A classification of finite \(\{0,1,2\}\)-inversive planes

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Abstract

In the present paper we shall classify the finite circular spaces \(\mathcal{C}\) with the property that for all points \(P\) of \(\mathcal{C}\) the residual space \(\mathcal{C}_P\) is a \(\{0,1,2\}\)-semi-affine plane. © 1999 Elsevier Science B.V. All rights reserved.

1. Introduction

A linear space is a geometry of points and lines satisfying the following two axioms:

(L1) Any two points are joined by a unique line.
(L2) Any line is incident with at least two points; there are at least two lines.

A linear space is called finite, if it contains finitely many points and lines. In such a case by degree of a line (resp. point) we mean the number of points on it (resp. lines through it). If \(n+1\) denotes the maximum between point-degrees then \(n\) is called the order of the linear space. Examples of linear spaces are affine and projective spaces.

A circular space is a geometry of points and circles fulfilling the following conditions:

(C1) Any three points are incident with a unique circle.
(C2) Any circle is incident with at least three points; there are at least two circles.

A finite circular space is a circular space with finitely many points and circles. In such a case by degree of a circle (resp. point) we mean the number of points on it (resp. circles through it).

Let \(\mathcal{C}\) be a circular space, and let \(P\) be a point of \(\mathcal{C}\). The residual space \(\mathcal{C}_P\) is the geometry of points and lines whose points are the points of \(\mathcal{C}\) except \(P\) and whose
lines are the circles of \( C \) through \( P \), except \( P \). The incidence relation of \( C_P \) is induced by \( C \). In view of Axioms (C1) and (C2) \( C_P \) is a linear space. If \( C \) and hence also \( C_P \) is finite, by order of \( C \) we mean the maximum between the orders of the linear spaces \( C_P \).

A circular space \( C \) is called degenerate if there is a circle containing all points but one and hence all other circles have degree 3.

The most important class of circular spaces are the so-called inversive planes. An inversive plane is a circular space with the property that for all points \( P \) of \( C \) the residual space \( C_P \) is an affine plane. The residual spaces of an inversive plane are called internal planes.

In the study of circular spaces the following question is of particular interest.

Given a family \( F \) of linear spaces classify all circular spaces \( C \) such that for all points \( P \) of \( C \) the residual space \( C_P \) belongs to \( F \).

A first result in this direction has been obtained by Buekenhout and Metz [3]. In order to state their results we need the notion of semi-affine planes.

Let \( \mathcal{H} \) be a set of non-negative integers. A linear space \( \mathcal{L} \) is called an \( \mathcal{H} \)-semi-affine plane if for any non-incident point-line-pair \((P, l)\) the number of lines through \( P \) disjoint to \( l \) belongs to \( \mathcal{H} \). The \( \{0\}\)-semi-affine planes and the \( \{1\}\)-semi-affine planes are exactly the generalized projective and the affine planes, respectively. The \( \{0,1\}\)-semi-affine planes are called semi-affine planes (see [4]).

The theorem of Buekenhout–Metz reads as follows.

**Theorem 1.1.** Let \( C \) be a finite circular space such that any residual space of \( C \) is a semi-affine plane. Then one of the following cases occurs:

1.1. \( C \) is an inversive plane.

1.2. \( C \) is either a 3-(8,4,1) or a 3-(22,6,1) design (\( \{0\}\)-inversive).

1.3. \( C \) is a 3-(8,4,1) or a 3-(22,6,1) design with one point deleted.

1.4. \( C \) has six points 1,2,3,4,5,6 with 1234, 1256 and 3456 as the only circles with more than three points.

1.5. \( C \) is the degenerate circular space on four or five points.

**Remark.** Buekenhout and Metz further obtain a 3-(112,12,1) design (with possibly one point deleted). But this would be the extension of a projective plane of order 10 and from Lam et al. [7] its known that such a plane does not exist.

Using the previous result Buekenhout and Metz [3] could also classify all finite circular spaces such that any two circles have at least one point in common.

Generalizing the \( \mathcal{H} \)-semi-affine planes Olanda [10] introduced \( \mathcal{H} \)-inversive planes as follows. Given a set \( \mathcal{H} \) of non-negative integers, a circular space \( C \) is called an \( \mathcal{H} \)-inversive plane, if for each point \( P \) of \( C \) the residual space \( C_P \) is \( \mathcal{H} \)-semi-affine.

So the theorem of Buekenhout and Metz is a classification of finite \( \{0,1\}\)-inversive planes. In his paper Olanda classifies \( \{1,2\}\)-inversive planes. His result reads as follows.
Theorem 1.2. Let \( \mathcal{C} \) be a \( \{1,2\} \) inversive plane of order \( n > 5 \). Then one of the following cases occurs.

1. \( \mathcal{C} \) is an inversive plane.
2. \( \mathcal{C} \) is an inversive plane of order \( n \) where one point has been deleted (punctured inversive plane).

Another result on \( \mathcal{H} \)-inversive planes has been obtained by Biondi [2], in the case \( \mathcal{H} = \{1,2,3\} \).

Furthermore an important result, related to the previous question, has been obtained by Thas [11] on inversive planes of odd order \( n \) and it is as follows.

Theorem 1.3. Let \( \mathcal{C} \) be an inversive plane of odd order \( n \). If for at least one point \( P \) of \( \mathcal{C} \) the internal plane \( \mathcal{C}_P \) is desarguesian then \( \mathcal{C} \) is classical (miquelian).

In the present paper we shall classify the \( \{0,1,2\} \)-inversive planes. Our main results read as follows.

Theorem 1.4. Let \( \mathcal{C} \) be a \( \{0,2\} \)-inversive plane that is not \( \{0\} \)-inversive. Then \( \mathcal{C} \) is one of the following.

1. \( \mathcal{C} \) has six points. All circles are of degree 3 (\( \{2\} \)-inversive).
2. \( \mathcal{C} \) has 16 points and every residual space is obtained from a projective plane of order 4 by deleting a 6-arc.
3. \( \mathcal{C} \) has 14 points, and every residual space is obtained from a projective plane of order 4 by deleting two lines with all points on these lines except their intersection point.
4. \( \mathcal{C} \) is the degenerate circular space on six points.
5. \( \mathcal{C} \) has 14 points, and the residual spaces are either obtained from a projective plane of order 4 by deleting two lines with all of its points on these lines except their intersection point or from the extended Nwankpa-Shrikhande plane.

Proposition 1.5. Let \( \mathcal{C} \) be a \( \{1,2\} \)-inversive plane of order \( n \leq 5 \) that is not an inversive plane nor \( \{2\} \)-inversive. Then one of the following cases occurs:

1. \( \mathcal{C} \) is a punctured inversive plane of order \( n \).
2. \( \mathcal{C} \) has six points, and the residual spaces are either obtained from the complete graph \( K_5 \) or from a linear space with five points, one line of degree 3 and 7 lines of degree two.

Observe that the previous result completes the classification of \( \{1,2\} \)-inversive planes.

Theorem 1.6. Let \( \mathcal{C} \) be a \( \{0,1,2\} \)-inversive plane. Then one of the following cases occurs.

1. \( \mathcal{C} \) is \( \{0,1\} \)-inversive, \( \{0,2\} \)-inversive or \( \{1,2\} \)-inversive.
6.2. There are some integers $n$ and $k$ with $n$ even and $2 \leq k \leq n$ such that any residual space of $\mathcal{C}$ is obtained from a projective plane of order $n$ by deleting a $k$-arc. Furthermore either $n = 4$, $k = 2$ and $\mathcal{C}$ is a $3$-$\begin{pmatrix} 22, 6, 1 \end{pmatrix}$ design with two points deleted or $k(k - 1)(k - 2) = 12n(n + 1)$.

6.3. $\mathcal{C}$ has 26 points, and every residual space is obtained from a projective plane of order 5 by deleting one line $l$ with all of its points except one and one point outside of $l$.

6.4. $\mathcal{C}$ has either 8, 22 or 112 points, and every residual space is obtained from an affine plane of order 3, 5 or 11, respectively, by deleting one line $l$ with all of its points except one.

6.5. Let $\mathcal{C}'$ be the $3$-$\begin{pmatrix} 8, 4, 1 \end{pmatrix}$ design, and let $x$ and $y$ be two disjoint circles of degree 4. Let $\mathcal{C}$ be the circular space constructed from $\mathcal{C}'$ by replacing each of the circles $x$ and $y$ by four circles of degree 3.

6.6. $\mathcal{C}$ is the extension of the Nwankpa-plane (see Definition 4.1).

6.7. $\mathcal{C}$ has 17 points, and the residual spaces are either obtained from a projective plane of order 4 by deleting a 5-arc or from an affine plane of order 4.

6.8. $\mathcal{C}$ has 22 points, and the residual spaces are either obtained from a projective plane of order 4 or from an affine plane of order 5 by deleting one line $l$ with all of its points except one.

6.9. $\mathcal{C}$ has 18 points, and the residual spaces are either obtained from an affine plane of order 4 with one point at infinity or from a projective plane of order 4 by deleting a 4-arc.

6.10. $\mathcal{C}$ has eight points, and the residual spaces are either obtained from a projective plane of order 2 or from the quasi-Fano plane.

6.11. $\mathcal{C}$ has seven points, and the residual spaces are either obtained from a projective plane of order 2 with one point deleted or from the quasi-Fano plane with one point of degree 3 deleted.

6.12. $\mathcal{C}$ has seven points, and the residual spaces are either obtained from an affine plane of order 3 by deleting one line with all of its points or from the quasi-Fano plane with one point of degree 3 deleted.

6.13. $\mathcal{C}$ has eight points, and the residual spaces are either obtained from an affine plane of order 3 by deleting one line with all of its points except one, or from the quasi-Fano plane.

6.14. $\mathcal{C}$ has eight points and the residual spaces are obtained from the Fano, plane or form an affine plane of order 3 by deleting one line with all of its points except one, or from the quasi-Fano plane.

6.15. $\mathcal{C}$ has six points, two circles of degree 4 having two points in common and all the other circles of degree 3.

