Development of fuzzy $\bar{X} - \bar{R}$ and $\bar{X} - \bar{S}$ control charts using $\alpha$-cuts

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**Abstract**

Statistical process control (SPC) is an approach that uses statistical techniques to monitor the process. Shewhart introduced the control charts that are one of the most important techniques of quality control to detect if assignable causes exist. The widely used control charts are $\bar{X}/C_0R$ and $\bar{X}/C_0S$. These are called traditional variable control charts. In the traditional variable control charts, center line, upper control limit and lower control limit are represented by numeric values. A process is either “in control” or “out of control” depending on numeric observation values. For many problems, control limits could not be so precise. Uncertainty comes from the measurement system including operators and gauges, and environmental conditions. In this context, fuzzy set theory is a useful tool to handle this uncertainty. Numeric control limits can be transformed to fuzzy control limits by using membership functions. If a sample mean is too close to the control limits and the used measurement system is not so sensitive, the decision may be faulty. Fuzzy control limits provide a more accurate and flexible evaluation. This study constructs the fuzzy $\bar{X}/C_0R$ and $\bar{X}/C_0S$ control charts with $\alpha$-cuts. An application is presented for fuzzy $\bar{X}/C_0R$ control charts. By using fuzzy $\bar{X}/C_0R$ and $\bar{X}/C_0S$ control charts, the flexibility of traditional control limits is increased.

1. Introduction

Control charts, proposed by Shewhart in 1920s, have a widespread application in especially production processes. These control charts were designed to monitor a process for shifts in mean and variance of a single quality characteristic. Two main types of control charts are variable control charts and attribute control charts. The fuzzy set theory is a more suitable tool for handling attribute data since these data may be expressed in linguistic terms such as “very good”, “good”, “medium”, “bad”, and “very bad”. The fuzzy set theory was first introduced by Zadeh [24]. Recently Zadeh [26] viewed fuzzy logic in a non-standard perspective. In this perspective, the cornerstones of fuzzy logic are: graduation, granulation, precisiation and the concept of a generalized constraint.

Many studies were done to combine statistical methods and fuzzy set theory. Zadeh [25] outlined the generalized theory of uncertainty (GTU) which represented a significant change both in perspective and direction in dealing with uncertainty and information. Watanabe and Imaizumi [23] introduced a testing method of a fuzzy hypothesis for random data. Romer and Kandel [16] investigated the impact of vague data on the statistical task of hypotheses testing. Grzegorzewski [6] proposed a method for testing hypothesis for fuzzy decision showing a grade of acceptability for the null and the alternative hypotheses. Buckley and Eslami [2] introduced a method of estimation of mean and variance in fuzzy statistics which uses a set of confidence interval to produce a triangular number as the estimator. Fuzzy regression model was first introduced by

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doi:10.1016/j.ins.2008.09.022
Tanaka et al. [19], where a fuzzy linear system was used as a regression model. Diomond [3] proposed several models for simple least-squares fitting of fuzzy-valued data. Lopez-Diaz and Gil [12] defined a fuzzy unbiased estimator of the sample mean in random sampling with replacement from a finite population. Garcia et al. [5] illustrated estimating the expected value of fuzzy random variables in the stratified random sampling. Trillas [20] gave a view that fuzzy sets are mathematical entities giving extension to the predicates. Fuzzy c-means (FCM) clustering methodology was investigated by Ozkan and Turksen [14] to determine the effective upper and lower boundaries of the level of fuzziness in order to capture the uncertainty generated by this parameter. Tuncer and Benli [21] defined c-statistical limit and c-statistical cluster points of a sequence of fuzzy numbers. Saavedra et al. [18] presented a proposal for decomposing large uncertainties associated with fuzzy numbers for solving non-linear fuzzy problems with crisp coefficients. There are few publications on fuzzy attribute control charts and their applications in literature: Raz and Wang [15], Wang and Raz [22], Kanagawa et al. [10], Gülbay et al. [7], Gülba and Kahraman [8,9]. Raz and Wang [15], Wang and Raz [22] proposed some approaches by assigning a fuzzy set to each linguistic term and then combining these for each sample using the rules of fuzzy arithmetic. Kanagawa et al. [10] introduced a control chart based on the probability density function for linguistic data. Gülbay et al. [7] suggested the x-cut fuzzy control charts for linguistic data. Gülba and Kahraman [8] developed fuzzy c control charts for determining the unnatural patterns. Gülba and Kahraman [9] proposed a direct fuzzy approach for c-charts.

There are few papers on fuzzy variable control charts and their applications. Rowlands and Wang [17] introduced fuzzy-SPC methods based on the application of fuzzy logic to the SPC-zone rules. El-Shal and Morris [4] modified SPC-zone rules to reduce false alarms and detect the real faults. Zarandi et al. [27] presented a new hybrid method based on a combination of fuzzified sensitivity criteria and fuzzy adaptive sampling rules to determine the sample size and sample interval of the control charts. However, many fuzzy studies on statistical analysis for continuous random variables are observed in the literature [1]. In a real system, it is assumed that there are no doubts about observations and their values. But when these observations include human judgments, evaluations and decisions, a continuous random variable (Xi) of a production process should include the variability caused by human subjectivity or measurement devices, or environmental conditions. These variability causes create vagueness in the measurement system. Thus, linguistic terms like “a range between 5.5 and 6.1” or “a range approximately equal to 5.8” can be used instead of an exact value of continuous random variable. In this situation, neither a deterministic model nor a probabilistic model does not reflect the real system adequately. Real situations are very often uncertain or vague in a number of ways. The fuzzy set theory provides a useful methodology for modeling these uncertain data. So, representing X, values by fuzzy numbers are a reasonable way to analyze and evaluate the process. Flexibility of control limits can be provided by fuzzy X, s. Thus, we can get rid of strict control limits.

Some measures of central tendency in descriptive statistics are used in variable control charts. These measures can be used to convert fuzzy sets into scalars which are fuzzy mode, x-level fuzzy midrange, fuzzy median, and fuzzy average. For the selection of these fuzzy measures, there is no theoretical basis. This selection is mainly based on the ease of computation or preference of the user [22].

