RESONANCE-LIKE PHENOMENA IN ARRAYS OF RÖSSLER OSCILLATORS UNDER CORRELATED PARAMETRIC FLUCTUATIONS

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The behavior of diffusively coupled Rössler oscillators parametrically perturbed with an Ornstein-Uhlenbeck noise is analyzed in terms of the degree of synchronization between the cells. A resonance-like behavior is found as a function of the noise correlation time, instead of the noise intensity as it occurs in the typical stochastic resonance. A power law scaling between the “optimum” correlation time with regard to synchronization and the deterministic time scale of the oscillators has been obtained, with an exponent that depends on the coupling strength.

1. Introduction

In the last decade, the phenomenon of Stochastic Resonance (SR) has awakened the interest of many researchers. Classical stochastic resonance occurs as a result of the interaction between a noisy signal and a periodic forcing on a bistable system, in such a way that the response of the system is enhanced for a certain value of the noise intensity (see [Gammaitoni et al., 1998] for an extensive review and references therein). In particular, the presence of stochastic resonance has been studied in a great number of models including spatiotemporal systems [Lindner et al., 1995, 1996; von Haefen et al., 2000]. Most of the work on stochastic resonance has considered systems with additive white Gaussian noise, without deeply considering the effect of parametric noise and colored noise. The effect of additive colored noise in periodically driven overdamped systems [Gammaitoni et al., 1989; Hänggi et al., 1993] has shown that the correlation time suppresses monotonically the stochastic resonance. However, in the last years, systems perturbed with additive and multiplicative colored noise have been considered. For different nonlinear dynamical systems with limit cycle behavior and perturbed parametrically with colored noise, Cabrera and De la Rubia [1997] and Cabrera et al. [1999] found a resonance-like behavior as a function of the correlation time of the noise.

Recently, studies on the behavior of arrays of nonlinear dynamical systems without periodic

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external forcing but submitted to a weak colored noise have been performed. They show a resonance-like effect in the degree of synchronization between the chaotic cells of the array as a function of the noise correlation time, instead of the noise intensity [Lorenzo & Pérez-Muñuzuri, 1999, 2000, 2001; Pérez-Muñuzuri & Lorenzo, 1999]. For given coupling and noise strength, the noise enables the system to explore the region of stable stationary dynamics and yield, on the average, an improvement of the synchronization. When this occurs, the transverse Lyapunov exponents become negative as expected.

Nevertheless, also the opposite behavior (i.e. a worsening of the synchronization) can be found for a given “resonant” value of the time correlation. In this case, even for large coupling between the cells, the colored noise perturbs the system to the extent that although the intrinsic dynamical properties of the oscillators are conserved, the synchrony of the array becomes poorer. The correlation time of the parametric random perturbation acts thus as a tuning parameter which controls (in a statistical sense) the synchronization of the system. The values of the correlation time for which the array attains its minimum degree of synchronization increase monotonically with the deterministic periods of the cells. An explicit relationship between the two magnitudes is thus to be expected.

In this paper we study an array of Rössler oscillators working in the quasiperiodic or chaotic regimes according to the value of the bifurcation parameter $c$. We moreover show that there is a power-law relationship between the stochastic and the deterministic time scales of the system and that the exponent turns out to be proportional to the coupling strength.

2. Model

The one-dimensional array consisting of diffusively coupled cells of the Rössler type is given by

$$\begin{align*}
\dot{x}_j &= P(-y_j - z_j) \\
\dot{y}_j &= P(x_j + ay_j) + D(y_{j+1} + y_{j-1} - 2y_j) \\
\dot{z}_j &= P(b + z_j(x_j - |c + \xi_j(t)|))
\end{align*}$$

(1)

where $\xi_j(t)$ is a time-correlated noise which is assumed to be uncorrelated in space ($\xi_i(t) \neq \xi_j(t)$ $\forall i, j = 1, \ldots, N$). Here $a = b = 0.1$, and $c$ is regarded as the bifurcation parameter. As $c$ is increased, there is a sequence of period-doubling bifurcations from a simple, period-one oscillation. Chaos develops at the accumulation point of the period-doubling sequence, just above $c_{\infty} = 9$. Since we are interested to obtain a resonant behavior between the time scale of the attractor and the time correlation of the colored noise, the time scale of the diffusionless system was scaled ($P = 10$) in order to use reasonable values of the correlation time. Going back to Eq. (1), $D$ accounts for the diffusive coupling between cells, $j$ runs from 1 to $N$ (number of cells in the array), and $\xi(t)$ is a colored Gaussian noise of zero mean and correlation function

$$\langle \xi(t)\xi(t') \rangle = \frac{A}{\tau} \exp \left(-\frac{|t - t'|}{\tau}\right),$$

(2)

where $\tau$ is the correlation time, $A$ the noise intensity and $\sigma = \sqrt{A/\tau}$ the noise dispersion. The absolute value in Eq. (1) indicates that no negative values of $c + \xi_j^m(t)$ are allowed since then the Rössler system would become unstable. To avoid these negative values the colored noise intensity is appropriately calculated in order to keep its Gaussian character and correlation [García-Ojalvo & Sancho, 1999].