Observe that the additional condition that $\mathcal{C}$ has at least 27 points reduces the above classification to the cases 1.1, 2.2, 6.2 and one of the cases in 6.4.

This paper is organized as follows. Section 2 is devoted to a survey of the classification of $\{0, 1, 2\}$-semi-affine planes. In Section 3 we shall classify finite $\{0, 1, 2\}$-inversive
planes admitting at least two residual spaces with different combinatorial properties. Finally, in Section 4 we shall prove Theorem 1.6.

2. $\mathcal{H}$-semi-affine planes

In the present section we shall survey the classification of the $\{0,1,2\}$-semi-affine planes. An $\mathcal{H}$-semi-affine plane $\mathcal{L}$ is called proper if for any value $h \in \mathcal{H}$ there is a non-incident point-line-pair $(P,l)$ of $\mathcal{L}$ such that there are exactly $h$ lines through $P$ disjoint from $l$. The linear spaces appearing in the classification are projective and affine planes, near pencils, complete graphs, $(h,k)$-crosses, the quasi-Fano plane, the plane of Nwankpa-Shrikhande and linear spaces derived from these structures.

We give a brief definition of some of them. The $(h,k)$-cross is the linear space whose points are on two lines of degree $h$ and $k$ that intersect on a point and all the other lines are of degree 2. In particular a $(2,n+1)$-cross is called a near-pencil of order $n$.

The Fano-quasi plane is the linear space obtained from the projective plane of order 2 by replacing one of its lines by three lines of degree 2.

The Nwankpa plane $N$ is the unique linear space on 11 points having one line of degree 5 and 15 lines of degree 3 which is obtained from the complete graph on six vertices $K_6$ by adding five points corresponding to the partitions of $K_6$ into three lines of degree 2. The six points of degree 5 can be seen as the real points of $N$, whereas the five points of degree 4 play the role of points at infinity. For two real points $x$ and $y$ we denote by $\infty(x,y)$ the point at infinity of the line through $x$ and $y$. Let us denote by $1,2,3,4,5,6$ the real points of $N$.

The Nwankpa-Shrikhande plane is a linear space on 12 points and 19 lines with constant point degree 5, each point being on one line of degree 4 and four lines of degree 3.

The extended Nwankpa-Shrikhande plane is the Nwankpa-Shrikhande plane with one additional point on all lines of degree 4.

We recall that a $k$-arc of a finite projective plane is a set of $k$ points, no three of them collinear. (Regarding the previous definitions see also [4,1].)

We start with the classification of the $\{0,1\}$-semi-affine planes due to [12,4].

**Theorem 2.1.** Let $\mathcal{L}$ be a finite $\{0,1\}$-semi-affine plane. Then $\mathcal{L}$ is one of the following:

1. a. $\mathcal{L}$ is a projective plane.
1. b. $\mathcal{L}$ is an affine plane.
1. c. $\mathcal{L}$ is an affine plane with one point at infinity.

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1 Totten and De Witte proved the theorem in the case where the semi-affine plane contains a line of degree two; Kuiper and Dembowski proved the general case, independently.
1.d. \( \mathcal{L} \) is a punctured projective plane, that is, a projective plane with one point deleted.
1.e. \( \mathcal{L} \) is a near pencil.

The classification of \( \{0,2\}\)-semi-affine planes is due to [6,8].

**Theorem 2.2.** Let \( \mathcal{L} \) be a finite \( \{0,2\}\)-semi-affine plane which is neither a projective plane nor a near pencil. Then \( \mathcal{L} \) is one of the following:
2.a. \( \mathcal{L} \) is obtained from a finite projective plane of even order \( n \geq 4 \) by deleting an \((n+2)\)-arc.
2.b. \( \mathcal{L} \) is obtained from a finite projective plane of order \( n \geq 3 \) by deleting two lines with all points on these lines except their intersection point.
2.c. \( \mathcal{L} \) is the complete graph \( K_5 \).
2.d. \( \mathcal{L} \) is the extended Nwankpa–Shrikhande plane.

The classification of the finite \( \{1,2\}\)-semi-affine planes is due to [9].

**Theorem 2.3.** Let \( \mathcal{L} \) be a finite proper \( \{1,2\}\)-semi-affine plane. Then \( \mathcal{L} \) is one of the following:
3.a. \( \mathcal{L} \) is a finite affine plane of order \( n \geq 3 \) with one point deleted.
3.b. \( \mathcal{L} \) is obtained from a finite affine plane of order \( n \geq 3 \) by deleting one line with all of its points.
3.c. \( \mathcal{L} \) is the Nwankpa–Shrikhande plane.
3.d. \( \mathcal{L} \) is a linear space with five points, one line of degree 3 and 7 lines of degree two.

Finally, the classification of finite proper \( \{0,1,2\}\)-semi-affine planes have been obtained by Hauptmann [6] and Lo Re and Olanda [8], independently.

**Theorem 2.4.** Let \( \mathcal{L} \) be a finite proper \( \{0,1,2\}\)-semi-affine plane. Then \( \mathcal{L} \) is one of the following:
4.a. \( \mathcal{L} \) is obtained from a finite projective plane of order \( n \geq 3 \) by deleting one line \( l \) with all of its points except one and one point outside of \( l \).
4.b. \( \mathcal{L} \) is obtained from a finite projective plane of order \( n \geq 3 \) by deleting one line \( l \) with all of its points except one point \( P \) and two points \( Q \) and \( Q' \) outside of \( l \) and collinear with \( P \).
4.c. \( \mathcal{L} \) is obtained from a finite projective plane of order \( n \geq 3 \) by deleting two lines \( l \) and \( l' \) with all of its points except two points \( P \) and \( P' \) on \( l \) and \( l' \), respectively. The points \( P \) and \( P' \) are both different from the intersection of \( l \) and \( l' \).
4.d. \( \mathcal{L} \) is obtained from a finite projective plane of order \( n \geq 3 \) by deleting a \( k \)-arc with \( 2 \leq k \leq n + 1 \).
Table 1
\{(0,1,2)\}-semi-affine planes

<table>
<thead>
<tr>
<th>Type</th>
<th>Number of points</th>
<th>Number of lines</th>
<th>Possible point degrees</th>
<th>Possible line degrees</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.a</td>
<td>$n^2 + n + 1$</td>
<td>$n^2 + n + 1$</td>
<td>$n + 1$</td>
<td>$n + 1$</td>
</tr>
<tr>
<td>1.b</td>
<td>$n^2$</td>
<td>$n^2 + n$</td>
<td>$n + 1$</td>
<td>$n$</td>
</tr>
<tr>
<td>1.c</td>
<td>$n^2 + 1$</td>
<td>$n^2 + n$</td>
<td>$n, n + 1$</td>
<td>$n + 1, n$</td>
</tr>
<tr>
<td>1.d</td>
<td>$n^2 + n$</td>
<td>$n^2 + n + 1$</td>
<td>$n + 1$</td>
<td>$n + 1, n$</td>
</tr>
<tr>
<td>1.e</td>
<td>$n + 2$</td>
<td>$n + 2$</td>
<td>$n + 1, 2$</td>
<td>$n + 1, 2$</td>
</tr>
<tr>
<td>2.a</td>
<td>$n^2 - 1$</td>
<td>$n^2 + n + 1$</td>
<td>$n + 1$</td>
<td>$n + 1, n - 1$</td>
</tr>
<tr>
<td>2.b</td>
<td>$n^2 - n + 1$</td>
<td>$n^2 + n - 1$</td>
<td>$n + 1, n - 1$</td>
<td>$n + 1, n - 1$</td>
</tr>
<tr>
<td>2.e</td>
<td>5</td>
<td>10</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>2.f</td>
<td>13</td>
<td>19</td>
<td>3, 5</td>
<td>3, 5</td>
</tr>
<tr>
<td>3.a</td>
<td>$n^2 - 1$</td>
<td>$n^2 + n$</td>
<td>$n + 1$</td>
<td>$n, n - 1$</td>
</tr>
<tr>
<td>3.b</td>
<td>$n^2 - n$</td>
<td>$n^2 + n - 1$</td>
<td>$n + 1$</td>
<td>$n, n - 1$</td>
</tr>
<tr>
<td>3.c</td>
<td>12</td>
<td>19</td>
<td>5</td>
<td>3, 4</td>
</tr>
<tr>
<td>3.d</td>
<td>5</td>
<td>8</td>
<td>3, 4</td>
<td>2, 3</td>
</tr>
<tr>
<td>4.a</td>
<td>$n^2$</td>
<td>$n^2 + n$</td>
<td>$n + 1, n$</td>
<td>$n + 1, n, n - 1$</td>
</tr>
<tr>
<td>4.b</td>
<td>$n^2 - 1$</td>
<td>$n^2 + n$</td>
<td>$n + 1, n$</td>
<td>$n + 1, n, n - 1$</td>
</tr>
<tr>
<td>4.c</td>
<td>$n^2 - n + 2$</td>
<td>$n^2 + n - 1$</td>
<td>$n + 1, n$</td>
<td>$n + 1, n, n - 1$</td>
</tr>
<tr>
<td>4.d</td>
<td>$n^2 + n + 1 - k$</td>
<td>$n^2 + n + 1$</td>
<td>$n + 1$</td>
<td>$n + 1, n, n - 1$</td>
</tr>
<tr>
<td>4.e</td>
<td>$n^2 - n + 1$</td>
<td>$n^2 + n - 1$</td>
<td>$n + 1, n$</td>
<td>$n, n - 1$</td>
</tr>
<tr>
<td>4.f</td>
<td>11</td>
<td>16</td>
<td>4, 5</td>
<td>3, 5</td>
</tr>
<tr>
<td>4.g</td>
<td>7</td>
<td>9</td>
<td>3, 4</td>
<td>2, 3</td>
</tr>
<tr>
<td>4.h</td>
<td>6</td>
<td>9</td>
<td>3, 4</td>
<td>2, 3</td>
</tr>
<tr>
<td>4.i</td>
<td>6</td>
<td>8</td>
<td>2, 3, 4</td>
<td>2, 3, 4</td>
</tr>
<tr>
<td>4.j</td>
<td>7</td>
<td>10</td>
<td>3, 4</td>
<td>2, 3, 4</td>
</tr>
</tbody>
</table>

4.e. $\mathcal{L}$ is obtained from a finite affine plane of order $n \geq 3$ by deleting one line $l$ with all of its points except one.

