\frac{\partial u}{\partial n} + \beta u = 0, \quad x \in \partial \Omega, \quad t > 0,$$

$$u(x, 0) = u_0(x), \quad x \in \Omega,$$

(1.1)

where $0 \leq \beta = \beta(x) \leq \infty$, which is $C^{1+\alpha}(\partial \Omega)$, $\alpha > 0$, whenever it is bounded and $p$ is a positive number. Moreover $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $f(s)$ is assumed to be a positive, increasing, convex function. $\lambda$ is the parameter of the problem and stands for a physical quantity, e.g. for $N = 1$ and $p = 2$ when problem (1.1) models Ohmic heating, $\lambda$ is proportional to the square of the potential difference applied across the ends of the circuit under consideration.

The motivation for studying this kind of problem is that (1.1) has important implications for a variety of technological processes. Such non-local problems arise, for example, in the analytical study of phenomena associated with the occurrence of shear bands in metals being deformed at high strain rates (see [2–4] and the references therein), in modelling Ohmic heating phenomena [11,12], in the theory of gravitational equilibrium of polytropic stars [10], in

* Corresponding author.

E-mail addresses: nkaval@math.ntua.gr (N.I. Kavallaris), dtzan@math.ntua.gr (D.E. Tzanetis).

0893-9659/$ - see front matter © 2005 Elsevier Ltd. All rights reserved.
the investigation of the fully turbulent behaviour of flows, using invariant measures for the Euler equation [5], and in modelling aggregation of cells via interaction with a chemical substance (chemotaxis); see [14].

Here we focus on finding some conditions under which blow-up occurs for the solution of problem (1.1). It is known that for \( f(s) \) increasing, if \( p \geq 1 \) and \( N = 1 \) then (1.1) has a global-in-time solution which is bounded; see [2]. The same result, at least for \( f(s) = e^s \), holds in the two-dimensional case; see also [2]. Thus when \( f(s) \) is an increasing function, blow-up should be expected to occur only if \( p < 1 \). Bebernes and Lacey in [2] have proved the occurrence of blow-up for the exponential function \( f(s) = e^s \) when \( 0 < p < 1 \) and \( N = 1, 2 \) using a method which is due to Fila [6]. This method has a serious limitation, namely that problem (1.1) should admit a Lyapunov functional; this is the case only for \( f(s) = e^s \). Some blow-up results for \( p = 1 \) can be found in [14], where only \( f(s) = e^s \) is considered, or in [9], where only Neumann boundary conditions and power-law functions are considered.

Using arguments similar to those used for the rigid-ignition (local) model, e.g. Kaplan’s method, we extend the blow-up result given in [2] to a general increasing function \( f(s) \) and to higher spatial dimensions. More precisely, in the next section we show that

\[
\int_b^\infty \frac{ds}{s^{1-p}(s)} < \infty, \quad \text{for} \quad 0 < p < 1 \quad \text{and any positive} \quad b, \tag{1.2}
\]

is a sufficient condition for the solution of (1.1) to perform blow-up. Actually (1.2) implies that \( f^{1-p}(s) \) is superlinear at infinity. Condition (1.2) is satisfied by \( f(s) = e^s \) and by \( f(s) = (1+s)^{1+k}, \quad k > p/(1-p) > 0, \) as well.

2. Main results

2.1. Blow-up for Dirichlet and Robin boundary conditions

We assume that \( 0 < \beta(x) \leq \infty \) and that there exists at least one \( x_0 \in \partial \Omega \) such that \( \beta(x_0) = \infty \), i.e. \( u(x_0) = 0 \).

Let us denote by \( \mu = \mu(\Omega) > 0 \) the principal (smallest) eigenvalue of the problem

\[
-\Delta \Psi(x) = \mu \Psi(x), \quad x \in \Omega, \quad \frac{\partial \Psi}{\partial n} + \beta \Psi(x) = 0, \quad x \in \partial \Omega, \tag{2.1}
\]

and by \( \Psi(x) \) the corresponding (first) eigenfunction, which is known to be of constant sign in \( \Omega \). Let \( \Psi(x) > 0 \) be normalized such that \( \|\Psi\| = \int_\Omega \Psi(x) \, dx = 1 \). Using now Kaplan’s technique, i.e. studying the behaviour of the first Fourier coefficient of \( u \) connected with this eigenfunction, we prove the following result.

