Some Localization Theorems on Hamiltonian Circuits

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Theorems on the localization of the conditions of G. A. Dirac (Proc. London Math. Soc. (3) 2, 1952, 69–81), O. Ore (Amer. Math. Monthly 67, 1960, 55), and Geng-hua Fan (J. Combin. Theory Ser. B 37, 1984, 221–227) for a graph to be hamiltonian are obtained. It is proved, in particular, that a connected graph $G$ on $p \geq 3$ vertices is hamiltonian if $d(u) \geq |M^3(u)|/2$ for each vertex $u$ in $G$, where $M^3(u)$ is the set of vertices $v$ in $G$ that are a distance at most three from $u$.

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1. INTRODUCTION

Our notation and terminology follows Harary [4]. Let $k$ be a positive integer. For each vertex $u$ of a graph $G = (V, X)$ we will denote by $M^k(u)$ and $N(u)$ the sets of all $v \in V$ with $d(u, v) \leq k$ and $d(u, v) = 1$, respectively. The subgraph of $G$ induced by $M^k(u)$ is denoted by $G_k(u)$. The degree in $G_k(u)$ of a vertex $v \in M^k(u)$ is denoted by $d_{G_k(u)}(v)$.

The closure $C(G)$ of $G$ is the graph obtained from $G$ by recursively joining pairs of nonadjacent vertices whose degree-sum is at least $|V|$, until no such pair remains.

The following results are known. A graph $G = (V, X)$ on $p \geq 3$ vertices is hamiltonian if:

$$d(v) \geq p/2 \quad \text{for each } v \in V \quad \text{(Dirac [2])}. \quad (1.1)$$

$$uv \notin X \Rightarrow d(u) + d(v) \geq p \quad \text{(Ore [6])}. \quad (1.2)$$

$$d(u) = k < (p - 1)/2 \Rightarrow |\{v \in V/d(v) \leq k\}| < k \quad \text{(Posa [7])}. \quad (1.3)$$
and
\[ d(u) = (p - 1)/2 \Rightarrow |\{v \in V | d(v) \leq (p - 1)/2\}| \leq (p - 1)/2. \]

\[ C(G) \text{ is a complete graph (Bondy and Chvátal [1]).} \] (1.4)

\[ G \text{ is 2-connected and } d(v) < p/2, d(u, v) = 2 \Rightarrow d(u) \geq p/2 \]

\[ (\text{Geng-hua Fan [3]}). \] (1.5)

In [5] the following theorem on a localization of condition (1.3) is proved:

**Theorem.** A connected graph \( G \) on \( p \geq 3 \) vertices is hamiltonian if
\[ d(u) = k < (p - 1)/2 \Rightarrow |\{v \in M^2(u) | d(v) \leq k\}| < k \]
and
\[ d(u) = (p - 1)/2 \Rightarrow |\{v \in M^2(u) | d(v) \leq (p - 1)/2\}| \leq (p - 1)/2. \]

In this paper we obtain the theorems on localizations of conditions (1.1), (1.2), and (1.5).

### 2. Results

**Lemma.** Let \( G \) be a graph with \( d(u, v) = 2 \), \( w \in N(u) \cap N(v) \), and \( d(u) + d(v) \geq |N(u) \cup N(v) \cup N(w)| \). Then \( |N(w) \setminus (N(u) \cup N(v))| \leq |N(u) \cap N(v)| \).

**Proof.**
\[
|N(w) \setminus (N(u) \cup N(v))| \\
= |N(w)| - |N(w) \cap (N(u) \cup N(v))| \\
= |N(w)| - (|N(w)| + |N(u) \cup N(v)| - |N(u) \cup N(v) \cup N(w)|) \\
= |N(u) \cup N(v) \cup N(w)| - (|N(u)| + |N(v)| - |N(u) \cap N(v)|) \\
= |N(u) \cap N(v)| - (d(u) + d(v) - |N(u) \cup N(v) \cup N(w)|) \\
\leq |N(u) \cap N(v)|.
\]

**Theorem 1.** Let \( G = (V, X) \) be a connected graph with at least three vertices. If
\[ d(u) + d(v) \geq |N(u) \cup N(v) \cup N(w)| \]
for each triple of vertices \( u, v, w \) with \( d(u, v) = 2 \) and \( w \in N(u) \cap N(v) \), then \( G \) is hamiltonian.
**Proof.** Let $G$ satisfy the hypothesis of Theorem 1. Clearly, $G$ contains a circuit; let $C$ be the largest one. If $G$ has no hamiltonian circuit, then there is a vertex $u$ outside of $C$ that is adjacent to at least one vertex in $C$. Let $\{w_1, \ldots, w_n\}$ be the set of vertices in $C$ that are adjacent to $u$, and for each $i = 1, \ldots, n$ let $v_i$ be the successor of $w_i$ in a fixed cyclic ordering of $C$.