4.f. $\mathcal{L}$ is the plane of Nwankpa.

4.g. $\mathcal{L}$ is the quasi-Fano plane.

4.h. $\mathcal{L}$ is the quasi-Fano plane with one point of degree 3 deleted.

4.i. $\mathcal{L}$ is the $(3, 4)$-cross.

4.j. $\mathcal{L}$ is the linear space on seven points and 10 lines with point degree 3 or 4 and line degree 2, 3, 4 and there is a unique line of degree 4. It is obtained from the punctured Fano plane by adding a new point $P$ on one of its lines, say $l$, of degree 3 and joining $P$ with the points off $l$ by lines of degree 2.

According to Theorems 2.1–2.4 we shall define the type of a $\{0,1,2\}$-semi-affine plane. For example, a projective plane is a linear space of type 1.a. Throughout this paper the linear spaces of type 2.c, 2.d, 3.c, 3.d, 4.f, 4.g, 4.h, 4.i and 4.j are called sporadic.

For the reader's convenience we include Table 1 containing the most important combinatorial properties of the $\{0,1,2\}$-semi-affine planes. The parameter $n$ appearing in the table is the order of the corresponding non-sporadic linear space.
3. Residual spaces of a \{0, 1, 2\}-inversive plane

In this section we shall prove the following theorem.

**Theorem 3.1.** Let \( \mathcal{C} \) be a finite \{0, 1, 2\}-inversive plane such that for any point \( X \) of \( \mathcal{C} \) the residual space \( \mathcal{C}_x \) is not a near-pencil. Then one of the following occurs, where \( n \) denotes the order of the first residual spaces in curly brackets.

1. We have type \((\mathcal{C}_{P})= (\mathcal{C}_{Q})\), for any pair of distinct points, \( P \) and \( Q \). In particular if type \((\mathcal{C}_{P})= 4.d_{k}\) and type \((\mathcal{C}_{Q})= 4.d_{k'}\), then \( k = k'\).
2. We have \( n = 4\), with type \( \mathcal{C}_X \in \{ 4.d_{n+1}, 1.b \} \). (Case 6.7 of Theorem 1.6).
3. We have \( n = 4\), with type \( \mathcal{C}_X \in \{ 1.a, 4.e \} \). (Case 6.8).
4. We have \( n = 4\), with type \( \mathcal{C}_X \in \{ 1.c, 4.d_{n} \} \). (Case 6.9).
5. We have \( n = 4\), with type \( \mathcal{C}_X \in \{ 2.b, 2.d \} \). (Case 4.5).
6. We have \( n = 3\), with type \( \mathcal{C}_X \in \{ 4.e, 4.g \} \). (Case 6.13).
7. We have \( n = 2\), with type \( \mathcal{C}_X \in \{ 1.a, 4.g \} \). (Case 6.10).
8. We have \( n = 2\), with type \( \mathcal{C}_X \in \{ 1.d, 4.h \} \). (Case 6.11).
9. We have \( n = 3\), with type \( \mathcal{C}_X \in \{ 3.b, 4.h \} \). (Case 6.12).
10. We have type \( \mathcal{C}_X \in \{ 2.c, 3.d \} \). (Case 5.2).
11. We have \( n = 2\), with type \( \mathcal{C}_X \in \{ 1.c, 3.d \} \). (Case 6.16).
12. We have \( n = 2\), with type \( \mathcal{C}_X \in \{ 1.a, 4.g, 4.e \} \). (Case 6.14).

We do not know whether the situations described in 3.1 (2),(3),(4),(5),(6),(12), (1) \( n \neq 4\) occur, whereas we have examples for the situations described in 3.1 (7),(8),(9),(10),(11) (see Lemma 3.20) and in (1) \( n = 4\) (see 4.4).

**Lemma 3.2.** Let \( \mathcal{C} \) be a finite circular space, and let \( P \) and \( Q \) be two points of \( \mathcal{C} \). Then the residual spaces \( \mathcal{C}_P \) and \( \mathcal{C}_Q \) have the same number of points.

**Lemma 3.3.** Let \( \mathcal{C} \) be a finite circular space, and let \( P \) and \( Q \) be two points. Let \( r \) be the degree of \( Q \) in \( \mathcal{C}_P \), and let \( s \) be the degree of \( P \) in \( \mathcal{C}_Q \). Then \( r = s \).

**Proposition 3.4.** Let \( \mathcal{C} \) be a finite \{0, 1, 2\}-inversive plane admitting a point \( P \) such that \( \mathcal{C}_P \) is a near pencil. Then \( \mathcal{C} \) is degenerate on four, five or six points.

**Proof.** Let \( v \) be the number of points of \( \mathcal{C} \). Since \( \mathcal{C}_P \) is a near pencil, there exists a circle \( x \) with \( v - 1 \) points and exactly one point \( Q \) not on \( x \). The residual spaces with respect to the points on \( x \) are near pencils of order \( v - 3 \). The residual space \( \mathcal{C}_Q \) is the complete graph \( K_{v-1} \). Since \( \mathcal{C}_Q \) is \{0, 1, 2\}-semi-affine, it follows that \( v \in \{ 4, 5, 6 \} \).

(Case 1.5 and 4.4). \( \square \)

Let \( \mathcal{L} \) be a finite linear space, and let \( k \) be the maximal line degree of \( \mathcal{L} \). We say that \( \mathcal{L} \) has property \((*)\), if any point of \( \mathcal{L} \) is incident with a line of degree \( k \).
Lemma 3.5. Let $\mathcal{C}$ be a finite $\{0,1,2\}$-inversive plane, and let $P$ and $Q$ be two points of $\mathcal{C}$. Suppose that $\mathcal{C}_P$ and $\mathcal{C}_Q$ both have property (*), and let $k_P$ and $k_Q$ be the maximal line degrees of $\mathcal{C}_P$ and $\mathcal{C}_Q$, respectively. Then $k_P = k_Q$.

Proof. The point $Q$ is a point of $\mathcal{C}_P$. By assumption, there is a line of degree $k_P$ through $Q$. Similarly there is a line through $P$ of degree $k_Q$ in $\mathcal{C}_Q$. Let $m$ be the maximal number of points of a circle through $P$ and $Q$. Since the lines of $\mathcal{C}_P$ through $Q$ and the lines of $\mathcal{C}_Q$ through $P$ correspond to the circles of $\mathcal{C}$ through $P$ and $Q$, it follows that $k_P + 1 = m = k_Q + 1$. □

From Lemmas 3.6 to 3.15 we shall suppose that for any point $X$ of $\mathcal{C}$ the residual space $\mathcal{C}_X$ is neither a near-pencil nor sporadic.

Lemma 3.6. Let $\mathcal{C}$ be a finite $\{0,1,2\}$-inversive plane, and let $P$ and $Q$ be two points of $\mathcal{C}$. Let $n$ and $m$ be the order of $\mathcal{C}_P$ and $\mathcal{C}_Q$, respectively. Then one of the following occurs.

1. We have type $(\mathcal{C}_P) = \text{type } (\mathcal{C}_Q)$. In particular if type $(\mathcal{C}_P) = 4.d_k$ and type $(\mathcal{C}_Q) = 4.d_{k'}$, then $k = k'$.
2. We have type $(\mathcal{C}_P) = 1.a$ and type $(\mathcal{C}_Q) = 2.b$.
3. We have type $(\mathcal{C}_P) = 1.a$ and type $(\mathcal{C}_Q) = 4.e$.
4. We have type $(\mathcal{C}_P) = 1.b$ and type $(\mathcal{C}_Q) = 4.a$.
5. We have type $(\mathcal{C}_P) = 1.b$ and type $(\mathcal{C}_Q) = 4.d_{n+1}$.
6. We have type $(\mathcal{C}_P) = 1.c$ and type $(\mathcal{C}_Q) = 4.d_e$.
7. We have type $(\mathcal{C}_P) = 1.d$ and type $(\mathcal{C}_Q) = 3.b$.
8. We have type $(\mathcal{C}_P) = 2.a$ and type $(\mathcal{C}_Q) = 3.a$.
9. We have type $(\mathcal{C}_P) = 2.a$ and type $(\mathcal{C}_Q) = 4.b$.
10. We have type $(\mathcal{C}_P) = 2.b$ and type $(\mathcal{C}_Q) = 4.e$.
11. We have type $(\mathcal{C}_P) = 3.a$ and type $(\mathcal{C}_Q) = 4.b$.
12. We have type $(\mathcal{C}_P) = 4.a$ and type $(\mathcal{C}_Q) = 4.d_{e+1}$.

Proof. As an example we shall consider the case where $\mathcal{C}_P$ is of type $1.a$.

Let $\mathcal{C}_P$ be of type $1.a$, and let $\mathcal{C}_Q$ be non-sporadic and neither a near pencil nor of type $1.a$. If $w$ is the number of points of $\mathcal{C}_Q$, it follows that

$$m^2 - m \leq w \leq m^2 + m \quad \text{(see Table 1)}.$$ 

On the other hand, by Lemma 3.2, we have $w = n^2 + n + 1$. In particular $n < m$, that is, $n \leq m - 1$. Hence

$$m^2 - m \leq w = n^2 + n + 1 \leq m^2 - m + 1.$$ 

If $w = m^2 - m + 1$, then $\mathcal{C}_Q$ is of type $2.b$ or $4.e$. The case $w = m^2 - m$ does not occur, since $m^2 - m = w = n^2 + n + 1$ would be at the same time an even and an odd number. □
Lemma 3.7. Let $C$ be a finite $\{0,1,2\}$-inversive plane, and let $P$ and $Q$ be two points of $C$.