Theorem 1. Let \( \Omega \) be a convex domain of \( \mathbb{R}^N \), \( f(s) \) a positive, increasing and convex function satisfying (1.2) and \( 0 < p < 1 \). Then the solution \( u(x,t) \) of (1.1) blows up in finite time for sufficiently large values of the parameter \( \lambda \), provided that \( u_0 \in L^2(\Omega) \).

Proof. Multiplying (1.1a) by \( \Psi(x) \) and integrating over \( \Omega \) we get

\[
\frac{dA(t)}{dt} = \int_\Omega \Psi(x) \Delta u(x,t) \, dx + \lambda \int_\Omega f(u(x,t)) \Psi(x) \, dx 
\left( \int_\Omega f(u(x,t)) \, dx \right)^p,
\]

where \( A(t) = \int_\Omega u(x,t) \Psi(x) \, dx \) \( (A(t) \leq ||u(.,t)||_\infty) \). \( A(t) \) is the first Fourier coefficient of \( u(x,t) \) in terms of the eigenfunctions of (2.1). Using Green’s identity and the boundary conditions of \( u \) and \( \Psi \) we obtain

\[
\frac{dA(t)}{dt} = -\mu \int_\Omega u(x,t) \Psi(x) \, dx + \lambda \int_\Omega f(u(x,t)) \Psi(x) \, dx 
\left( \int_\Omega f(u(x,t)) \, dx \right)^p,
\]

recalling that \( (\mu, \Psi) \) is a principal eigenpair of problem (2.1). Due to the convexity of \( \Omega \) and the fact that \( f \) is increasing we can construct, by using the method of moving parallel planes, see [8,13], a relative compact set \( \Omega_0 \subset \Omega \), \((\Omega_0 \subset \Omega) \) such that

\[
\int_\Omega f(u) \, dx \leq (k + 1) \int_{\Omega_0} f(u) \, dx, \tag{2.3}
\]
for some \( k \in \mathbb{N} \). Let \( m = \inf_{x \in \Omega} \Psi(x) \); then by using the fact that \( \Omega_0 \subset \Omega \) and the maximum principle for problem (2.1), we have \( m > 0 \); thus (2.3) implies that

\[
\int_{\Omega} f(u) \, dx \leq \frac{k+1}{m} \int_{\Omega_0} f(u) \Psi(x) \, dx \leq \frac{k+1}{m} \int_{\Omega} f(u) \Psi(x) \, dx
\]

and so

\[
\left( \int_{\Omega} f(u) \, dx \right)^{-p} \geq \left( \frac{m}{k+1} \right)^p \left( \int_{\Omega} f(u) \Psi(x) \, dx \right)^{-p}.
\]

Combining (2.2) and (2.4), we get \( \frac{dA}{dt} \geq -\mu A(t) + \lambda C \left( \int_{\Omega} f(u) \Psi(x) \, dx \right)^{1-p} \), and using Jensen’s inequality (for this inequality to hold it is essential to have \( \Psi > 0 \) in \( \Omega \) normalized by the \( L^1 \)-norm) we finally obtain that \( A(t) \) satisfies

\[
\frac{dA(t)}{dt} \geq \lambda C f^{1-p} (A(t)) - \mu A(t),
\]

where \( C = (m/(k+1))^p \). If we choose \( \lambda > \mu B/C \) (note that \( 0 < B = \sup_{s \geq A(0)} s/f^{1-p}(s) < \infty \), since \( f \) satisfies (1.2)), then (2.5) implies

\[
t \leq \int_{A(0)}^{A(t)} \frac{ds}{\lambda C f^{1-p}(s) - \mu s} \leq \frac{1}{\lambda} \int_{A(0)}^{A(t)} \frac{ds}{f^{1-p}(s)} < \frac{1}{\lambda} \int_{A(0)}^{\infty} \frac{ds}{f^{1-p}(s)} < \infty,
\]

for \( 0 < \lambda \leq \lambda C - \mu B < \infty \). Hence \( A(t) \) blows up in finite time, i.e.

\[ A(t) \to \infty \quad \text{as} \quad t \to T^* \leq \int_{A(0)}^{\infty} \frac{ds}{\lambda C f^{1-p}(s) - \mu s} < \frac{1}{\lambda} \int_{A(0)}^{\infty} \frac{ds}{f^{1-p}(s)} < \infty.
\]