Note, that if $1 \leq i < j \leq n$ then $v_i v_j \notin X$. Otherwise delete the edges $w_i v_i, w_j v_j$ from $C$ and add the edges $v_i v_j, w_i u, u v_j$. In this way we obtain a circuit longer than $C$, which is a contradiction.

For each $i = 1, \ldots, n$ let $F_i$ and $T_i$ denote the sets $N(u) \cap N(v_i)$ and $N(w_i) \setminus (N(u) \cup N(v_i))$, respectively. Since $C$ is the longest circuit, $uw_i \notin X$, $i = 1, \ldots, n$. Then $d(u, v_i) = 2$ and from the Lemma we have $|T_i| \leq |F_i|$ for each $i = 1, \ldots, n$.

We shall show that there is a vertex $u'$ such that $u' \notin C$ and $u' \in F_i$ for some $i, 1 \leq i \leq n$.

Consider the following iterated algorithm.

**Step 1.** $k := 1$, $m := 1$, and $Z_i^1 := \{u, v_i\}$, $Y_i^1 := \{w_i\}$ for each $i = 1, \ldots, n$.

**Step 2.** If the set $F_m \setminus Y_m^k$ contains a vertex $u' \notin C$, then stop. Otherwise, choose an arbitrary vertex $w$ in $F_m \setminus Y_m^k$. Clearly, $w = w_i$ for some $i, 1 < i < n$.

Set

$$Y_m^{k+1} := Y_m^k \cup \{w_i\}; \quad Z_m^{k+1} := Z_m^k;$$

$$Y_r^{k+1} := Y_r^k; \quad Z_r^{k+1} := Z_r^k \cup \{v_m\};$$

$$Y_i^{k+1} := Y_i^k; \quad Z_i^{k+1} := Z_i^k \quad \text{for} \quad i \neq m, r \text{ and } 1 \leq i \leq n.$$

**Step 3.** $k := k + 1$, $m := r$ and go to Step 2.

It is not difficult to see that before the $k$th iteration of the algorithm we have

(a) $Z_i^k \subseteq T_i, Y_i^k \subseteq F_i, |Z_i^k| \geq |Y_i^k|$ for each $i, 1 \leq i \leq n$.

(b) $F_m \setminus Y_m^k \neq \emptyset$, because $|T_m| \leq |F_m|$ and $|Z_m^k| > |Y_m^k|$.

(c) $Y_i^k \subseteq \{w_1, \ldots, w_n\}$ for each $i, 1 \leq i \leq n$.

(d) $|Y_m^k| = |Y_m^{k-1}| + 1$ if $k > 2$.

From (b), (c), (d) it follows that if $k \geq 2$ then $\sum_{i=1}^n |Y_i^{k-1}| < \sum_{i=1}^n |Y_i^k| \leq n^2$. Hence there exists $k$ such that $1 \leq k \leq n^2$ and the set $F_m \setminus Y_m^k$ contains a vertex $u' \notin C$. Delete the edge $w_m v_m$ from $C$ and add the edges $w_m u, uu', u' v_m$. In this way we obtain a circuit longer than $C$, which is a contradiction. The proof is complete.
Note that for every \( t \geq 5 \) there exists a graph \( G_t = (V_t, X_t) \) with 
\[ V_t = \{v_1, v_2, \ldots, v_{2t}\} \]
and 
\[ X_t = \bigcup_{k=0}^{t-2} \{v_i v_j / 2k + 1 \leq i < j \leq 2k + 4\} , \]
which fulfills the condition in Theorem 1 and does not fulfill conditions (1.1)-(1.5). Clearly, \( C(G_t) = G_t \), \( |V_t| = 2t \), and \( |X_t| = 5t - 4 = (5/2) \cdot |V_t| - 4 \).

**Corollary 1.** Let \( G \) be a connected graph on \( p \geq 3 \) vertices. If 
\[ d(u) + d_G(u, v) \geq |M^2(u)| \]
for each pair of vertices \( u, v \) with \( d(u, v) = 2 \), then \( G \) is hamiltonian.