(a) If type $(C_P) = 1.d$, then type $(C_Q) \neq 3.b$.
(b) If type $(C_P) = 1.a$, then type $(C_Q) \neq 2.b$.
(c) If type $(C_P) = 2.b$, then type $(C_Q) \neq 4.e$.

Proof. Let $n$ be the order of $C_P$.

(a) Assume that $C_P$ and $C_Q$ are of type $1.d$ and $3.b$, respectively. By 3.2, $C_Q$ is of order $n + 1$. It follows that any point of $C_P$ is of degree $n + 1$ and any point of $C_Q$ is of degree $n + 2$, a contradiction (see Lemma 3.3).

(b) The proof is similar to the proof of (a).

(c) Assume that $C_P$ and $C_Q$ are of type $2.b$ and $4.e$, respectively. By Lemma 3.2, $C_Q$ is of order $n + 1$. The maximal line degrees of $C_P$ and $C_Q$ are $n + 1$ and $n$, respectively. Since $C_P$ and $C_Q$ both have property ($\ast$), we get a contradiction. \qed

Lemma 3.8. Let $C$ be a finite $\{0,1,2\}$-inversive plane, and let $P$ and $Q$ be two points of $C$. Let $C_P$ be of type $4.d_k$ and let $n$ be the order of $C_P$.

(a) If $n \neq 4$, then $C_Q$ is not of type $1.b$.
(b) If $n \neq 3$, then $C_Q$ is not of type $4.a$.

Proof. Let $C_P$ be of type $4.d_k$. Assume that $C_Q$ is of type $1.b$ or $4.a$. By Lemma 3.2, $C_Q$ is of order $n$, and $k = n + 1$. Let $\mathcal{P}$ be a projective plane of order $n$, and let $K$ be a $(n + 1)$-arc of $\mathcal{P}$ such that $C_P$ is obtained from $\mathcal{P}$ by deleting $K$.

Step 1: Let $C_Q$ be of type $1.b$. Then $n$ is even and $Q$ is the nucleus of $K$. Indeed there are $n + 1$ lines of degree $n$ through $P$ in $\mathcal{P}$. By Lemma 3.3, there are $n + 1$ lines of degree $n$ through $Q$ in $C_P$. Since the $n$-lines of $C_P$ are the tangent lines of $K$ in $\mathcal{P}$, it follows that any line of $C_P$ through $Q$ is a tangent line. Hence $n$ is even and $Q$ is the nucleus of $K$.

Step 2: Let $C_Q$ be of type $4.a$. For $n \neq 3$, $n$ is even and $Q$ is the nucleus of $K$. Indeed, there are $n + 1$ lines through $Q$ in $C_P$. By Lemma 3.3, there are $n + 1$ lines through $P$ in $C_Q$. $C_Q$ is a linear space that has been obtained from a projective plane by deleting one of its lines except a point $Z$ and by deleting a further point not on this line. It follows $P \neq Z$. In particular there are at least $n - 1$ lines of degree $n$ through $P$ in $C_Q$. Again by Lemma 3.3, it follows that there are at least $n - 1$ lines of degree $n$ through $Q$ in $C_P$. Since $n \geq 4$, we have $n - 1 \geq 3$. It follows that $n$ is even and that $Q$ is the nucleus of $K$.

Step 3: If $n \geq 4$, then for any point $X \neq Q$ the residual space $C_X$ is of type $4.d_{n+1}$. If $n = 3$, then there is no residual space of type $1.b$. This follows immediately from Steps 1, 2 and the fact that the nucleus of an $(n + 1)$-arc is unique.

Step 4: Let $n \neq 4$. Then $C_Q$ is not of type $1.b$. Assume that $C_Q$ is of type $1.b$. By Step 3, $n \neq 3$ and any residual spaces $C_X$ with $X \neq Q$ is of type $4.d_{n+1}$. Let $b_{n+2}$ be
the number of circles of degree \( n + 2 \). In \( \mathcal{C}_Q \) there are no lines of degree \( n + 1 \), hence there are no circles of degree \( n + 2 \) through \( Q \). On the other hand in \( \mathcal{C}_P \) there are \((n^2 - n)/2\) lines of degree \( n + 1 \). By Step 3, any point of \( \mathcal{C} \) distinct from \( Q \) is incident with \((n^2 - n)/2\) circles of degree \( n + 2 \). Counting the incident point-circle-pairs \((X,x)\), where \( x \) is of degree \( n + 2 \) we get

\[
\frac{n^2n^2 - n}{2} = b_{n+2}(n + 2).
\]

Therefore

\[
b_{n+2} = \frac{1}{2}(n^3 - 3n^2 + 6n - 12) + \frac{12}{n+2}.
\]

Since \( n \) is even, the first term is an integer. So \( 12/(n+2) \) has to be an integer, too. In view of \( n \geq 5 \) we get \( n = 10 \). Since there is no projective plane of order 10, this is a contradiction.

**Step 5**: Let \( n \neq 3 \). Then \( \mathcal{C}_Q \) is not of type \( 4.a \). Assume that \( \mathcal{C}_Q \) is of type \( 4.a \). By Step 3, all residual spaces \( \mathcal{C}_X \) are of type \( 4.d_{n+1} \) except \( \mathcal{C}_Q \). Let \( b_{n+1} \) be the number of circles of degree \( n + 1 \). A similar reasoning as in Step 4, and using that there are \( n^2 - n + 1 \) circles of degree \( n + 1 \) through \( Q \), yields that

\[
b_{n+1} = n^2 + n - 2 + \frac{3}{n+1}.
\]

a contradiction.

**Remark.** By Lemma 3.8, it follows that residual spaces of type \( 4.d_k \), \( 4.a \) and \( 1.b \) cannot occur simultaneously. Furthermore, residual spaces of type \( 4.d_k \) and \( 4.a \) can only occur simultaneously, if \( n = 3 \), residual spaces of \( 4.d_k \) and \( 1.b \) can only occur simultaneously, if \( n = 4 \). In the latter case we know that exactly one residual space is of type \( 1.b \) and all the other residual spaces are of type \( 4.d \).

**Lemma 3.9.** Let \( \mathcal{C} \) be a finite \( \{0,1,2\}\)-inversive plane, and let \( P \) and \( Q \) be two points of \( \mathcal{C} \). If \( \mathcal{C}_P \) is of type \( 4.d_k \), then \( \mathcal{C}_Q \) is not of type \( 4.a \).

**Proof.** Let \( \mathcal{C}_P \) and \( \mathcal{C}_Q \) of type \( 4.d_k \) and \( 4.a \), respectively. Let \( n \) be the order of \( \mathcal{C}_P \). In view of Lemma 3.8 and the above remark we can suppose that \( n = 3 \) and that for any for any point \( X \) of \( \mathcal{C} \) the residual space \( \mathcal{C}_X \) is of type \( 4.d_k \) or \( 4.a \). Let \( a \) and \( c \) be the number of points \( X \) of \( \mathcal{C} \) such that \( \mathcal{C}_X \) is of type \( 4.d_k \) and \( 4.a \), respectively. It follows that \( a + c = 10 \). Let \( b_5 \) and \( b_4 \) be the number of circles of \( \mathcal{C} \) of degree 5 and 4, respectively. Then we get

\[
b_5 = \frac{3a + 2c}{5} = a/5 + 4.
\]

If follows that \( a = 5 = c \). Then from

\[
4b_4 = 4a + 7c = 55.
\]

we have a contradiction. 

\[\Box\]
Lemma 3.10. Let \( \mathcal{C} \) be a finite \( \{0,1,2\} \)-inversive plane, and let \( P \) and \( Q \) be two points of \( \mathcal{C} \). If \( \mathcal{C}_P \) is of type 4.a, then \( \mathcal{C}_Q \) is not of type 1.b.

Proof. Assume that \( \mathcal{C} \) admits two points \( P \) and \( Q \) such that \( \mathcal{C}_P \) is of type 4.a and \( \mathcal{C}_Q \) is of type 1.b. Let \( n \) be the common order of \( \mathcal{C}_P \) and \( \mathcal{C}_Q \). Note that \( n \geq 3 \). Then \( \mathcal{C}_P \) is a linear space that has been obtained from a projective plane of order \( n \) by deleting one of its lines except a point \( Z \) and by deleting a further point not on this line. We denote by \( g \) the unique line of \( \mathcal{C}_P \) of degree \( n \) through \( Z \). One easily sees that \( Q \) has to be a point on \( g \) different from \( Z \).

Let \( a \) and \( c \) be the number of points \( X \) of \( \mathcal{C} \) such that \( \mathcal{C}_X \) is of type 4.a and 1.b, respectively. It follows that \( 1 \leq c \leq n-1 \). In view of \( a + c = n^2 + 1 \) we obtain

\[
 n^2 \geq a \geq n^2 - n + 2. 
\]

Let \( b_{n+1} \) and \( b_{n+2} \) be the number of circles of \( \mathcal{C} \) of degree \( n+1 \) and \( n+2 \), respectively. Then we get

\[
 b_{n+1} = a \cdot n + c \cdot n - 2a + \frac{3a}{n+1}, \\
 b_{n+2} = a - \frac{3a}{n+2}. 
\]

Since \( \gcd(n+1,n+2) = 1 \), it follows that \((n+1)(n+2)\) is a divisor of \( 3a \), that is, there exists an integer \( i \) with

\[
 3a = i(n+1)(n+2). 
\]

Because of \( n^2 \geq a > 0 \), it follows that \( i \in \{1,2\} \). Assuming \( i = 1 \) we obtain

\[
 n^2 - n + 2 \leq a = \frac{1}{2}(n+1)(n+2) < n^2 - n + 2 
\]

(note that \( n \geq 3 \)), a contradiction. Hence \( i = 2 \). It follows that

\[
 3a = 2(n+1)(n+2). 
\]

From

\[
 3(n^2 - n + 2) \leq 3a = 2(n+1)(n+2) \leq 3n^2 
\]

follows that \( n \in \{6,7,8\} \). Since \((n+1)(n+2)\) must be divisible by 3, we have \( n \in \{7,8\} \).