The aim of this study is first to introduce the framework of fuzzy $X - \bar{R}$ and $X - \bar{S}$ control charts with x-cuts by using x-level fuzzy midrange. First of all, we transform the traditional $X - \bar{R}$ and $X - \bar{S}$ control charts to fuzzy control charts. To obtain fuzzy $X - \bar{R}$ and $X - \bar{S}$ control charts, triangular fuzzy numbers (a, b, c) are used. Secondly, x-cut fuzzy $X - \bar{R}$ control charts and x-cut fuzzy $X - \bar{S}$ control charts are developed by using an x-cut approach. Thirdly, x-level fuzzy midranges for fuzzy $X - \bar{R}$ and $X - \bar{S}$ control charts are calculated by using x-level fuzzy midrange transformation techniques. Finally, an application is made for fuzzy $X - \bar{R}$ control charts.

This paper is organized as follows: x-level fuzzy midrange is shortly introduced in the second section. Fuzzy $X - \bar{R}$ control charts are developed in Section 3. Fuzzy $X - \bar{S}$ control charts are developed in Section 4. An application for fuzzy $X - \bar{R}$ control charts is given in Section 5. The conclusions are presented in the final section.

2. Fuzzy transformation techniques

There are four fuzzy transformation techniques, which are similar to the measures of central tendency used in descriptive statistics: x-level fuzzy midrange, fuzzy median, fuzzy average, and fuzzy mode [22]. In this study, the x-level fuzzy midrange transformation technique is used for fuzzy $X - \bar{R}$ and $X - \bar{S}$ control charts.

2.1. x-Level fuzzy midrange

The x-level fuzzy midrange $f_{mr}^x$ is defined as the midpoint of the ends of the x-level cuts. An x-level cut, denoted by $A^x$, is a nonfuzzy set that comprises all elements whose membership is greater than or equal to x. If $A^x$ and $B^x$ are the end points of $A^x$, then

$$f_{mr}^x = \frac{1}{2} (a^x + c^x).$$

(1)

In fact the fuzzy mode is a special case of x-level fuzzy midrange when $x = 1$.

x-level fuzzy midrange of sample $j$, $S_{mrj}^x$, is determined by

$$S_{mrj}^x = (a_j + c_j + x((b_j - a_j) - (c_j - b_j)))/2.$$  

(2)
3. Fuzzy \( \bar{X} \) and \( \bar{R} \) control charts

In the traditional approach, the control of process averages or mean quality levels is usually done by \( \bar{X} \) charts. The process variability or dispersion can be controlled by either a control chart for the range, called \( \bar{R} \) chart, or a control chart for the standard deviation, called \( S \) chart. In this section, fuzzy \( \bar{X} - \bar{R} \) control charts are introduced. The fuzzy \( \bar{X} - \bar{S} \) control charts are presented in the next section.

The formulation of traditional \( \bar{X} \) control charts based on sample ranges is given as follows [13]:

\[
\begin{align*}
\text{UCL}_\bar{X} &= \bar{X} + A_2 \bar{R}, \\
\text{CL}_\bar{X} &= \bar{X}, \\
\text{LCL}_\bar{X} &= \bar{X} - A_2 \bar{R},
\end{align*}
\]

(3)

where \( A_2 \) is a control chart coefficient [13] and \( \bar{R} \) is the average of \( R_i \)'s that are the ranges of samples.

In the fuzzy case, each sample, or subgroup, is represented by a triangular fuzzy number \((a, b, c)\) as shown in Fig. 1.

In this study, triangular fuzzy numbers are represented as \((\bar{X}_a, \bar{X}_b, \bar{X}_c)\) for each fuzzy observation. The center line, \( \bar{C}L \) is the arithmetic mean of fuzzy samples, and it is shown as \((\bar{X}_a, \bar{X}_b, \bar{X}_c)\) where \( \bar{X}_a, \bar{X}_b, \bar{X}_c \) are called general means and calculated as follows:

\[
\begin{align*}
\bar{X}_b &= \frac{\sum_{i=1}^{n} X_{bj}}{n}; \quad k = a, b, c; \quad i = 1, 2, \ldots, n; \quad j = 1, 2, \ldots, m, \\
\bar{X}_k &= \frac{\sum_{j=1}^{m} X_{kj}}{m}; \quad k = a, b, c; \quad j = 1, 2, \ldots, m, \\
\bar{C}L &= (\bar{X}_a, \bar{X}_b, \bar{X}_c) = \left(\frac{\sum_{j=1}^{m} X_{aj}}{m}, \frac{\sum_{j=1}^{m} X_{bj}}{m}, \frac{\sum_{j=1}^{m} X_{cj}}{m}\right),
\end{align*}
\]

(4) (5) (6)

where \( n \) is the fuzzy sample size and \( m \) is the number of fuzzy samples and \( \bar{C}L \) is a center line for fuzzy \( \bar{X} \) control chart.

3.1. Fuzzy \( \bar{X} \) control chart based on ranges

The fuzzy \( \bar{X} \) control charts based on ranges are calculated by using \( \bar{C}L \). Where \( \text{UCL}_\bar{X} \) and \( \text{LCL}_\bar{X} \) are the upper and lower control limits of fuzzy \( \bar{X} \) control charts with range. They are calculated as follows:

\[
\begin{align*}
\text{UCL}_\bar{X} &= \bar{C}L + A_2 \bar{R} = (\bar{X}_a, \bar{X}_b, \bar{X}_c) + A_2 (\bar{R}_a, \bar{R}_b, \bar{R}_c) = (\bar{X}_a, A_2 \bar{R}_a, \bar{X}_b, A_2 \bar{R}_b, \bar{X}_c, A_2 \bar{R}_c) = (\text{LCL}_1, \text{LCL}_2, \text{LCL}_3), \\
\text{CL}_\bar{X} &= (\bar{X}_a, \bar{X}_b, \bar{X}_c) = (\bar{C}L_1, \bar{C}L_2, \bar{C}L_3), \\
\text{LCL}_\bar{X} &= \bar{C}L - A_2 \bar{R} = (\bar{X}_a, \bar{X}_b, \bar{X}_c) - A_2 (\bar{R}_a, \bar{R}_b, \bar{R}_c) = (\bar{X}_a - A_2 \bar{R}_a, \bar{X}_b - A_2 \bar{R}_b, \bar{X}_c - A_2 \bar{R}_c) = (\text{LCL}_1, \text{LCL}_2, \text{LCL}_3),
\end{align*}
\]