**Proof.** Clearly,
\[ d(u) + d(v) - |N(v) \setminus M^2(u)| = d(u) + d_G(u, v) \geq |M^2(u)|. \]

Then
\[ d(u) + d(v) \geq |M^2(u)| + |N(v) \setminus M^2(u)| \geq |N(u) \cup N(v) \cup N(w)| \]
for each vertex \( w \in N(u) \cap N(v) \). Hence, Corollary 1 follows from Theorem 1.

**Corollary 2.** Let \( G \) be a connected graph on \( p \geq 3 \) vertices. If 
\[ d(u) + d(v) \geq |M^3(u)| \]
for each pair of vertices \( u, v \) with \( d(u, v) = 2 \), then \( G \) is hamiltonian.

Corollary 2 follows from Theorem 1 because \( |M^3(u)| \geq |N(u) \cup N(v) \cup N(w)| \) for each vertex \( w \in N(u) \cap N(v) \).

**Corollary 3.** Let \( G \) be a connected graph on \( p \geq 3 \) vertices. If 
\[ d(u) \geq |M^3(u)|/2 \]
for every vertex \( u \) in \( G \) then \( G \) is hamiltonian.

**Proof.** Let \( G \neq K_3 \), \( d(u, v) = 2 \), and \( d(u) \leq d(v) \). Since \( d(u) \geq |M^3(u)|/2 \), then \( d(u) + d(v) \geq |M^3(u)| \geq |N(u) \cup N(v) \cup N(w)| \) for each vertex \( w \in N(u) \cap N(v) \). Therefore Corollary 3 follows from Theorem 1.

**Corollary 4.** Let \( G \) be a connected graph on \( p \geq 3 \) vertices. If 
\[ d_G(u, v) + d_G(u, v) \geq |M^1(w)| \]
or
\[ d(u) + d(v) \geq |M^2(w)| \]
for each triple of vertices \( u, v, w \) with \( d(u, v) = 2 \) and \( w \in N(u) \cap N(v) \), then \( G \) is hamiltonian.
Proof. Let \( d(u, v) = 2 \) and \( w \in N(u) \cap N(v) \).

If \( d(u) + d(v) \geq |M^2(w)| \), then \( d(u) + d(v) \geq |N(u) \cup N(v) \cup N(w)| \) because \( |M^2(w)| = |N(u) \cup N(v) \cup N(w)| \).

Suppose that \( d_{G_1(w)}(u) + d_{G_1(w)}(v) \geq |M^1(w)| \). Clearly, \( d_{G_1(w)}(u) = d(u) - |N(u) \setminus M^1(w)| \) and \( d_{G_1(w)}(v) = d(v) - |N(v) \setminus M^1(w)| \). Hence
\[
d(u) + d(v) \geq |M^1(w)| + |N(u) \setminus M^1(w)| + |N(v) \setminus M^1(w)|
\]
and Corollary 4 follows from Theorem 1.

**Corollary 5.** Let \( G \) be a connected graph on \( p \geq 3 \) vertices. If for each vertex \( u \) in \( G \) at least one of the graphs \( G_1(u) \) or \( G_2(u) \) satisfies Ore's condition, then \( G \) is hamiltonian.

Corollary 5 follows from Corollary 4.

**Theorem 2.** Let \( G = (V, X) \) be a 2-connected graph on \( p \geq 3 \) vertices and let \( v \) and \( u \) be distinct vertices of \( G \). If
\[
d(u) < \frac{p}{2}, \quad d(u, v) = 2 \Rightarrow d(v) \geq |M^3(u)|/2,
\]
then \( G \) is hamiltonian.

Proof. Let \( A = \{P^1, \ldots, P^h\} \) be the set of all longest paths in \( G \). For each \( i = 1, \ldots, h \) let \( P^i = v_0^i v_1^i \cdots v_m^i \) and \( f(P^i) \) be the smallest \( r \) from \( \{0, 1, \ldots, m-1\} \) such that \( v_r, v_r^i \in X \). We denote by \( A_1 \) the set of all \( P^i \in A \) with \( d(v_0^i) = \max_{1 \leq j \leq h} d(v_0^j) \).

Suppose that \( G \) is a graph satisfying the condition of Theorem 2 and that \( G \) has no hamiltonian circuit. We shall arrive at a contradiction.