We shall use the following observation. Let \( X \) be a point such that \( \mathcal{C}_X \) is of type 4.a, and let \( Y \) be the unique point of \( \mathcal{C}_X \) of degree \( n \). Then \( \mathcal{C}_Y \) is also of type 4.a and \( X \) is the unique point of degree \( n \) in \( \mathcal{C}_Y \). We say that \( X \) and \( Y \) are married if \( Y \) is the unique point of degree \( n \) in \( \mathcal{C}_X \).

Case 1: Let \( n = 7 \). Then \( a = 48 \), \( c = 2 \) and \( b_{n+2} = 32 \). We count the pairs \((P,x)\), where \( P = \{X,Y\} \) is a set of two married points, \( x \) is a circle of degree 9 and \( X \) and \( Y \) are both incident with \( x \). Any pair of married points is incident with exactly six circles of degree 9. Because of \( a = 48 \) there are 24 pairs of married points. Finally, any circle of degree 9 is incident with at most four pairs of married points. Hence

\[
 24 \cdot 6 \leq 32 \cdot 4, 
\]

a contradiction.
Case 2: Let \( n = 8 \). Then \( a = 60 \) and \( c = 5 \). Let \( Q_1, \ldots, Q_5 \) be the five points of \( \mathcal{C} \) such that the residual space is of type 1.\( b \). Then \( Q_1, \ldots, Q_5 \) are incident with \( g \). Let \( x \) be the circle corresponding to \( g \), and let \( R \) be a point of \( \mathcal{C} \) not on \( x \). Then the residual space \( \mathcal{C}_R \) is of type 4.\( a \). Let \( Z' \) be the unique point of degree \( n \) in \( \mathcal{C}_R \), and let \( g' \) be the unique line of degree \( n \) through \( Z' \). As above it follows that the points \( Q_1, \ldots, Q_5 \) are incident with \( g' \). Denoting by \( x' \) the circle corresponding to \( g' \), we get \( x = x' \) contradicting the fact that \( R \) is incident with \( x' \) but not with \( x \). \( \square \)

**Lemma 3.11.** Let \( \mathcal{C} \) be a finite \( \{0,1,2\} \)-inversive plane, and let \( P \) and \( Q \) be two points of \( \mathcal{C} \).

(a) If \( \mathcal{C}_P \) is of type 2,\( a \), then \( \mathcal{C}_Q \) is not of type 4,\( b \).

(b) If \( \mathcal{C}_P \) is of type 2,\( a \), then \( \mathcal{C}_Q \) is not of type 3,\( a \).

**Proof.** (a) Assume that \( \mathcal{C} \) admits two points \( P \) and \( Q \) such that \( \mathcal{C}_P \) is of type 2,\( a \) and \( \mathcal{C}_Q \) is of type 4,\( b \). Since \( \mathcal{C}_P \) is obtained by deleting an \((n+2)\)-arc from a projective plane of order \( n \), there are \((n+2)/2\) lines of degree \( n-1 \) and \( n/2 \) lines of degree \( n+1 \) through any point of \( \mathcal{C}_P \). It follows that there must be \((n+2)/2\) lines of degree \( n-1 \) and \( n/2 \) lines of degree \( n+1 \) through \( P \) in \( \mathcal{C}_Q \), as well, a contradiction.

(b) The proof is similar to the proof of (a). \( \square \)

**Lemma 3.12.** Let \( \mathcal{C} \) be a finite \( \{0,1,2\} \)-inversive plane, and let \( P \) and \( Q \) be two points of \( \mathcal{C} \). If \( \mathcal{C}_P \) is of type 3,\( a \), then \( \mathcal{C}_Q \) is not of type 4,\( b \).

**Proof.** The proof is similar to the one of Lemma 3.10. \( \square \)

**Lemma 3.13.** Let \( \mathcal{C} \) be a finite \( \{0,1,2\} \)-inversive plane, and let \( P \) and \( Q \) be two points of \( \mathcal{C} \). Let \( \mathcal{C}_P \) be of type 1,\( a \) and let \( n \) be the order of \( \mathcal{C}_P \). If \( n \neq 4 \), then \( \mathcal{C}_Q \) is not of type 4,\( e \).

**Proof.** Assume that there exist two points \( P \) and \( Q \) with residual spaces of type 1,\( a \) and 4,\( e \), respectively. Let \( \mathcal{C}_P \) be of order \( n \). Then, by Lemma 3.2, \( \mathcal{C}_Q \) is of order \( n+1 \). Let \( T \) be the unique point of degree \( n+1 \) in \( \mathcal{C}_Q \). Since any point of \( \mathcal{C}_P \) is of degree \( n+1 \), it follows that \( P = T \). In particular there is exactly one residual space of type 1,\( a \) and \( n^2+n+1 \) residual spaces of type 4,\( e \) by 3.5 and 3.7. Let \( b_{n+2} \) be the number of circles of degree \( n+2 \). Counting the incident point-circle-pairs \((U,x)\), where \( x \) is of degree \( n+2 \) we obtain

\[
b_{n+2} = 2n^2 + 4 - \frac{6}{n+2}.
\]

It follows that \( n+2 \) is a divisor of 6. Hence \( n = 4 \). \( \square \)

**Lemma 3.14.** Let \( \mathcal{C} \) be a finite \( \{0,1,2\} \)-inversive plane, and let \( P \) and \( Q \) be two points of \( \mathcal{C} \). Let \( \mathcal{C}_P \) be of type 1,\( c \) and let \( n \) be the order of \( \mathcal{C}_P \). If \( n \neq 4 \), then \( \mathcal{C}_Q \) is not of type 4,\( d_k \).
Assume that there exist two points \( P \) and \( Q \) with residual spaces of type 1, \( c \) and 4, \( d_n \), respectively. By Lemma 3.6, \( C_Q \) is of order \( n \), and \( k = n \). Let \( Z \) be the unique point of degree \( n \) of \( C_P \). Then \( P \) is of degree \( n \) in \( C_Z \), hence \( C_Z \) is of type 1, \( c \). \( C_Q \) is obtained from a projective plane of order \( n \) by deleting some \( n \)-arc \( K \). In \( C_P \), the point \( Q \) is incident with one line of degree \( n+1 \) and \( n \) lines of degree \( n \). Therefore \( P \) and \( Z \) are points of \( C_Q \) with exactly the same properties, that is, they are incident with \( n \) tangent lines of \( K \) and one line missing \( K \). Assuming the existence of a third point \( R \) such that \( C_R \) is of type 1, \( c \), we get that \( R \) (as a point of \( C_Q \)) is incident with \( n \) tangent lines of \( K \). This contradicts the fact that \( K \) admits only \( 2n \) tangent lines. It follows that there are two points of \( C \) whose residual spaces are of type 1, \( c \) and \( n \). Points whose residual spaces are of type \( 4, d_n \). Let \( b_{n+2} \) be the number of circles of degree \( n + 2 \). It follows that

\[
2b_{n+2} = n^3 - 3n^2 + 8n - 12 + \frac{24}{n + 2}.
\]

Because of \( n \geq 3 \) we get \( n \in \{4, 6, 10, 22\} \). Since there are no projective planes of order 6, 10 or 22, it follows that \( n = 4 \). \( \square \)

So far we have seen what happens if \( C_X \) is neither a near pencil nor sporadic for any point \( X \) of \( C \) and there exist at least two points \( P \) and \( Q \) such that \( C_P \) and \( C_Q \) are of different type. More precisely we have proved the following.

**Theorem 3.15.** Let \( C \) be a finite \( \{0, 1, 2\} \)-inversive plane, such that for any point \( X \) of \( C \) the residual space \( C_X \) is neither a near-pencil nor sporadic. If moreover there exist two points \( P \) and \( Q \) such that the residual spaces \( C_P \) and \( C_Q \) are of different type. Then one of the following occurs, where \( n \) denotes the order of \( C_P \):

1. We have \( n = 4 \) with type \( C_P = 4, d_{n+1} \) and type \( C_Q = 1, b \). (Case 6.7).
2. We have \( n = 4 \) with type \( C_P = 1, a \) and type \( C_Q = 4, e \). (Case 6.8).
3. We have \( n = 4 \) with type \( C_P = 1, c \) and type \( C_Q = 4, d_n \). (Case 6.9).

We do not know if there exists a \( \{0, 1, 2\} \)-inversive plane admitting two points \( P \) and \( Q \) such that \( C_P \) is sporadic and the residual spaces are not all of the same type.

It remains to figure out which \( \{0, 1, 2\} \)-inversive planes can occur if there is at least one point \( P \) such that \( C_P \) is sporadic and the residual spaces are not all of the same type.

Let \( C_P \) be sporadic and let \( P_2, \ldots, P_v \) be the points of \( C \) different from \( P = P_1 \).

We can exclude many hypothetical cases by applying the following technique:

1. Check whether \( C_P \) and \( C_{P_i} \) have the same number of points for any \( i = 2, \ldots, v \).
2. Let \( \{r_1, \ldots, r_a\} \) and \( \{s'_1, \ldots, s'_b\} \) be the point degrees of the points in \( C_P \) and \( C_{P_i} \), respectively. Check whether \( \{r_1, \ldots, r_a\} \cap \{s'_1, \ldots, s'_b\} \neq \emptyset \) for any \( i = 2, \ldots, v \). (It follows from Lemma 3.3 that this intersection is non-empty.)
3. The lines of \( C_P \) through \( P_i \) and the lines of \( C_{P_i} \) through \( P \) both correspond to the circles of \( C \) through \( P \) and \( P_i \). In particular, if \( \{l_1, \ldots, l_r\} \) and \( \{m'_1, \ldots, m'_s\} \)
denote the line degrees of the lines of \( \mathcal{C}_P \) through \( P \) and of \( \mathcal{C}_{P_i} \) through \( P \), then \( \{l_1, \ldots, l_r\} = \{m_1', \ldots, m_d'\} \). Check whether this condition is fulfilled.