(7) (8) (9)

where \( \bar{R}_a, \bar{R}_b, \) and \( \bar{R}_c \) are the arithmetic means of the least possible values, the most possible values, and the largest possible values, respectively. Firstly, \( \bar{R}_a, \bar{R}_b, \) and \( \bar{R}_c \) are calculated as follows:

\[
\begin{align*}
\bar{R}_a &= X_{\text{max}, a} - X_{\text{min}, a}; \quad \bar{R}_b &= X_{\text{max}, b} - X_{\text{min}, b}; \quad \bar{R}_c &= X_{\text{max}, c} - X_{\text{min}, c}; \quad j = 1, 2, \ldots, m,
\end{align*}
\]

where \( (X_{\text{max}, a}, X_{\text{max}, b}, X_{\text{max}, c}) \) is the maximum fuzzy number in the sample and \( (X_{\text{min}, a}, X_{\text{min}, b}, X_{\text{min}, c}) \) is the minimum fuzzy number in the sample.

![Fig. 1. Representation of a sample by triangular fuzzy numbers.](image)
\[ R_a = \frac{\sum R_{ai}}{m}, \]  
\[ R_b = \frac{\sum R_{bj}}{m}, \]  
\[ R_c = \frac{\sum R_{cj}}{m}. \]  

3.2. \( \alpha \)-Cut fuzzy \( \tilde{X} \) control charts based on ranges

An \( \alpha \)-cut is a nonfuzzy set which comprises of all elements whose membership degrees are greater than or equal to \( \alpha \). Applying \( \alpha \)-cuts of fuzzy sets, the values of \( \hat{\bar{X}}_a \) and \( \hat{\bar{X}}_c \) are determined as follows:

\[ \hat{\bar{X}}_a = \bar{X}_a + \alpha(\bar{X}_b - \bar{X}_a) \]  
\[ \hat{\bar{X}}_c = \bar{X}_c - \alpha(\bar{X}_c - \bar{X}_b). \]

Similarly, \( \alpha \)-cut fuzzy \( \tilde{X} \) control chart limits based on ranges can be stated as follows:

\[ \hat{UCL}_a = (\hat{\bar{X}}_a, \bar{X}_b, \hat{\bar{X}}_c) + A_2(R_{aj}, R_{bj}, R_{cj}) = (\hat{\bar{X}}_a + A_2R_{aj} = A_2R_{aj} + A_2R_{bj} + A_2R_{cj}) = (\hat{UCL}_a, \hat{UCL}_b, \hat{UCL}_c), \]  
\[ \hat{CL}_a = (\hat{\bar{X}}_a, \bar{X}_b, \hat{\bar{X}}_c) = (\hat{CL}_a, \hat{CL}_b, \hat{CL}_c), \]  
\[ \hat{LCL}_a = (\hat{\bar{X}}_a, \bar{X}_b, \hat{\bar{X}}_c) - A_2(R_{aj}, R_{bj}, R_{cj}) = (\hat{\bar{X}}_a - A_2R_{aj} - A_2R_{bj} - A_2R_{cj}) = (\hat{LCL}_a, \hat{LCL}_b, \hat{LCL}_c). \]

where
\[ R_{al} = R_a + \alpha(R_b - R_a), \]  
\[ R_{cl} = R_c - \alpha(R_c - R_b). \]

\( \alpha \)-Cut fuzzy \( \tilde{X} \) control chart limits based on ranges (\( \hat{CL}, \hat{LCL} \) and \( \hat{UCL} \)) are shown in Fig. 2.

3.3. \( \alpha \)-Level fuzzy midrange for \( \alpha \)-cut fuzzy \( \tilde{X} \) control chart based on ranges

\( \alpha \)-Level fuzzy midrange is one of four transformation techniques used to determine the fuzzy control limits. These control limits are used to give a decision such as in-control or out-of-control for a process. In this study \( \alpha \)-level fuzzy midrange is used as the fuzzy transformation method while calculating the control limits

\[ \hat{UCL}_{\alpha \tilde{X}} = CL_{\alpha \tilde{X}} + A_2\left(\frac{R_{al} + R_{cl}}{2}\right), \]  
\[ \hat{CL}_{\alpha \tilde{X}} = f_{\alpha \tilde{X}}(\hat{CL}) = \frac{CL_{\tilde{X}} + CL_{\tilde{X}}}{2}, \]  
\[ \hat{LCL}_{\alpha \tilde{X}} = CL_{\alpha \tilde{X}} - A_2\left(\frac{R_{al} + R_{cl}}{2}\right). \]

![Fig. 2. \( \alpha \)-Cut fuzzy \( \tilde{X} \) control chart limits based on ranges (\( \hat{CL}, \hat{LCL} \) and \( \hat{UCL} \)).](image-url)
The definition of $\alpha$-level fuzzy midrange of sample $j$ for fuzzy $\bar{X}$ control chart is

$$ S^x_{mr\_j} = \frac{(\bar{X}_j + \bar{X}_j) + \alpha((\bar{X}_j + \bar{X}_j) - (\bar{X}_j + \bar{X}_j))]}{2} $$

Then, the condition of process control for each sample can be defined as

$$ \text{Process control} = \begin{cases} \text{in-control} & \text{for } LCL^x_{mr\_R} \leq S^x_{mr\_R} \leq UCL^x_{mr\_R} \\ \text{out-of control} & \text{for otherwise} \end{cases} $$

3.4. Fuzzy $R$ control chart

Shewhart’s traditional $R$ control chart is given by the following equations:

$$ UCL_R = D_4 \bar{R}, $$
$$ CL_R = \bar{R}, $$
$$ LCL_R = D_3 \bar{R}, $$

where $D_4$ and $D_3$ are control chart coefficients [11].