Let \( P = v_0 v_1 \cdots v_m \) be some longest path in \( G \) of length \( m \), chosen so that \( f(P) = \min_{P^i \in A_1} f(P^i) \). Clearly, \( d(v_0) \geq d(v_m) \). If \( d(v_0) + d(v_m) \geq p \) then there are at least two consecutive vertices on \( P \), \( v_i, v_{i+1} \), such that \( v_i, v_{i+1} \in X \) and \( v_{i+1} v_i \in X \), and so we obtain a circuit of length \( m + 1 \). By the connectedness of \( G \), we have either a hamiltonian circuit or a path of length \( m + 1 \). Each leads to contradictions. Consequently \( d(v_0) + d(v_m) < p \).

From the proof above we can also suppose that

(a) \( G \) has no circuit of length \( m + 1 \).

Since \( G \) is 2-connected, \( d(v_m) \geq 2 \). Let \( N(v_m) = \{v_j, \ldots, v_{j_h}\} \) and \( j_1 < \cdots < j_h \). Clearly, \( j_1 \geq 1 \), otherwise \( G \) has a circuit of length \( m + 1 \), which is contrary to (a). We show now that
(b) if \(v_m v_i \in X\) and \(j_1 \leq i \leq m - 1\) then \(N(v_{i+1}) \subseteq \{v_{j_1}, \ldots, v_m\}\),
(c) \(v_m v_i \notin X\) for some \(i, j_1 < i < m\),
(d) \(v_{j_i - 1} v_{j_i + 1} \notin X\) for every \(i, 1 \leq i \leq t\).

**Proof.** (b) Clearly, we have \(N(v_{i+1}) \subseteq \{v_0, v_1, \ldots, v_m\}\) otherwise \(G\) has a path of length \(m + 1\). Suppose that there is \(s\) such that \(1 \leq s < j_1\) and \(v_s v_{j_1} \in X\). Then

\[
P' = v_0 v_1 \cdots v_{j_1 - 1} v_{j_1} v_{j_1 + 1} \cdots v_m v_{j_1 - 1} \cdots v_{j_1}
\]

is the longest path in \(G\) with \(f(P') < f(P)\). This contradicts the choice of \(P\). Therefore \(N(v_{i+1}) \subseteq \{v_{j_1}, v_{j_1 + 1}, \ldots, v_m\}\).

(c) If \(v_m v_i \in X\) for every \(i, j_1 \leq i < m\), then \(\bigcup_{i=j_1}^{m-1} N(v_{i+1}) \subseteq \{v_{j_1}, v_{j_1 + 1}, \ldots, v_m\}\). This contradicts the 2-connectedness of \(G\).

(d) It is obvious that (d) follows from (b).

From (c) it follows that there is a \(k\) such that \(j_1 < k < m - 1\), \(v_m v_k \notin X\), and \(v_m v_i \in X\) for every \(i, j_1 \leq i < m - 1\). Thus we have

(e) there is no \(i\) such that \(j_1 < i < m - 1\), \(v_j v_i \in X\), and \(v_k v_{i+1} \in X\).

Indeed, if \(v_j v_i \in X\) and \(v_k v_{i+1} \in X\) then from (d) it follows that \(i > k\). Then \(G\) has the longest path \(P'\),

\[
P' = v_0 v_1 \cdots v_{i-1} v_i v_{i+1} \cdots v_k v_{k+1} \cdots v_m v_{m-1} \cdots v_{j_1}
\]

with \(f(P') < f(P)\). This contradicts the choice of \(P\).

Clearly, \(d(v_k, v_m) = 2\) and \(d(v_{j_1 - 1}, v_m) = 2\). Since \(d(v_m) < p/2\), it follows from (2.1) that \(d(v_{j_1 - 1}) > |M^3(v_m)|/2\) and \(d(v_k) > |M^3(v_m)|/2\).

Since \(v_k v_{j_1 - 1} \notin X\) and the degree-sum of vertices \(v_k\) and \(v_{j_1 - 1}\) in \(G_3(v_m)\) is at least \(|M^3(v_m)|\), \(d(v_k, v_{j_1 - 1}) = 2\).

From (d) it follows that \(d(v_{j_1 - 1}) < |M^3(v_m)| - d(v_m)\). Since \(d(v_{j_1 - 1}) \geq |M^3(v_m)|/2\), then \(d(v_m) < |M^3(v_m)|/2\). Therefore \(d(v_{j_1 - 1}) > d(v_m)\) and \(d(v_k) > d(v_m)\).