(4) Let \( h \) be the number of residual spaces of different type and suppose \( P, P_2, \ldots, P_h \)
are all of different type. Let \( a_1, a_2, \ldots, a_h \) be the number of residual spaces of the same type as \( \mathcal{C}_P, \mathcal{C}_{P_2}, \ldots, \mathcal{C}_{P_h} \), respectively. For \( j \in \mathbb{N} \), let \( b_j \) be the number of circles of degree \( j \), let \( c_1^j \) and \( c_2^j \) be the number of lines of degree \( j - 1 \) in \( \mathcal{C}_P \) and \( \mathcal{C}_{P_i} \), respectively. Then

\[
a_1 + a_2 + \cdots + a_h = v \quad \text{and} \quad b_j \cdot j = a_1 c_1^j + a_2 c_2^j + \cdots + a_h c_h^j.
\]

Check whether these equations, for \( j \in \mathbb{N} \), yield a contradiction.

If there is a point \( P \) such that \( \mathcal{C}_P \) is sporadic then applying the previous process it is straightforward to prove the following.

**Lemma 3.16.** Let \( \mathcal{C} \) be a finite \( \{0, 1, 2\}\)-inversive plane such that no residual space is a near-pencil, and let \( P \) a point of \( \mathcal{C} \) such that \( \mathcal{C}_P \) is sporadic. Let \( P_2, \ldots, P_e \) be the points of \( \mathcal{C} \) different from \( P \) and suppose that type \( \mathcal{C}_{P_2} \) is different from type \( \mathcal{C}_P \). If \( h \) denotes the number of different types between the residual spaces \( \mathcal{C}_X \) then one of the following occurs:

1. If type \( \mathcal{C}_P = 2.c \) then \( h = 2 \) and type \( \mathcal{C}_X \in \{2.c, 3.d\} \) for all \( X \).
2. If type \( \mathcal{C}_P = 2.d \) then \( h = 2 \) and type \( \mathcal{C}_X \in \{2.b, 2.d\} \) for all \( X \).
3. If type \( \mathcal{C}_P = 3.d \) then \( h = 2 \) and type \( \mathcal{C}_X \in \{1.c, 3.d\} \) for all \( X \).
4. If type \( \mathcal{C}_P = 4.g \) then \( h \leq 4 \) and type \( \mathcal{C}_X \in \{1.a, 4.e, 4.g, 4.j\} \) for all \( X \).
5. If type \( \mathcal{C}_P = 4.h \) then \( h \leq 4 \) and type \( \mathcal{C}_X \in \{1.d, 3.b, 4.h, 4.i\} \) for all \( X \).
6. If type \( \mathcal{C}_P = 4.i \) then \( h \leq 3 \) and type \( \mathcal{C}_X \in \{3.b, 4.h, 4.i\} \) for all \( X \).
7. If type \( \mathcal{C}_P = 4.j \) then \( h \leq 5 \) and type \( \mathcal{C}_X \in \{1.a, 2.b, 4.e, 4.g, 4.j\} \) for all \( X \).

We investigate with a more careful analysis cases from (4) to (7) of the previous lemma.

**Lemma 3.17.** There is no \( \{0, 1, 2\}\)-inversive plane having a residual space \( \mathcal{C}_P \) of type \( 4.i \).

**Proof.** By Lemma 3.16, we have type \( \mathcal{C}_X \in \{4.i, 4.h, 3.b\} \) for any point \( X \) of \( \mathcal{C} \). Let type \( \mathcal{C}_P = 4.i \) then \( v = 7 \), there is one circle \( C \) of degree 5 and for any point \( X \) of \( C \) type \( \mathcal{C}_X = 4.i \). If \( Q \) and \( Q' \) are the points of \( \mathcal{C} \) off \( C \) then type \( \mathcal{C}_Q = 4.h \) or type \( \mathcal{C}_Q = 3.b \) so through \( Q \) there are at least 4 circles of degree 4 that is a contradiction. \( \square \)

**Lemma 3.18.** There is no \( \{0, 1, 2\}\)-inversive plane having a residual space \( \mathcal{C}_P \) of type \( 4.j \).

**Proof.** By Lemma 3.16, we have type \( \mathcal{C}_X \in \{4.j, 1.a, 2.b, 4.e, 4.g\} \) for any point \( X \) of \( \mathcal{C} \). Let type \( \mathcal{C}_P = 4.j \). We start by showing that \( 4.g \) is not possible. Let \( Q \) be a point such that type \( \mathcal{C}_Q = 4.g \) then there is a circle \( C \) of degree 5 such that for any point \( X \in C \) we have type \( \mathcal{C}_X = 4.j \). It follows \( Q \notin C \). Through \( Q \) there are six circles of
degree 4 but because outside $C$ there are only three points we get at most five circles of degree 4 through $Q$, a contradiction.

Then $v = 8$ and there is one circle $C$ of degree $5$. If there is a point $Q$ such that the residual space in $Q$ is of type $2.b$ then through any point of $C$ there is a circle of degree 5 so every residual space is of type $4.j$ or $2.b$. Let $a_1$ and $a_2$ be the number of residual spaces of type $4.j$ and $2.b$, respectively then

$$a_1 + a_2 = 8 \quad \text{and} \quad 5b_5 = a_1 + 2a_2 = 16 - a_1,$$

it follows $5|a_1 - 1$ so $a_1 = 6, a_2 = 2$ or $a_1 = 1, a_2 = 7$. Moreover $4b_4 = 3a_1 = 18$ or $4b_4 = 3a_1 = 3$, that is a contradiction.

So there is no point $Q$ such that type $C_Q = 2.b$. Observe that there is also no point $Q$ such that the residual space in $Q$ is of type $1.a$. Indeed if type $C_Q = 1.a$ then through $Q$ in $C_P$ there would be three lines of degree 3, that is a contradiction. Let $a_1$ and $a_2$ be the number of residual spaces of type $4.j$ and $4.e$, respectively, then

$$a_1 + a_2 = 8 \quad \text{and} \quad 5b_5 = a_1,$$

it follows $a_1 = 5, a_2 = 3$. Moreover $4b_4 = 3a_1 + 5a_2 = 30$ that is a contradiction. \qed

So we get the following.

**Lemma 3.19.** Let $\mathcal{C}$ be a finite $\{0,1,2\}$-inversive plane such that no residual space is a near pencil, and let $P$ a point of $\mathcal{C}$ such that $\mathcal{C}_P$ is sporadic. If there exists $P_2$ such that type $\mathcal{C}_{P_2} \neq \mathcal{C}_P$. Then for any point $X \in \mathcal{C}$ we have

1. type $\mathcal{C}_X \in \{2c, 3d\}$,
2. type $\mathcal{C}_X \in \{2d, 2b\}$,
3. type $\mathcal{C}_X \in \{3d, 1c\}$,
4. type $\mathcal{C}_X \in \{4.g, 1.a, 4.e\}$,
5. type $\mathcal{C}_X \in \{4.h, 1.d, 3.b\}$.

**Lemma 3.20.** There exist $\{0,1,2\}$-inversive planes such that for any point $X \in \mathcal{C}$ one of the following occurs:

1. We have type $\mathcal{C}_X \in \{4.g, 1.a\}$.
2. We have type $\mathcal{C}_X \in \{4.h, 1.d\}$.
3. We have type $\mathcal{C}_X \in \{4.h, 3.b\}$.
4. We have type $\mathcal{C}_X \in \{3.d, 2.e\}$.
5. We have type $\mathcal{C}_X \in \{3.d, 1.c\}$.

**Proof.** (1) Let $\mathcal{C}$ be the affine geometry $AG(3, 2)$ with circles being the planes of $AG(3, 2)$ and one circle being replaced by four circles of length 3.

One easily verifies that $\mathcal{C}$ admits four residual spaces of type $1.a$ and four residual spaces of type $4.g$.

(2) Let $\mathcal{C}$ be the circular space constructed in (1) and remove one point of $X$ of $\mathcal{C}$ with the property that $\mathcal{C}_X$ is a projective plane of order 2 (type $1.a$). The so-constructed circular space has the required properties.
(3) We endow the set \{1, 2, 3, 4, 5, 6, 7\} with five circles 1234, 1567, 2456, 2367 and 3457 of degree 4 and 15 circles of degree 3. The so-obtained circular space \( \mathcal{C} \) has one residual space of type 3.b, namely \( \mathcal{C}_1 \), and six residual spaces of type 4.h.

(4) We endow the set \{1, 2, 3, 4, 5, 6\} with one circle 1234 of degree 4 and 16 circles of degree 3. The so-obtained circular space \( \mathcal{C} \) admits two residual spaces of type 2.c (namely \( \mathcal{C}_3 \) and \( \mathcal{C}_5 \)) and four residual spaces of type 3.d.

(5) We endow the set \{1, 2, 3, 4, 5, 6\} with two circles 1234 and 1256 of degree 4 and 12 circles of degree 3. The so-obtained circular space \( \mathcal{C} \) admits two residual spaces of type 1.c (namely \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \)) and four residual spaces of type 3.d.

Lemma 3.21. There is no circular space such that type \( \mathcal{C}_X \in \{4.h, 1.d, 3.b\} \) having residual of all three types.

Proof. Let \( a_1, a_2, a_3 \) be the number of residual spaces of type 4.h, 1.d and 3.b, respectively, and let \( P_1, P_2, P_3 \) be three points such that type \( \mathcal{C}_{P_1} = 4.h \), type \( \mathcal{C}_{P_2} = 1.d \) and type \( \mathcal{C}_{P_3} = 3.b \).

Since \( P_2 \in \mathcal{C}_{P_1} \), then through \( P_2 \) and \( P_3 \) there is a unique circle of length 4. On the other hand since \( P_3 \in \mathcal{C}_{P_2} \), then through \( P_3 \) and \( P_2 \) there are at least two circles of length 4, that is a contradiction.

This completes the proof of Theorem 3.1.

Finally we give more informations on case (4) of Lemma 3.19.

Let \( a_1, a_2, a_3 \) be the number of points such that residual spaces are of type 4.g, 1.a and 4.e respectively.