Fuzzy $R$ control chart limits can be obtained in a similar way to traditional $R$ control charts but they are represented by triangular fuzzy numbers as follows:

$$ UCL^x_R = D_4 \bar{R}^x = D_4(\bar{R}_a, \bar{R}_b, \bar{R}_c), $$
$$ CL^x_R = \bar{R}^x = (\bar{R}_a, \bar{R}_b, \bar{R}_c), $$
$$ LCL^x_R = D_3 \bar{R}^x = D_3(\bar{R}_a, \bar{R}_b, \bar{R}_c). $$

3.5. $\alpha$-Cut fuzzy $\bar{R}$ control chart

Control limits of $\alpha$-cut fuzzy $\bar{R}$ control chart can be stated as follows by using Eqs. (26)–(28)

$$ UCL^x_{\bar{R}} = D_4 \bar{R}^x = D_4(\bar{R}_a, \bar{R}_b, \bar{R}_c), $$
$$ CL^x_{\bar{R}} = \bar{R}^x = (\bar{R}_a, \bar{R}_b, \bar{R}_c), $$
$$ LCL^x_{\bar{R}} = D_3 \bar{R}^x = D_3(\bar{R}_a, \bar{R}_b, \bar{R}_c). $$

3.6. $\alpha$-Level fuzzy midrange for $\alpha$-cut fuzzy $\bar{R}$ control chart

Control limits of $\alpha$-level fuzzy midrange for $\alpha$-cut fuzzy $\bar{R}$ control chart can be calculated as follows:

$$ UCL^x_{mr\_R} = D_4 f^x_{mr\_R}(CL), $$
$$ CL^x_{mr\_R} = f^x_{mr\_R}(CL) = \frac{\bar{R}_a + \bar{R}_c}{2}, $$
$$ LCL^x_{mr\_R} = D_3 f^x_{mr\_R}(CL). $$

The definition of $\alpha$-level fuzzy midrange of sample $j$ for fuzzy $\bar{R}$ control chart is

$$ S^x_{mr\_R} = \frac{(R_{aj} + R_{dj}) + \alpha((R_{aj} + R_{dj}) - (R_{aj} + R_{bj}))}{2}. $$

The condition of process control for each sample can be defined as

$$ \text{Process control} = \begin{cases} \text{in-control} & \text{for } LCL^x_{mr\_R} \leq S^x_{mr\_R} \leq UCL^x_{mr\_R} \\ \text{out-of control} & \text{for otherwise} \end{cases} $$

4. Fuzzy $\bar{X}$ and $S$ control charts

The $R$ chart is a very popular control chart used to monitor the dispersion associated with a quality characteristic. Its simplicity of construction and maintenance makes the $R$ chart very commonly used and the range is a good measure of variation for small subgroup sizes. When the sample size increases ($n > 10$), the utility of the range measure as a measure of dispersion falls off and the standard deviation measure is preferred [13].
The traditional $X$ control chart based on standard deviation is given as follows:

\[
\begin{align*}
UCL_X &= \bar{X} + A_3 \bar{S}, \\
CL_X &= \bar{X}, \\
LCL_X &= \bar{X} - A_3 \bar{S},
\end{align*}
\]  

(37)

where $A_3$ is a control chart coefficient [11].

$\bar{S}$ is calculated by the following equations:

\[
\begin{align*}
S_j &= \sqrt{\frac{\sum_{i=1}^{n}(X_{ij} - \bar{X})^2}{n-1}}, \\
\bar{S} &= \frac{\sum_{j=1}^{m}S_j}{m},
\end{align*}
\]  

(38)

(39)

where $S_j$ is the standard deviation of sample $j$ and $\bar{S}$ is the average of $S_j$.

### 4.1. Fuzzy $\bar{X}$ control chart based on standard deviation

The fuzzy $\bar{S}_j$ is a standard deviation of sample $j$ and it is calculated as follows:

\[
\bar{S}_j = \sqrt{\frac{\sum_{i=1}^{n}(X_{ij} - \bar{X})^2}{n-1}}.
\]  

(40)

And the fuzzy average $\bar{\bar{S}}$ is calculated as

\[
\bar{\bar{S}} = \left(\frac{\sum_{j=1}^{m}S_{jh}}{m}, \frac{\sum_{j=1}^{m}S_{jb}}{m}, \frac{\sum_{j=1}^{m}S_{js}}{m}\right) = (\bar{s}_h, \bar{s}_b, \bar{s}_s),
\]  

(41)

The control limits of fuzzy $\bar{X}$ control chart based on standard deviation are obtained as follows:

\[
\begin{align*}
U\bar{C}L_x &= \bar{CL} + A_3 \bar{S} = (\bar{X}_d, \bar{X}_b, \bar{X}_c) + A_3(\bar{S}_d, \bar{S}_b, \bar{S}_c) = (\bar{X}_d + A_3 \bar{S}_d, \bar{X}_b + A_3 \bar{S}_b, \bar{X}_c + A_3 \bar{S}_c) = (L\bar{C}L_1, L\bar{C}L_2, L\bar{C}L_3), \\
\bar{CL}_x &= (\bar{X}_d, \bar{X}_b, \bar{X}_c) = (C\bar{L}_1, C\bar{L}_2, C\bar{L}_3), \\
L\bar{C}L_x &= \bar{CL} - A_3 \bar{S} = (\bar{X}_d, \bar{X}_b, \bar{X}_c) - A_3(\bar{S}_d, \bar{S}_b, \bar{S}_c) = (\bar{X}_d - A_3 \bar{S}_d, \bar{X}_b - A_3 \bar{S}_b, \bar{X}_c - A_3 \bar{S}_c) = (L\bar{C}L_1, L\bar{C}L_2, L\bar{C}L_3).
\end{align*}
\]  