Equation (1) was numerically integrated using an explicit Euler method with a time step of $10^{-4}$. Besides, free ends were considered for the $y$ variable. The exponentially correlated noise was numerically calculated as suggested in [García-Ojalvo & Sancho, 1999]. Random initial conditions for all the variables were assigned to each cell in the array.

In order to characterize the degree of synchronization between the cells of the array, two different definitions have been used here:

1. The function $K$ defined as a time- and space-averaged difference between the simultaneous values of the variables $(x, y, z)$ at two neighboring cells in the array [Lorenzo & Pérez-Muñuzuri, 1999],

$$K = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \left( \frac{1}{N-1} \sum_{j=2}^{N} \|u_j^t - u_j^t\| \right)$$

(3)

where $u = (x, y, z)$ and $\| \cdot \|$ denotes the Euclidean distance. This function is positive definite and vanishes when all the cells in the array are globally synchronized. We shall call $K_0$ the value of $K$ in the absence of noise.

2. The minimum of the similarity function $S$ is defined as a time-average of the delayed difference between the values of a given variable at
neighboring cells (with the delay \( t_d \) regarded as an optimization parameter) [Rosenblum et al., 1997],

\[
S^2(t_d) = \frac{\langle [x_j(t + t_d) - x_{j+1}(t)]^2 \rangle}{\langle x_j^2(t) \rangle \langle x_{j+1}^2(t) \rangle} \tag{4}
\]

averaged again over the whole array. If the variables at neighboring cells are independent, their difference is of the same order as the variables themselves, namely \( S(t_d) \approx 1 \forall t_d \). In the case of complete synchronization \( S(t_d) \) attains its minimum for \( t_d = 0 \), while for phase synchronization \( S(t_d) \) has a minimum for a nonzero time delay \( t_d \).

As in the case of \( K \), we shall call \( \min(S_0(t_d)) \) the value of \( \min(S(t_d)) \) corresponding to the noise-free case.

For each value of \( \tau \) and \( \sigma \) used in our numerical simulations, roughly 20 realizations of the noise were generated to reduce statistical dispersion.

3. Results

Figure 1 shows the effect of time-correlated noise on the synchronization of a chain of \( N = 4 \) chaotic Rössler oscillators. The most remarkable effect is the occurrence, for some intermediate value \( \tau_R \) of the correlation time and for constant noise dispersion \( \sigma \), of a maximum value of both measures \( K - K_0 \) and \( \min(S(t_d)) - \min(S_0(t_d)) \). The positive values of these measures indicate that the synchronization worsens around \( \tau_R \), whereas the fact that this resonance-like effect appears in both measures suggests that it does not depend on the measuring tools. This seems to be the signature of a resonant behavior between the intrinsic time scale of the Rössler oscillators and the noise correlation time, similar to the effect found in arrays of Lorenz and Chua oscillators [Lorenzo & Pérez-Muñuzuri, 1999, 2000, 2001; Pérez-Muñuzuri & Lorenzo, 1999]. Thus, the correlation time of the noise acts as a tuning parameter which controls the degree of synchronization of the array.

Note from Fig. 1(a) that for \( c = 10 \) the measure \( K - K_0 \) becomes negative for both small and large values of \( \tau \), indicating that an overall improvement of the synchronization between chaotic cells can be achieved by using colored noise. Besides, while \( K_0(c = 4) \ll K_0(c = 10) \), after forcing the cells with colored noise the overall synchronization between chaotic cells (\( c = 10 \)) is much better than for quasiperiodic ones (\( c = 4 \)). On the other hand, the difference between the minima of the similarity functions \( S \) and \( S_0 \) [Fig. 1(b)] is positive for any value of \( \tau \), so this measure is not sensible to the overall improvement in phase synchronization. For \( \tau \) not extremely small it does however indicate (like
the K-measure) the fact that a better synchronization can be achieved in the chaotic regime (c = 10) than in the quasiperiodic one (c = 4).