Actually the blow-up behaviour of \( A(t) \) defines the blow-up behaviour of \( u(x, t) \). Thus since \( A(t) \leq \|u(\cdot, t)\|_\infty \) we deduce that \( u(x, t) \) blows up in finite time as well, i.e. \( \|u(\cdot, t)\|_\infty \to \infty \) as \( t \to T^* \). \( \square \)

Let us suppose now that \( f(s) \) grows faster than any power of \( s \) in the sense that \( s f'(s)/f(s) \to \infty \) as \( s \to \infty \). This condition implies that \( f(s) \) satisfies condition (1.2) and so the previous blow-up result is valid in this case as well. This last result is expected, at least for the one-dimensional case, since it has been proved in [2] for the associated steady problem

\[
w'' + \frac{\lambda f(w)}{\left( f_{1-p}(w) \, dx \right)^p} = 0, \quad -1 < x < 1, \quad w'(\pm 1) \pm \beta w(\pm 1) = 0,
\]

for \( \beta > 0 \) and for \( 0 < p < 1 \), that there exists a critical value of parameter \( \lambda \), say \( \lambda^* \), such that problem (2.6) has no solution for \( \lambda > \lambda^* \).

 Blow-up occurs for big enough initial data as well; again the convex structure of \( \Omega \) is crucial for the proof. More precisely the following result is valid.

**Theorem 2.** Let \( \Omega \) be a convex domain of \( \mathbb{R}^N \), and \( f(s) \) a positive, increasing and convex function satisfying (1.2) and \( 0 < p < 1 \). Then the solution \( u(x, t) \) of (1.1) blows up in finite time for sufficiently large initial data \( u_0(x) \in L^2(\Omega) \).

**Proof.** Following the same steps as in the proof of Theorem 1, see also [13], we get

\[
\frac{dA(t)}{dt} \geq \lambda C f^{1-p} (A(t)) - \mu A(t) := g(A(t)).
\]

Let \( \delta_0 = \delta_0(\lambda) > 0 \) be the largest root (otherwise \( g(s) > 0 \)) of the equation \( g(s) = \lambda C f^{1-p}(s) - \mu s = 0 \); then \( g(s) \geq 0 \) for all \( s \geq \delta_0 \) and furthermore

\[
\int_{A_0}^{\infty} \frac{ds}{g(s)} \leq \frac{1}{\lambda} \int_{A_0}^{\infty} \frac{ds}{f^{1-p}(s)} < \infty, \quad A_0 > \delta_0, \quad 0 < \lambda \leq \lambda C - \mu B < \infty.
\]

Now for positive initial data \( u_0(x) \in L^2(\Omega) \) such that \( A_0 = \int_{\Omega} u_0(x) \Psi(x) \, dx > \delta_0, (1.2) \) and (2.7) imply that the solution of (1.1) blows up at finite time \( t^* \leq T^* \leq \int_{A_0}^{\infty} ds/g(s) < \infty. \) \( \square \)
2.2. Blow-up for Neumann boundary conditions

Let us assume now that $\beta(x) = 0$ for every $x \in \partial \Omega$, i.e. $u$ satisfies Neumann boundary conditions. In this case the associated steady problem
\[ \Delta u + \frac{\lambda}{f(w)} \left( \frac{f(w)}{f(w)} \right)^p = 0, \quad x \in \Omega, \quad \frac{\partial u}{\partial n} = 0, \quad x \in \partial \Omega, \]
(2.8)
does not admit any solution for every $\lambda > 0$. Actually, if we integrate the equation of problem (2.8) we get that $0 = \lambda f(w)/f(w)^{p-1}$ which is a contradiction. We can prove the same result using maximum principle arguments (Hopf’s lemma). This fact is an indication that time-dependent solutions $u(x, t)$ are unbounded in this case for every $\lambda > 0$. More precisely, in the following we prove that $u(x, t)$ blows up in finite time under again the condition (1.2). In this case the hypothesis concerning the convex structure of the domain $\Omega$ is not necessary.