(42)

(43)

(44)

### 4.2. $\alpha$-Cut fuzzy $\bar{X}$ control chart based on standard deviation

$\alpha$-Cut fuzzy $\bar{X}$ control chart limits based on standard deviation can be obtained as follows:

\[
\begin{align*}
UCL^\alpha_x &= (\bar{X}_d, \bar{X}_b, \bar{X}_c) + A_3(\bar{S}_d^\alpha, \bar{S}_b^\alpha, \bar{S}_c^\alpha) = (\bar{X}_d + A_3 \bar{S}_d^\alpha, \bar{X}_b + A_3 \bar{S}_b^\alpha, \bar{X}_c + A_3 \bar{S}_c^\alpha) = (U\bar{C}L^\alpha_1, U\bar{C}L^\alpha_2, U\bar{C}L^\alpha_3), \\
CL^\alpha_x &= (\bar{X}_d, \bar{X}_b, \bar{X}_c) = (C\bar{L}^\alpha_1, C\bar{L}^\alpha_2, C\bar{L}^\alpha_3), \\
L\bar{C}L^\alpha_x &= (\bar{X}_d, \bar{X}_b, \bar{X}_c) - A_3(\bar{S}_d^\alpha, \bar{S}_b^\alpha, \bar{S}_c^\alpha) = (\bar{X}_d - A_3 \bar{S}_d^\alpha, \bar{X}_b - A_3 \bar{S}_b^\alpha, \bar{X}_c - A_3 \bar{S}_c^\alpha) = (L\bar{C}L^\alpha_1, L\bar{C}L^\alpha_2, L\bar{C}L^\alpha_3),
\end{align*}
\]  

(45)

(46)

(47)

where

\[
\begin{align*}
\bar{S}_d^\alpha &= \bar{s}_d + \alpha(\bar{s}_b - \bar{s}_d), \\
\bar{S}_b^\alpha &= \bar{s}_b - \alpha(\bar{s}_b - \bar{s}_d), \\
\bar{S}_c^\alpha &= \bar{s}_c - \alpha(\bar{s}_b - \bar{s}_c).
\end{align*}
\]  

(48)

(49)

### 4.3. $\alpha$-Level fuzzy midrange for $\alpha$-cut fuzzy $\bar{X}$ control chart based on standard deviation

The control limits and center line for $\alpha$-cut fuzzy $\bar{X}$ control chart based on standard deviation using $\alpha$-level fuzzy mid-range are

\[
\begin{align*}
UCL^\alpha_{mr-\bar{X}} &= \alpha CL^\alpha_{mr-\bar{X}} + A_3 \left(\frac{\bar{S}_d^\alpha + \bar{S}_b^\alpha}{2}\right), \quad (50) \\
CL^\alpha_{mr-\bar{X}} &= f^\alpha_{mr-\bar{X}}(CL) = \frac{CL^\alpha_{C0} + CL^\alpha_{\bar{C}0}}{2}, \quad (51) \\
LCL^\alpha_{mr-\bar{X}} &= CL^\alpha_{mr-\bar{X}} - A_3 \left(\frac{\bar{S}_d^\alpha + \bar{S}_b^\alpha}{2}\right). \quad (52)
\end{align*}
\]
\[ \bar{S}_{\text{mr.}} = \text{is calculated by using Eq. (23) for each sample mean.} \]

The condition of process control for each sample can be defined as

\[
\text{Process control} = \begin{cases} 
\text{in-control} & \text{for } LCL_{\text{mr.}} \leq \bar{S}_{\text{mr.}} \leq UCL_{\text{mr.}} \\
\text{out-of control} & \text{for otherwise}
\end{cases}
\]

4.4. Fuzzy \( \bar{S} \) control chart

The traditional \( \bar{S} \) control chart is given by

\[
\begin{align*}
UCL_s &= B_4 \bar{S}, \\
CL_s &= \bar{S}, \\
LCL_s &= B_3 \bar{S},
\end{align*}
\]

where \( B_i \) is a control chart coefficient [11]. Fuzzy \( \bar{S} \) control chart limits can be obtained as follows:

\[
\begin{align*}
UCL_{\bar{S}} &= B_4 \bar{S} = B_4(\bar{S}_a, \bar{S}_b, \bar{S}_c), \\
CL_{\bar{S}} &= \bar{S} = (\bar{S}_a, \bar{S}_b, \bar{S}_c), \\
LCL_{\bar{S}} &= B_3 \bar{S} = B_3(\bar{S}_a, \bar{S}_b, \bar{S}_c).
\end{align*}
\]

4.5. \( \alpha \)-Cut fuzzy \( \bar{S} \) control chart

The control limits of \( \alpha \)-cut fuzzy \( \bar{S} \) control chart can be obtained as follows:

\[
\begin{align*}
UCL_{\alpha \bar{S}} &= B_4 \bar{S}^a = B_4(\bar{S}_a^a, \bar{S}_b^a, \bar{S}_c^a), \\
CL_{\alpha \bar{S}} &= \bar{S}^a = (\bar{S}_a^a, \bar{S}_b^a, \bar{S}_c^a), \\
LCL_{\alpha \bar{S}} &= B_3 \bar{S}^a = B_3(\bar{S}_a^a, \bar{S}_b^a, \bar{S}_c^a).
\end{align*}
\]

4.6. \( \alpha \)-Level fuzzy midrange for \( \alpha \)-cut fuzzy \( \bar{S} \) control chart

The control limits of \( \alpha \)-level fuzzy midrange for \( \alpha \)-cut fuzzy \( \bar{S} \) control chart can be obtained in a similar way to \( \alpha \)-cut fuzzy \( R \) control chart

\[
\begin{align*}
UCL_{\alpha \text{mr.} \bar{S}} &= B_4 \delta_{\text{mr.} \bar{S}}(CL), \\
CL_{\alpha \text{mr.} \bar{S}} &= f_{\text{mr.} \bar{S}}(CL) = \frac{\bar{S}_a^a + \bar{S}_c^a}{2}, \\
LCL_{\alpha \text{mr.} \bar{S}} &= B_3 \delta_{\text{mr.} \bar{S}}(CL).
\end{align*}
\]