**Case 1.** \(d(v_k) < p/2\). Since \(d(v_k, v_m) = d(v_k, v_{j_1 - 1}) = 2\), it follows from (2.1) that \(d(v_m) > |M^3(v_k)|/2\) and \(d(v_{j_1 - 1}) > |M^3(v_k)|/2\). Together with \(d(v_k) > d(v_m)\) this implies that

\[
d(v_k) + d(v_{j_1 - 1}) > |M^3(v_k)|. \quad (2.2)
\]

From (d) it follows that \(v_i v_{j_1 - 1} \notin X\) for each \(i, 1 + j_1 \leq i < k\). From (e) it follows that \(v_i v_{j_1 - 1} \notin X\) for every \(i\) such that \(i > j_1\) and \(v_k v_{i+1} \in X\). Besides, \(v_m v_{j_1 - 1} \notin X\) and \(v_m, v_{j_1 - 1} \in M^3(v_k)\). Thus \(d(v_{j_1 - 1}) \leq |M^3(v_k)| - d(v_k) - 1\). This contradicts (2.2).

**Case 2.** \(d(v_k) > p/2\). If \(d(v_{j_1 - 1}) < p/2\) then (2.1) and \(d(v_{j_1 - 1}, v_m) = \)
$d(v_{j-1}, v_k) = 2$ imply that $d(v_m) \geq |M^3(v_{j-1})|/2$ and $d(v_k) \geq |M^3(v_{j-1})|/2$. Since $d(v_{j-1}) > d(v_m)$, we have

$$d(v_{j-1}) + d(v_k) \geq |M^3(v_{j-1})|.$$  \hfill (2.3)

If $d(v_{j-1}) \geq p/2$ then $d(v_{j-1}) + d(v_k) \geq p \geq |M^3(v_{j-1})|$, so (2.3) holds again.

From (b) it follows that $v_k v_{j-1} \notin X$ and $v_k$ is not adjacent to every vertex $v \in N(v_{j-1}) \setminus \{v_{j-1}, v_{j-1}^1, \ldots, v_{j-1}^m\}$.

From (e) it follows that $v_k v_{1+i} \notin X$ for every $i$ such that $v_i \in N(v_{j-1}) \cap \{v_{1+j}, v_{2+j}, \ldots, v_m\}$. Besides, we have $v_{j-1}, v_k \in M^3(v_{j-1})$. Therefore $d(v_k) \leq |M^3(v_{j-1})| - d(v_{j-1}) - 1$.

This contradicts (2.3). The proof is complete.

Note that for every $r \geq 2$ there exists a graph $G_r = (V_r, X_r)$ with $V_r = \{w_1, w_2\} \cup \{v_1, \ldots, v_{3r-1}\}$ and $X_r = \{w_1 v_1, w_2 v_{i-1}, \ldots, 2r\} \cup \{v_{i+j}, v_{i+j+1}, \ldots, v_{i+j}\}$ that satisfies the condition of Theorem 2 and does not satisfy the condition (1.5).

Besides, for every $n \geq 5$ there exists a graph $G_n = (V_n, X_n)$ with $V_n = \{v_1, \ldots, v_n\}$ and $X_n = \{v_{i+j}, v_{i+j+1}, \ldots, v_{i+j}\}$ that satisfies the condition of Theorem 2 and does not satisfy the condition of Theorem 1.

Let $G = (V, X)$. It is shown in [1] (by paraphrasing Ore's proof [6]) that if $G + uv$ is hamiltonian and $d(u) + d(v) \geq |V|$ then $G$ itself is hamiltonian.

**Theorem 3.** If $G + uv$ is hamiltonian, $d(u, v) = 2$, and

$$d(u) + d_{G}(v) \geq |M^2(u)|,$$  \hfill (2.4)

then $G$ itself is hamiltonian.

**Proof.** Suppose $G + uv$ is hamiltonian but $G$ is not. Then $G$ has a hamiltonian path $u_1, u_2, \ldots, u_p$ with $u_1 = v$ and $u_p = u$. Let $N(u) = \{u_{i_1}, \ldots, u_{i_t}\}$. If $u_{i_{j+1}} \notin X$ for every $j$, $1 \leq j \leq t$, then $d_{G}(v) \geq |M^2(u)| - d(u)$. This contradicts (2.4). Hence there is $m$ such that $1 \leq m \leq t-1$, $v u_{i+m} \in X$, and $u_{i+m} \in X$.

But then $G$ has the hamiltonian circuit

$$u_1 u_{i+m} u_2 + u_{i+m} \cdots u_p u_{i+m} u_{i+m-1} \cdots u_1.$$  

This contradicts the hypothesis.

**Corollary 6.** If $G + uv$ is hamiltonian, $d(u, v) = 2$, and $d(u) + d(v) \geq |M^3(v)|$, then $G$ itself is hamiltonian.
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