Then \( a_1 \geq 1, \ a_2 + a_3 \geq 1 \). Because of the previous lemma we may assume \( a_3 \geq 1 \).

\[ \begin{array}{ccc}
\text{type } \mathcal{C}_{P_1} = 4.g & \text{type } \mathcal{C}_{P_2} = 1.a & \text{type } \mathcal{C}_{P_3} = 4.e.
\end{array} \]

\[ ^2 \text{In the pictures we do not draw lines of degree 2.} \]
If \( a_2 \geq 1 \), then denote by \( P_1, P_2, P_3 \) three points such that type \( \mathcal{C}_{P_1} = 4.2 \), type \( \mathcal{C}_{P_2} = 1.a \) and type \( \mathcal{C}_{P_3} = 4.e \). We have
\[
a_1 + a_2 + a_3 = 8 \quad \text{and} \quad 4b_4 = 6a_1 + 7a_2 + 5a_3 = 6a_1 + 7(8 - a_1 - a_3) + 5a_3,
\]
it follows that \( b_4 = 14 - (a_1 + 2a_3)/4 \). So \( a_1 + 2a_3 = 0 \) (mod 4). But \( 3b_3 = 3a_1 + 6a_3 \) that is \( b_3 = a_1 + 2a_3 \) (there are six circles of degree 3 through \( P_3 \) so \( a_1 + 2a_3 > 8 \). But in \( \mathcal{C}_{P_3} \), there is only one point of degree 3 and it is \( P_2 \) hence \( a_2 = 1 \); so \( a_1 = 6 \), \( a_3 = 1 \). Moreover \( b_4 = 12 \), \( b_3 = 8 \).

If \( a_2 = 0 \), then in a similar way we obtain \( a_1 = a_3 = 4 \) and \( b_4 = 11 \), \( b_3 = 12 \).

### 4. The classification

We start with the definition of the extension of the Nwankpa plane. We shall use the Nwankpa plane \( \mathcal{N} \) defined in Section 2.

**Definition 4.1.** Let \( \mathcal{C} \) be the following circular space. The 12 points of \( \mathcal{C} \) are denoted by \( 1, 2, 3, 4, 5, 6, 1', 2', 3', 4', 5', 6' \). There are two circles of length 6, namely 123456 and 1'2'3'4'5'6'. For the definition of the 45 circles of length 4 we say that the points 1, 1' of \( \mathcal{C} \) correspond to the point \( \bar{1} \) of \( \mathcal{N} \). In the same way the points \( 2, 2', \ldots, 6, 6' \) correspond to \( 2, 3, 4, 5, 6, \) respectively. For a point \( x \) of \( \mathcal{C} \) let \( \bar{x} \) be the point of \( \mathcal{N} \) corresponding to \( x \). Let \( \{x, y, x', y'\} \) be a set of four points of \( \mathcal{C} \) with \( x, y \in \{1, 2, 3, 4, 5, 6\} \) and \( x', y' \in \{1', 2', 3', 4', 5', 6'\} \). By definition, \( \{xyx'y'\} \) is a circle if and only if \( \infty(x, y) = \infty(x', y') \). \( \mathcal{C} \) is called the extension of the Nwankpa plane.

One of the most difficult cases is to classify all finite circular spaces such that any residual space is of type 4.d.

**Proposition 4.2.** Let \( \mathcal{C} \) be a finite circular space such that every residual space is of type 4.d for some \( k \). Let \( n \) be the order of one of the residual spaces.

(a) Then \( n \) is even.

(b) We have either \( n = 4 \), \( k = 2 \) and \( \mathcal{C} \) is a 3-(22, 6, 1) design with two points deleted or \( 12(n + 1) = k(k - 1)(k - 2) \) and \( 4 \leq k \leq n \).

**Proof.** Let \( \mathcal{C} \) be a circular space such that all residual spaces are of type 4.d. Let \( P \) be a point of \( \mathcal{C} \), and let \( n \) be the order of \( \mathcal{C}_P \). Then there exists an integer \( k \) with \( 2 \leq k \leq n + 1 \) such that \( \mathcal{C}_P \) is of type 4.d_\( k \). By Lemma 3.6, it follows that for any point \( Q \) of \( \mathcal{C} \) the residual space \( \mathcal{C}_Q \) is of type 4.d_\( k \).

(a) Consider the residual space \( \mathcal{C}_Q \) for some point \( Q \) of \( \mathcal{C} \). Since \( \mathcal{C}_Q \) is of type 4.d_\( k \), there exists a finite projective plane \( \mathcal{P} \) of order \( n \) admitting a \( k \)-arc \( \mathcal{H} \) such that \( \mathcal{C}_Q \) is obtained from \( \mathcal{P} \) by deleting the points of \( \mathcal{H} \). Since \( k \leq n + 1 \), there is a line \( l \) in \( \mathcal{P} \) missing \( \mathcal{H} \), that is, there is a line \( l \) of \( \mathcal{C}_Q \) of degree \( n + 1 \). Let \( P \) be a point of \( \mathcal{C}_Q \) which is not on \( l \), and let \( x \) be the circle defined by \( l \). Then the points of \( x \) form an \((n + 2)\)-arc in \( \mathcal{C}_P \). By a well-known result on arcs, this implies that \( n \) is even.
(b) We first show that \( k \neq n+1 \). Assume that \( k = n+1 \), let \( P \) be a point of \( \mathcal{C} \), let \( \mathcal{P} \) be a projective plane of order \( n \) admitting an \((n+1)\)-arc \( \mathcal{K} \) such that \( \mathcal{C}_P \) is obtained from \( \mathcal{P} \) by deleting \( \mathcal{K} \). Since \( n \) is even, \( \mathcal{K} \) admits a nucleus \( Q \).

In \( \mathcal{C}_Q \) we have the same situation as in \( \mathcal{C}_P \) with a nucleus \( P' \). Since the lines of \( \mathcal{C}_Q \) of degree \( n \) are exactly the lines through \( P' \), it follows that \( P = P' \). Therefore the point set of \( \mathcal{C} \) splits into pairs. In particular the number \( v \) of points of \( \mathcal{C} \) is even. But

\[
v = n^2 + n + 2 - k = n^2 + n + 2 - (n + 1) = (n - 1)(n + 1) + 2\]

is odd, since \( n \) is even, a contradiction.

Let \( b_n, b_{n+1} \) and \( b_{n+2} \) be the numbers of circles of length \( n, n + 1 \) and \( n + 2 \), respectively. Considering the incident point–circle pairs \((P, x)\), where \( x \) is a circle of length \( n, n + 1 \) and \( n + 2 \), respectively, we get the following three equations:

\[
2b_n = (n + 1)k^2 - (n + 1)k - \frac{k(k - 1)(k - 2)}{n},
\]

\[
b_{n+1} = -(n + 1)k^2 + (n^2 + 2n + 2)k + \frac{k(k - 1)(k - 2)}{n + 1},
\]

\[
2b_{n+2} = (n + 1)k^2 - (2n^2 + 3n + 3)k + 2n^3 + 8n - 10 - \frac{k(k - 1)(k - 2) - 24}{n + 2}.
\]

Let \( c := k(k - 1)(k - 2) \). From the above equations it follows that \( n \mid c, n + 1 \mid c \) and \( n + 2 \mid c - 24 \). Since \( n \) and \( n + 1 \) are relatively prime, we have \( n(n + 1) \mid c \). Hence there are two integers \( a \) and \( b \) such that \( c = an(n + 1) \) and \( c = b(n + 2) + 24 \).

It follows that \( an(n + 1) = b(n + 2) + 24 \) or, equivalently, \((a(n + 1) - b)n = 2b + 24 \). Let \( d := a(n + 1) - b \). Then \( d \) is an integer, and we have \( dn = 2b + 24 \), that is, \( b = dn/2 - 12 \). It follows that

\[
an(n + 1) = \left( \frac{dn}{2} - 12 \right)(n + 2) + 24.
\]

Hence \((2a - d)(n + 2) = 2a - 24 \). Let \( e := 2a - d \). Then \( e \) is an integer with \( e(n + 2) = 2a - 24 \), that is, \( a = (e(n + 2) + 24)/2 \). It follows that

\[
c = k(k - 1)(k - 2) = an(n + 1) = \frac{e(n + 2) + 24}{2}n(n + 1)
\]

\[
= \frac{1}{2}(en^3 + (3e + 24)n^2 + (2e + 24)n).
\]

Since \( 2 \leq k \leq n + 1 \), it follows from \( c = k(k - 1)(k - 2) \) that \( c \leq (n + 1)n(n - 1) = n^3 - n \). We get

\[
en^3 + (3e + 24)n^2 + (2e + 24)n = 2c \leq 2n^3 - 2n.
\]

Hence \( e \leq 1 \).

On the other hand, we have

\[
0 \leq c = \frac{1}{2}(en^3 + (3e + 24)n^2 + (2e + 24)n)
\]

\[
= \frac{1}{2}n(n + 1)(en + 24).
\]
Hence $e(n+2) \geq -24$. Because of $n \geq 4$ ($n$ is even and $n \geq 3$) it follows that $e \geq -4$. We shall treat the cases $e = -4, -3, \ldots, 1$ separately. Recall that $k(k - 1)(k - 2) = e = \frac{1}{2}n(n + 1)(e(n + 2) + 24)$.

Case 1: Let $e = -4$. From $e(n + 2) \geq -24$ and $n \geq 4$ follows $n = 4$. The above equation yields $k = 2$.

Case 2: Let $e = -3$. Again in view of $e(n + 2) \geq -24$ we get $n \leq 6$. Since $n$ is even and since there is no projective plane of order 6, we get $n = 4$. It follows that $k = 5$. But we have already seen that $k \neq n + 1$, so Case 2 does not occur.

Case 3: Let $e = -2$. As above it follows that $n \in \{4, 8\}$. If $n = 4$, then $k = 6$ contradicting $k \leq n + 1$. For $n = 8$ there is no integer $k$ fulfilling the above equation.