The definition of \( \alpha \)-level fuzzy midrange of sample \( j \) for fuzzy \( \bar{S} \) control chart is

\[
\bar{S}_{\text{mr.} \bar{S}, j} = \frac{(S_{aj} + S_{cj}) + \alpha(S_{aj} - S_{cj}) - (S_{aj} - S_{cj})}{2}.
\]

The condition of process control for each sample can be defined as

\[
\text{Process control} = \begin{cases} 
\text{in-control} & \text{for } LCL_{\alpha \text{mr.} \bar{S}} \leq \bar{S}_{\text{mr.} \bar{S}, j} \leq UCL_{\alpha \text{mr.} \bar{S}} \\
\text{out-of control} & \text{for otherwise}
\end{cases}
\]

5. An application for fuzzy \( \bar{X} \) and \( \bar{R} \) control charts

The application was made on controlling piston inner diameters in compressors. Fifteen samples with a sample size of 5 (the total measurement number is \( 5 \times 15 = 75 \)) were taken from the production process by different operators. Quality experts evaluated each value by a fuzzy number because the variability of a measurement system includes operators and gauge variability. For example, the numeric observed value 5.73 can be measured by various operators between 5.71 and 5.76. The fuzzy measurement values and their arithmetic means and the fuzzy ranges for each sample and their arithmetic means are given in Table 1. Fuzzy control limits are calculated according to the equations given in the previous sections.

For \( n = 5 \), \( A_2 = 0.577 \). \( A_2 \) is obtained from the coefficients table for variable control charts.
5.1. Fuzzy $\bar{X}$ control chart based on ranges

$$UCL_\bar{X} = (\bar{X}_{a, b, c}) + A_2(\bar{R}_a, \bar{R}_b, \bar{R}_c) = (UCL_{\bar{X}}_1, UCL_{\bar{X}}_2, UCL_{\bar{X}}_3) = (5.54, 5.63, 5.72),$$
$$CL_\bar{X} = (\bar{X}_{a, b, c}) = (CL_{\bar{X}}_1, CL_{\bar{X}}_2, CL_{\bar{X}}_3) = (5.39, 5.39, 5.43),$$
$$LCL_\bar{X} = (\bar{X}_{a, b, c}) - A_2(\bar{R}_a, \bar{R}_b, \bar{R}_c) = (LCL_{\bar{X}}_1, LCL_{\bar{X}}_2, LCL_{\bar{X}}_3) = (5.06, 5.15, 5.24).$$

(64)

5.2. $\alpha$-Cut fuzzy $\bar{X}$ control chart based on ranges

$\alpha$-Cuts in control limits provide the ability of determining the tightness of the sampling process. $\alpha$-Level can be selected according to the nature of the production process. $\alpha$-Level was defined as 0.65 for piston inner diameter production process. $\bar{X}_{a, b, c}^{0.65}$ and $\bar{R}_{a, b, c}^{0.65}$ were calculated by using Eqs. (13), (14) and $\bar{X}_{a, b, c}^{0.65}$ and $\bar{R}_{a, b, c}^{0.65}$ were found by using Eqs. (18), (19)

$$\bar{X}_{a, b, c}^{0.65} = 5.38, \quad \bar{R}_{a, b, c}^{0.65} = 0.39,$$
$$\bar{X}_{c}^{0.65} = 5.40, \quad \bar{R}_{c}^{0.65} = 0.45,$$
$$UCL_{\bar{X}}^{0.65} = \left(\bar{X}_{a, b, c}^{0.65}, \bar{X}_{a, b, c}^{0.65}, \bar{X}_{a, b, c}^{0.65}\right) + A_2(\bar{R}_{a, b, c}^{0.65}, \bar{R}_{a, b, c}^{0.65}, \bar{R}_{a, b, c}^{0.65}) = (UCL_{\bar{X}}^{0.65}, UCL_{\bar{X}}^{0.65}, UCL_{\bar{X}}^{0.65}) = (5.61, 5.63, 5.66),$$
$$CL_{\bar{X}}^{0.65} = \left(\bar{X}_{a, b, c}^{0.65}, \bar{X}_{a, b, c}^{0.65}, \bar{X}_{a, b, c}^{0.65}\right) = (CL_{\bar{X}}^{0.65}, CL_{\bar{X}}^{0.65}, CL_{\bar{X}}^{0.65}) = (5.38, 5.39, 5.40),$$
$$LCL_{\bar{X}}^{0.65} = \left(\bar{X}_{a, b, c}^{0.65}, \bar{X}_{a, b, c}^{0.65}, \bar{X}_{a, b, c}^{0.65}\right) - A_2(\bar{R}_{a, b, c}^{0.65}, \bar{R}_{a, b, c}^{0.65}, \bar{R}_{a, b, c}^{0.65}) = (LCL_{\bar{X}}^{0.65}, LCL_{\bar{X}}^{0.65}, LCL_{\bar{X}}^{0.65}) = (5.12, 5.155, 18).$$

(65)

5.3. $\alpha$-Level fuzzy midrange for $\alpha$-cut fuzzy $\bar{X}$ control chart based on ranges

Control limits using $\alpha$-level fuzzy midrange for $\alpha$-cut fuzzy $\bar{X}$ control chart have been obtained as given in Eq. (66)

$$UCL_{\alpha-\bar{X}}^{0.65} = 5.63,$$
$$CL_{\alpha-\bar{X}}^{0.65} = f_\alpha^{\alpha}(CL_{\bar{X}}) = 5.39,$$
$$LCL_{\alpha-\bar{X}}^{0.65} = 5.15.$$  

(66)

5.4. Fuzzy $\bar{R}$ control chart

$$UCL_{\bar{R}} = D_4 \bar{R} = D_4(\bar{R}_a, \bar{R}_b, \bar{R}_c) = (0.70, 0.89, 1.08),$$
$$\tilde{C}_\bar{R} = \bar{R} = (\bar{R}_a, \bar{R}_b, \bar{R}_c) = (0.33, 0.42, 0.51),$$
$$LCL_{\bar{R}} = D_3 \bar{R} = D_3(\bar{R}_a, \bar{R}_b, \bar{R}_c) = (0.0, 0),$$

(67)

where $D_4 = 2.114$ for $n = 5$. $D_4$ and $D_3$ are obtained from the coefficients table for variable control charts.