**Proposition 3.** Let $\Omega$ be a smooth bounded domain of $\mathbb{R}^N$, and $f(s)$ a positive increasing and convex function satisfying (1.2) and $0 < p < 1$. Then the solution $u(x, t)$ of (1.1) blows up in finite time for every $\lambda > 0$ provided that $u_0 \in L^2(\Omega)$.

**Proof.** We introduce the energy functional $\Gamma = \int_{\Omega} u(x, t) \, dx/|\Omega|$, $t > 0$, and, integrating (1.1a) by parts over $\Omega$, taking $\frac{\partial u}{\partial n} = 0$ also into consideration, we get
\[ \frac{d\Gamma}{dt} = \frac{\lambda}{|\Omega|} \left( \int_{\Omega} f(u) \, dx \right)^{1-p}. \]
(2.9)
Using Jensen’s inequality
\[ \int_{\Omega} f(u(x, t)) \, dx = |\Omega| \int_{\Omega} \frac{1}{|\Omega|} f(u(x, t)) \, dx \geq |\Omega| f(\Gamma(t)), \]
and (2.9) we obtain
\[ \frac{d\Gamma(t)}{dt} \geq \lambda |\Omega|^{-p} f^{-1-p}(\Gamma(t)), \quad \text{for } t > 0. \]
(2.10)
Therefore by (1.2) and (2.10), it follows that the energy $\Gamma(t)$ and so $u(x, t)$ is defined and bounded only on the bounded time interval $[0, t^*)$, where $t^* \leq T^* \leq \lambda^{-1} |\Omega|^p \int_{t=0}^{\infty} f(s) \, ds < \infty$. Therefore $\|u(\cdot, t)\|_\infty \to \infty$ as $t \to t^*$. □

A result complementary to Proposition 3 (Neumann problem) is valid when $\int_0^{\infty} ds/f(s) = \infty$ for every $b > 0$. In fact, if we note $M(t) = \max_{x \in \Omega} u(x, t)$ then, recalling that $f(s)$ is increasing and positive, we obtain that $M(t)$ satisfies $\dot{M}(t) = dM/dt \leq \lambda f(M)/\left( \int_{\Omega} f(u) \, dx \right)^p \leq \lambda f(M)/(f(0) |\Omega|)^p$, which leads to $\int_{t=0}^{M(t)} ds/f(s) \leq \lambda |\Omega|/(f(0) |\Omega|)^p$. The latter implies that $u(x, t)$ cannot blow up in finite time. We claim that $u(x, t)$ is unbounded; otherwise there would be a constant $K$ such that $M(t) < K$ for every $t > 0$ and thus $m(t) = \min_{x \in \Omega} u(x, t)$ would satisfy $\dot{m}(t) = dM/dt \geq \lambda f(m)/(f(K) |\Omega|)^p > 0$, or $\int_{t=0}^{M(t)} ds/f(s) \geq \lambda |\Omega|/(f(K) |\Omega|)^p$, implying that $m(t) \to \infty$ and so $M(t) \to \infty$ as $t \to \infty$, which is a contradiction. Consequently, in this case problem (1.1) has a global-in-time unbounded solution.

Considering now the structure of the blow-up set, it has been proved in [4] that in the radial symmetric case when $f(s) = e^s$ and $N = 1, 2$, blow-up takes place only at the origin $r = 0$. We believe that the same can be proved, following a Friedman–McLeod type of approach (see [7]), for more general functions and higher dimensions.

Also using formal asymptotics we have some evidence concerning the growth of $u(x, t)$ near the origin as we reach the blow-up time. For the exponential case $f(s) = e^s$ we conjecture that for $N = 1$ we have $u(r, t) \sim -[(p/2) (1 - p)] \ln(t^* - t) + v(r/\sqrt{(t^* - t)})$ as $t \to t^*$ for $r > 0$, which seems to be in agreement with the upper estimate that was proved in [4]. The two-dimensional case seems to be more delicate. For $N \geq 3$ we conjecture that $u(x, t)$ grows like in the “standard” reaction–diffusion problem; see [1]. For the power-law case $f(s) = (1 + s)^{\frac{1}{2}}$, $k > 0$, we have to distinguish more cases. In our view, all this asymptotic analysis is interesting enough to be explained in another work.
Acknowledgments

The authors would like to thank Professor A.A. Lacey for several fruitful discussions. Also, N.I. Kavallaris was supported by the Greek State Scholarship Foundation (I.K.Y.).

References