The two limit cases $\tau \to 0$ and $\tau \to \infty$ can be easily explained. For $\tau \to 0$ and finite $\sigma^2$, the white Gaussian noise limit with zero noise intensity is recovered. Noise fluctuates so fast compared with the time scales of the Rössler oscillator that in average the cell behaves as in the noise-free situation, namely $\xi = \langle \xi \rangle = 0$. Naturally (as Fig. 1(b) shows), the phase synchronization is more readily achieved in the quasiperiodic regime than in the chaotic one. For $\tau \to \infty$ the noise becomes quasistatic and stops interfering with the synchronization process.

Analyzing the behavior of $\tau_R$ as a function of the bifurcation parameter $c$ (see Fig. 2) we have observed that it closely follows the $c$-dependence of $T_z$, the lapse of time during which $z(t) > 0$. This period was calculated for the middle oscillator of the array as a mean value over all the realizations of the noise. On the other hand, the mean period $T$ of the oscillator smoothly increases with $c$. Both periods were found not to depend on the value of the coupling strength $D$.

Regarding the dependence of $\tau_R$ on the diffusive coupling $D$, two well differentiated zones can be identified in Fig. 2: one for values of $c$ within the period-doubling cascade, where $\tau_R$ decreases with $c$ at a rate that is an increasing function of $D$, and another above the accumulation point of the period-doubling sequence, where $\tau_R$ remains approximately the same for any value of $c$. Small differences with the coupling strength were observed in the latter case, probably because of the limited number of realizations used; nonetheless, these differences were less pronounced than in the arrays of Lorenz cells that we have considered in a previous work [Lorenzo & Pérez-Muñuzuri, 1999].

Restricting to the behavior within the period-doubling region and recalling that $T$ increases monotonically with $c$, we have sought for a power-law relationship between $\tau_R$ and the oscillator period $T$. Following Cabrera and De la Rubia [1997], we fitted $\log(\tau_R)$ versus $\log(T - T_H)$, $T_H$ being the minimum period of the limit cycle, occurring in a nongeneric codimension-2 bifurcation that can be defined for the Rössler oscillator. Figure 3 shows an example of such fitting for $D = 0.4$. Thus, within a range of values of the bifurcation parameter $c$ lying inside the period-doubling sequence, it is possible to obtain a straight line whose slope $\alpha$ has been found to depend linearly on the coupling strength $D$, as it is shown in Fig. 4.

The following relationship summarizes the obtained results:

$$
\tau_R \approx \begin{cases} 
(T - T_H)^{-\alpha} & c < c_{\infty} \\
\beta & c \geq c_{\infty},
\end{cases}
$$

with $\alpha = (0.91 \pm 0.07)D + (0.58 \pm 0.05)$ and $\beta \approx 0.2$.

Equation (5) seems to indicate that in an array of cells performing in a periodic or quasiperiodic regime, the value of $\tau$ for which the minimum degree of synchronization is attained for the whole
array varies with the period $T$ of the isolated cell in a way that is characterized by a single exponent, and that moreover this exponent depends linearly on the diffusion coefficient $D$. It is worth remarking on the fact that for the limit case $D = 0$ (uncoupled cells) we recover the same exponent ($\alpha \approx 1/2$) that Cabrera et al. [1999] obtained for different nonlinear dynamical systems with limit cycle behavior, and perturbed parametrically with a colored noise. For the second region, $c \geq c_{\infty}$, $\tau_R$ was found not to depend on the coupling strength $D$.

We do not have yet a definitive explanation for this power law and in particular for the exponent obtained with $D = 0$. In the previously mentioned works this exponent is related to a noise-mediated coherence enhancement, while here it is linked with a synchronization worsening. The finding of the same exponent describing the coupling of two characteristic time scales in such different phenomena may be an indication of an universality shared by systems perturbed with parametric correlated noise.

4. Conclusions

In summary, we have presented numerical evidence of a worsening of the synchronization between diffusively coupled Rössler oscillators around a specific correlation time $\tau_R$ of a multiplicative Ornstein-Uhlenbeck process. We have also found a power-law relationship between this value $\tau_R$ and the intrinsic period $T$ of the Rössler oscillators when the cells are in the quasiperiodic regime. Moreover, the exponent in this relation has been shown to depend linearly on the diffusion coefficient $D$, and for the limit of uncoupled cells we reproduce the results by Cabrera et al. [1999]. On the other hand, no power-law behavior was obtained for coupled cells in the chaotic regime.

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References