Table 1

<table>
<thead>
<tr>
<th>Sample</th>
<th>$X_a$</th>
<th>$X_b$</th>
<th>$X_c$</th>
<th>$R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1-1</td>
<td>5.71</td>
<td>5.73</td>
<td>5.76</td>
<td>$R_{a1} = 0.43$</td>
</tr>
<tr>
<td>S1-2</td>
<td>5.50</td>
<td>5.57</td>
<td>5.62</td>
<td>$R_{a1} = 0.43$</td>
</tr>
<tr>
<td>S1-3</td>
<td>5.43</td>
<td>5.45</td>
<td>5.47</td>
<td>$R_{a1} = 0.48$</td>
</tr>
<tr>
<td>S1-4</td>
<td>5.20</td>
<td>5.25</td>
<td>5.28</td>
<td>$R_{a1} = 0.56$</td>
</tr>
<tr>
<td>S1-5</td>
<td>5.51</td>
<td>5.53</td>
<td>5.56</td>
<td>$R_{a1} = 0.43$</td>
</tr>
<tr>
<td>S2-1</td>
<td>5.41</td>
<td>5.43</td>
<td>5.45</td>
<td>$R_{a1} = 0.43$</td>
</tr>
<tr>
<td>S2-2</td>
<td>5.52</td>
<td>5.57</td>
<td>5.59</td>
<td>$R_{a1} = 0.43$</td>
</tr>
<tr>
<td>S2-3</td>
<td>5.25</td>
<td>5.29</td>
<td>5.31</td>
<td>$R_{a1} = 0.43$</td>
</tr>
<tr>
<td>S2-4</td>
<td>5.51</td>
<td>5.53</td>
<td>5.58</td>
<td>$R_{a1} = 0.43$</td>
</tr>
<tr>
<td>S2-5</td>
<td>5.65</td>
<td>5.69</td>
<td>5.71</td>
<td>$R_{a1} = 0.43$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>S15-1</td>
<td>5.25</td>
<td>5.29</td>
<td>5.33</td>
<td>$R_{a1} = 0.43$</td>
</tr>
<tr>
<td>S15-2</td>
<td>5.31</td>
<td>5.33</td>
<td>5.39</td>
<td>$R_{a1} = 0.43$</td>
</tr>
<tr>
<td>S15-3</td>
<td>5.00</td>
<td>5.13</td>
<td>5.19</td>
<td>$R_{a1} = 0.43$</td>
</tr>
<tr>
<td>S15-4</td>
<td>5.20</td>
<td>5.25</td>
<td>5.29</td>
<td>$R_{a1} = 0.43$</td>
</tr>
<tr>
<td>S15-5</td>
<td>5.31</td>
<td>5.33</td>
<td>5.39</td>
<td>$R_{a1} = 0.43$</td>
</tr>
</tbody>
</table>

$\bar{X}_a = 5.35$  $\bar{X}_b = 5.39$  $\bar{X}_c = 5.43$  $R_a = 0.33$  $R_b = 0.42$  $R_c = 0.51$

Sevil Senturk, Nihal Erginel / Information Sciences 179 (2009) 1542–1551
5.5. $x$-Cut fuzzy $R$ control chart

$$UCR_{0.65}^{0.65} = D_4 R_{0.65}^{0.65} = D_4 (R_{a.65}^{0.65}, R_{b.65}^{0.65}, R_{c.65}^{0.65}) = (0.82, 0.89, 0.95),$$

$$LBR_{0.65}^{0.65} = \frac{R_{0.65}^{0.65}}{C_0 r_{0.65}} = (R_{a.65}^{0.65}, R_{b.65}^{0.65}, R_{c.65}^{0.65}) = (0.39, 0.42, 0.45),$$

$$LBR_{0.65}^{0.65} = D_3 (R_{a.65}^{0.65}, R_{b.65}^{0.65}, R_{c.65}^{0.65}) = (0, 0, 0).$$

(68)

5.6. $x$-Level fuzzy midrange for $x$-cut fuzzy $R$ control chart

Control limits using $x$-level fuzzy midrange for $x$-cut fuzzy $R$ control chart have been obtained as given in Eq. (69)

$$UCL_{mr_{x-R}}^{0.65} = 0.89,$$

$$CL_{mr_{x-R}}^{0.65} = f_{mr_{x-R}}^{0.65}(CL) = 0.42,$$

$$LCL_{mr_{x-R}}^{0.65} = 0.$$

$S_{mr_{x-R}}^{x}$ and $S_{mr_{x-R}}^{x}$ have been calculated for 15 samples by using Eqs. (23)–(35), respectively, and are given in Table 2. As shown in Table 2, the process was in control with respect to $S_{mr_{x-R}}^{x}$ and $S_{mr_{x-R}}^{x}$ for each sample. So, these control limits can be used to control the process.

After determining if the process was in control, nine samples with a sample size of 5 were taken from the process for monitoring the process. The values of samples are given in Table 3.