Case 4: Let $e = -1$. Using the Theorem of Bruck and Ryser we get $n \in \{4, 8, 12, 16, 18, 20\}$. The only integral solution of the above equation is $n = 8$ and $k = 9$. Since $k \neq n + 1$, this case cannot occur.

Case 5: Let $e = 0$. Then we have $a = (e(n + 2) + 24)/2 = 12$. It follows that

$$k(k - 1)(k - 2) = 12n(n + 1).$$

Case 6: Let $e = 1$. Then we have $a = \frac{1}{2}(n + 26)$. Since $n$ is even, there is an integer $m$ with $n = 2m$. It follows that $a = m + 13$. We also have

$$2b_n = (n + 1)k^2 - (n + 1)k - \frac{k(k - 1)(k - 2)}{n}$$

$$= (n + 1)(k^2 - k) - a(n + 1)$$

$$= (n + 1)(k^2 - k) - (m + 13)(n + 1).$$

Since $2b_n$ and $k^2 - k$ are even and since $n + 1$ is odd, it follows that $m + 13$ is even, hence $m$ is odd.

On the other hand, from $a = \frac{1}{2}(n + 26)$, $d = 2a - e$, $e = 1$ and $b = dn/2 - 12$ follows that $b = (n + 25)m - 12$. Since $m$ is odd and $n$ is even, $b$ is odd too. From

$$2b_{n+2} = (n + 1)k^2 - (2n^2 + 3n + 3)k + 2n^3 + 8n - 10 - \frac{k(k - 1)(k - 2) - 24}{n + 2}$$

$$= nk^2 - (2n^2 + 3n + 2)k + 2n^3 + 8n - 10 + k^2 - k - b$$

follows that $b$ is even, a contradiction. So the case $e = 1$ cannot occur.

By using Lemma 3.4.5 in [5] on $\{1, s\}$-inversive planes, for the case $s = 2$, we get that there is no circular space such that all residual spaces are of type 3.b. For sake of completeness we give the proof in our case.

**Proposition 4.3.** There is no finite circular space $\mathcal{C}$ such that every residual space is of type 3.b.

**Proof.** By way of contradiction, let $\mathcal{C}$ be a circular space, of order $n$, such that all derived structures are of type 3.b. Then $|\mathcal{C}| = n^2 - n + 1$. Through any point there are
\( n - 1 \) circles of degree \( n + 1 \) and \( n^2 \) circles of degree \( n \). In any derived structure \( \mathcal{C}_p \) lines of degree \( n \) give a partition, that is in \( \mathcal{C} \) for any pair of distinct points there is exactly one circle of degree \( n + 1 \). This means that the set of points of \( \mathcal{C} \), together with the set of circles of degree \( n + 1 \) form a linear space. In such a linear space all lines have degree \( n + 1 \) and all points have degree \( n - 1 \), that is a contradiction. \( \square \)

**Theorem 4.4.** Let \( \mathcal{C} \) be a finite \( \{0, 1, 2\} \)-inversive plane such that all residual spaces are of the same type. Then one of the following occurs:

(a) \( \mathcal{C} \) is an inversive plane. (Cases 1.1 and 1.2).

(b) \( \mathcal{C} \) is an inversive plan, where one point has been deleted. (Cases 2.2 and 5.1).

(c) \( \mathcal{C} \) is either a 3-(8,4,1) or a 3-(22,6,1) design. (Case 1.2).

(d) \( \mathcal{C} \) is a 3-(8,4,1) or a 3-(22,6,1) design with one point deleted. (Case 1.3).

(e) \( \mathcal{C} \) has six points, say 1,2,3,4,5,6 with three circles 1234, 1256 and 3456 of degree 4 and 11 circles of degree 3. (Case 1.4).

(f) \( \mathcal{C} \) is degenerate on four points. (Case 1.5).

(g) \( \mathcal{C} \) has six points. All circles are of degree 3. (Case 4.1).

(h) We have \( n = 4 \), and all residual spaces are of type 2.a. An example is obtained from a 3-(22,6,1) design by deleting a circle. (Case 4.2).

(i) We have \( n = 4 \), and all residual spaces are of type 2.b. (Case 4.3).

(l) We have \( n = 5 \), and all residual spaces are of type 4.a. (Case 6.3).

(m) We have \( n = 3, 5, 11 \), and all residual spaces are of type 4.e. If \( n = 3 \) then an inversive plane of order 3 with two points deleted gives an example. (Case 6.4).

(\( n \)) Let \( \mathcal{C}' \) be the 3-(8,4,1) design, and let \( x \) and \( y \) be two disjoint circles of degree 4. Let \( \mathcal{C} \) be the circular space constructed from \( \mathcal{C}' \) by replacing each of the circles \( x \) and \( y \) by four circles of degree 3. (Case 6.5).

(o) \( \mathcal{C} \) is the circular space described in Definition 4.1. (Case 6.6).

(p) Every residual space is of type 4.\( d_k \), that is, it is obtained from a projective plane of even order \( n \) by deleting a \( k \)-arc, where \( k \) and \( n \) fulfill the following conditions:

\[(k - 2)(k - 1)k = 12n(n + 1)\quad \text{and} \quad 4 \leq k \leq n \quad \text{(Case 6.2)}\]

or \( n = 4 \), \( k = 2 \) and \( \mathcal{C} \) is a doubly punctured 3-(22,6,1) design.

**Proof.** Let \( \mathcal{C} \) be a finite \( \{0, 1, 2\} \)-inversive plane such that all residual spaces are of the same type. Let \( P \) be a point of \( \mathcal{C} \). Let \( n \) be the order of \( \mathcal{C}_p \). Throughout the proof we denote by \( b_j \) the number of circles of length \( j \).

Case 1: \( \mathcal{C}_p \) is of type from 1.a to 1.e. It follows that \( \mathcal{C} \) is \( \{0, 1\} \)-inversive and from 1.1 we get the circular spaces described in (a),(c),(d),(e),(f).

Case 2: \( \mathcal{C}_p \) is of type 2.a. Let \( S \) be the set of all pairs \((P,x)\), where \( P \) is a point and \( x \) is a circle of length \( n + 2 \) such that \( P \) and \( x \) are incident. Then we get

\[(n + 2)b_{n+2} = \frac{1}{2}n^2(n^2 - n).

It follows that \( n + 2 \) is a divisor of 12, that is, \( n \in \{2, 4, 10\} \). Since residual spaces of type 2.a are only defined for \( n \geq 3 \) and since there are no projective planes of order
10, we have \( n = 4 \). This yields the case described in (h).

Case 3: \( \mathcal{C}_P \) is of type 2.b. As in the proof of Case 6, we obtain \( n \in \{2, 4, 6, 10, 22\} \).

By the theorem of Bruck and Ryser, the cases \( n = 6 \) and 22 are excluded. As before it follows that \( n = 4 \). This yields the case described in (i).

Case 4: \( \mathcal{C}_P \) is of type 2.c. Obviously any circle is of length 3, that is, \( \mathcal{C} \) consists of six points and any three of them form a circle (see (g)).

Case 5: \( \mathcal{C}_P \) is of type 2.d. It follows that \( b_5 = 16 \), a contradiction.

Case 6: \( \mathcal{C}_P \) is of type 2.e. It follows that \( b_5 = 14 \), a contradiction.

Case 7: \( \mathcal{C}_P \) is of type 3.a. Observe that the circles of \( \mathcal{C} \) are either of degree \( n \) or \( n + 1 \). We add a point \( \infty \) to \( \mathcal{C} \) such that \( \infty \) is incident with exactly the circles of degree \( n \). Let \( \mathcal{C} \) be the resulting circular spaces. One easily verifies that \( \mathcal{C} \) is an inversive plane, thus \( \mathcal{C} \) is the circular space described in (b).

Case 8: \( \mathcal{C}_P \) is of type 3.c. It follows that \( b_5 = \frac{13}{5} \), a contradiction.

Case 9: \( \mathcal{C}_P \) is of type 3.d. It follows that \( b_4 = \frac{4}{3} \), a contradiction.

Case 10: \( \mathcal{C}_P \) is of type 4.a. Considering the circles of length \( n + 1 \), one obtains that \( n + 1 \) is a divisor of 6, that is, \( n = 5 \). This yields the case described in (l).

Case 11: \( \mathcal{C}_P \) is of type 4.b. It follows that \( n + 1 \) is a divisor of 3, a contradiction.

Case 12: \( \mathcal{C}_P \) is of type 4.c. Considering the circles of length \( n + 1 \) and \( n + 2 \), one obtains the following

\[
 b_{n+2} = n - 3 + \frac{9}{n+2}, \quad \text{hence } n = 7
\]

and

\[
 b_{n+1} = 3n^2 - 9n + 21 - \frac{30}{n+1}, \quad \text{hence } n \neq 7
\]

a contradiction.

Case 13: \( \mathcal{C}_P \) is of type 4.d.k. This case has been considered in Proposition 4.2.

Case 14: \( \mathcal{C}_P \) is of type 4.e. Considering the circles of length \( n + 1 \) we get

\[
 b_{n+1} = 2n^2 - 5n + 10 - \frac{12}{n+1}.
\]

It follows that \( n \in \{3, 5, 11\} \). This yields the case described in (m).

Case 15: \( \mathcal{C}_P \) is of type 4.f. One easily verifies that \( \mathcal{C} \) is the circular space described in (o).

Case 16: \( \mathcal{C}_P \) is of type 4.g. One easily verifies that \( \mathcal{C} \) is the circular space described in (n).

Case 17: \( \mathcal{C}_P \) is of type 4.h. It follows that \( b_4 = \frac{73}{7} \), a contradiction.

Case 18: \( \mathcal{C}_P \) is of type 4.i. It follows that \( b_5 = \frac{73}{5} \), a contradiction.

Case 19: \( \mathcal{C}_P \) is of type 4.j. It follows that \( b_5 = \frac{8}{5} \), a contradiction.

Note that it is not clear whether the circular spaces described in (i), (l), (m) (two cases), (p) \( (n \neq 4) \) exist.
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References