<table>
<thead>
<tr>
<th>Sample no.</th>
<th>$S_{mr_{x-R}}^{x}$</th>
<th>$5.15 \leq S_{mr_{x-R}}^{x} \leq 5.63$</th>
<th>$S_{mr_{x-R}}^{x}$</th>
<th>$0 \leq S_{mr_{x-R}}^{x} \leq 0.89$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5.51</td>
<td>In control</td>
<td>0.49</td>
<td>In control</td>
</tr>
<tr>
<td>2</td>
<td>5.50</td>
<td>In control</td>
<td>0.40</td>
<td>In control</td>
</tr>
<tr>
<td>3</td>
<td>5.38</td>
<td>In control</td>
<td>0.63</td>
<td>In control</td>
</tr>
<tr>
<td>4</td>
<td>5.41</td>
<td>In control</td>
<td>0.39</td>
<td>In control</td>
</tr>
<tr>
<td>5</td>
<td>5.23</td>
<td>In control</td>
<td>0.59</td>
<td>In control</td>
</tr>
<tr>
<td>6</td>
<td>5.28</td>
<td>In control</td>
<td>0.38</td>
<td>In control</td>
</tr>
<tr>
<td>7</td>
<td>5.45</td>
<td>In control</td>
<td>0.49</td>
<td>In control</td>
</tr>
<tr>
<td>8</td>
<td>5.48</td>
<td>In control</td>
<td>0.52</td>
<td>In control</td>
</tr>
<tr>
<td>9</td>
<td>5.28</td>
<td>In control</td>
<td>0.15</td>
<td>In control</td>
</tr>
<tr>
<td>10</td>
<td>5.49</td>
<td>In control</td>
<td>0.37</td>
<td>In control</td>
</tr>
<tr>
<td>11</td>
<td>5.53</td>
<td>In control</td>
<td>0.40</td>
<td>In control</td>
</tr>
<tr>
<td>12</td>
<td>5.40</td>
<td>In control</td>
<td>0.60</td>
<td>In control</td>
</tr>
<tr>
<td>13</td>
<td>5.22</td>
<td>In control</td>
<td>0.09</td>
<td>In control</td>
</tr>
<tr>
<td>14</td>
<td>5.35</td>
<td>In control</td>
<td>0.39</td>
<td>In control</td>
</tr>
<tr>
<td>15</td>
<td>5.30</td>
<td>In control</td>
<td>0.41</td>
<td>In control</td>
</tr>
</tbody>
</table>

Table 2

Control limits using $x$-level fuzzy midrange for $x$-cut fuzzy $R$ control chart based on ranges and $x$-level fuzzy midrange for $x$-cut fuzzy $R$ control chart.

<table>
<thead>
<tr>
<th>Sample no.</th>
<th>$S_{mr_{x}}^{x}$</th>
<th>$5.15 \leq S_{mr_{x}}^{x} \leq 5.63$</th>
<th>$S_{mr_{x}}^{x}$</th>
<th>$0 \leq S_{mr_{x}}^{x} \leq 0.89$</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1-1</td>
<td>5.43</td>
<td>5.49</td>
<td>5.55</td>
<td>$R_a = 0.34$</td>
</tr>
<tr>
<td>S1-2</td>
<td>5.30</td>
<td>5.35</td>
<td>5.46</td>
<td>$R_a = 0.34$</td>
</tr>
<tr>
<td>S1-3</td>
<td>5.51</td>
<td>5.55</td>
<td>5.63</td>
<td>$R_a = 0.54$</td>
</tr>
<tr>
<td>S1-4</td>
<td>5.32</td>
<td>5.35</td>
<td>5.38</td>
<td>$R_a = 0.60$</td>
</tr>
<tr>
<td>S1-5</td>
<td>5.80</td>
<td>5.89</td>
<td>5.90</td>
<td></td>
</tr>
<tr>
<td>S2-1</td>
<td>5.29</td>
<td>5.33</td>
<td>5.36</td>
<td></td>
</tr>
<tr>
<td>S2-2</td>
<td>5.24</td>
<td>5.29</td>
<td>5.33</td>
<td>$R_a = 0.287$</td>
</tr>
<tr>
<td>S2-3</td>
<td>5.45</td>
<td>5.47</td>
<td>5.54</td>
<td>$R_a = 0.36$</td>
</tr>
<tr>
<td>S2-4</td>
<td>5.61</td>
<td>5.65</td>
<td>5.70</td>
<td>$R_a = 0.46$</td>
</tr>
<tr>
<td>S2-5</td>
<td>5.41</td>
<td>5.45</td>
<td>5.47</td>
<td></td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>S9-1</td>
<td>5.51</td>
<td>5.57</td>
<td>5.60</td>
<td></td>
</tr>
<tr>
<td>S9-2</td>
<td>5.42</td>
<td>5.43</td>
<td>5.46</td>
<td>$R_a = 0.21$</td>
</tr>
<tr>
<td>S9-3</td>
<td>5.21</td>
<td>5.25</td>
<td>5.30</td>
<td>$R_a = 0.32$</td>
</tr>
<tr>
<td>S9-4</td>
<td>5.29</td>
<td>5.33</td>
<td>5.35</td>
<td>$R_a = 0.39$</td>
</tr>
<tr>
<td>S9-5</td>
<td>5.22</td>
<td>5.25</td>
<td>5.29</td>
<td></td>
</tr>
</tbody>
</table>

Table 3

Nine samples with a sample size of 5 for piston inner diameter.
fuzzy midrange for $\alpha$-cut fuzzy $\bar{X}$ control chart based on ranges. It meant that the process was out of control. The assignable causes should be determined and some corrective actions should be made.

6. Conclusions

With this study, it is shown that the fuzzy set theory is applicable on the traditional variable control charts. Since the observations that are close to the control limits may cause false alarms with traditional control charts, fuzzy observations, and fuzzy control charts can provide more flexibility for controlling a process. Fuzzy control charts for attribute data were introduced by several papers. Fuzzy variable control charts based on fuzzy rules exist in the literature. $\alpha$-cuts fuzzy $\bar{X}$ and $\bar{R}$ and $\bar{X}$ and $\bar{R}$ control charts are first time proposed in this paper. $\alpha$-cuts fuzzy $\bar{X}$ and $\bar{R}$ and $\bar{X}$ and $\bar{S}$ control charts are developed together with $\alpha$-level fuzzy midrange transformation techniques. For further research, trapezoidal fuzzy numbers can be used together with different transformation techniques and the obtained results can be compared with the ones presented in this study.

References