Apart from hyperovals and their duals there are only three classes of maximal arcs known in Desarguesian projective planes. Two classes are due to J. A. Thas and one to R. H. F. Denniston. In this paper collineation stabiliser and isomorphism problems for those maximal arcs in Desarguesian projective planes are examined. The full collineation stabilisers of the known maximal arcs are calculated, and it is shown that all of one class of Thas maximal arcs and those of the second class of Thas maximal arcs in Desarguesian projective planes arising from elliptic quadrics are isomorphic to those of Denniston. The final result is to classify maximal arcs in Desarguesian projective planes whose collineation stabilisers are transitive on the points of the maximal arcs.

1. INTRODUCTION

In a finite projective plane of order $q$, a $\{k; n\}$-arc $\mathcal{K}$ is a non-empty proper subset of $k$ points of the plane such that some line of the plane meets $\mathcal{K}$ in $n$ points, but no line meets $\mathcal{K}$ in more than $n$ points [18]. For a given $q$ and $n$, the size $k$ can not exceed $q(n - 1) + n$. If equality occurs the set is called a maximal arc [6], and $n$ is called the degree of $\mathcal{K}$. Equivalently, a maximal arc can be defined as a non-empty, proper subset of

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points such that every line meets the set in 0 or \( n \) points, for some \( n \). For example, any point of a projective plane of order \( q \) is a maximal \( \{ 1; 1 \} \)-arc in that plane, and the complement of any line is a maximal \( \{ q^2; q \} \)-arc. We shall refer to these two types of maximal arcs as **trivial**.

If \( \mathcal{K} \) is a maximal \( \{ q(n-1)+n; n \} \)-arc, the set of lines external to \( \mathcal{K} \) is a maximal \( \{ q(q-n+1)/n; q/n \} \)-arc in the dual plane called the dual of \( \mathcal{K} \) [6]. It follows that a necessary condition for the existence of a maximal \( \{ q(n-1)+n; n \} \)-arc in a projective plane of order \( q \) is that \( n \) divides \( q \). R. H. F. Denniston has shown that in the Desarguesian projective plane \( PG(2, q) \) of order \( q \) that this necessary condition is sufficient when \( q \) is even [12]. Ball et al. have shown that no non-trivial maximal arcs exist in \( PG(2, q) \) when \( q \) is odd [4, 5].

The purpose of this paper is to examine the collineation stabilisers of maximal arcs in Desarguesian projective planes. Apart from hyperovals and their duals, which have a literature of their own, there are three classes of non-trivial maximal arcs known in \( PG(2, q) \), they are those due to R. H. F. Denniston and two classes due to J. A. Thas. In Section 2 the collineation stabilisers of the Denniston maximal arcs are calculated (Theorem 2.3), and the dual maximal arcs are identified (Theorem 2.4). In Section 3, the maximal arcs of the first class of Thas arising from elliptic quadrics and all of the second class in \( PG(2, q) \) are shown to be isomorphic to certain (Corollary 3.2) of the Denniston maximal arcs (Theorem 3.6). The stabilisers and isomorphism classes of those Thas maximal arcs in \( PG(2, q) \) that arise from Tits ovoids are also calculated (Corollary 3.1). In Section 4 maximal arcs in \( PG(2, q) \) whose collineation stabilisers are transitive on the points of the maximal arc are classified (Theorem 4.2).

### 2. STABILISERS OF DENNISTON MAXIMAL ARCS

In this section the collineation stabilisers of the maximal arcs constructed by Denniston are calculated. The additive subgroups of the dual maximal arcs are also given. In the following \( \text{trace}_{GF(2^d)/GF(2^e)} \) will denote the usual trace function from \( GF(2^d) \) to \( GF(2^e) \), i.e., for \( \lambda \in GF(2^d) \),

\[
\text{trace}_{GF(2^d)/GF(2^e)}(\lambda) = \lambda + \lambda^2 + \cdots + \lambda^{(2^d-1)/2^e}.
\]

In the following the Desarguesian plane \( PG(2, 2^e) \) is represented via homogeneous coordinates over the Galois field \( GF(2^e) \); i.e., represent the points of \( PG(2, 2^e) \) by \( \langle (x, y, z) \rangle \), \( x, y, z \in GF(2^e) \) and \( (x, y, z) \neq (0, 0, 0) \), and similarly lines by \( \langle [a, b, c] \rangle \), \( a, b, c \in GF(2^e) \) and \( [a, b, c] \neq [0, 0, 0] \).
Incidence is given by the dot product $\langle (x, y, z) I [a, b, c] \rangle \iff ax + by + cz = 0$. To avoid awkward notation the angle brackets will be dropped in the following.

Let $\xi^2 + \eta^2 + 1$ be an irreducible polynomial over $GF(2^e)$, and let $\mathcal{F}_{std}$ be the set of conics given by the pencil

$$F_\lambda : x^2 + axy + y^2 + \lambda z^2 = 0, \quad \lambda \in GF(2^e) \cup \{ \infty \}.$$ 

Then $F_0$ is the point $(0, 0, 1)$, $F_\infty$ is the line $z^2 = 0$ (which we shall call the line at infinity). Every other conic in the pencil is non-degenerate and has nucleus $F_0$. Further, the pencil is a partition of the points of the plane. For convenience, this pencil of conics will be referred to as the standard pencil.

In it is well known (see for instance [1]) that the pencil $\mathcal{F}_{std}$ is stabilised by the following cyclic group of order $2^e + 1$,

$$C_{2^e+1} = \left\{ \begin{pmatrix} a+zb & b & 0 \\ b & a & 0 \\ 0 & 0 & 1 \end{pmatrix} : a^2 + zab + b^2 = 1 \right\}.$$ 

Note that all cyclic subgroups of $PGL(3, 2^e)$ of order $2^e + 1$ are conjugate in $PGL(3, 2^e)$ (see for instance [1, Lemma 6]). Hence, up to isomorphism, the choice of pencil of conics is arbitrary (as long as the pencil is stabilised by a cyclic linear group of order $2^e + 1$).

In 1969, R. H. F. Denniston proved the following theorem.

**Theorem 2.1 (Denniston [12]).** If $A$ is an additive subgroup of $GF(2^e)$ of order $n$, then the set of points of all $F_\lambda$ for $\lambda \in A$ form a maximal $\{2^e(n-1)+n; n\}$-arc in $PG(2, 2^e)$.

These maximal arcs were then characterised by Abatangelo and Larato in the following theorem.

**Theorem 2.2 (Abatangelo and Larato [1]).** If a maximal arc $\mathcal{K}$ in $PG(2, 2^e)$, is invariant under a linear collineation group of $PG(2, 2^e)$ which is cyclic and has order $2^e + 1$, then $\mathcal{K}$ is a Denniston maximal arc.

In the next theorem the full collineation stabilisers of the Denniston maximal arcs are calculated. The theorem does not hold for Denniston maximal arcs that are (regular) hyperovals or duals of (regular) hyperovals, so we first state some well known results for these two cases.

The full collineation group of a regular hyperoval in $PG(2, 2^e)$ (and so the dual of a regular hyperoval) is isomorphic to $PGL(2, 2^e)$ for $e > 2$,
and is isomorphic to $S_4$ and $S_6$ for $e = 1$ and 2, respectively [18, 8.4.2 Corollary 6; 25]. A maximal arc formed from the dual of a regular hyperoval has the interesting property that the full collineation stabiliser of the maximal arc is transitive on the points of the maximal arc (see Section 4.)

**Theorem 2.3.** In $PG(2, 2^e)$, $e > 2$, let $\mathcal{K}$ be a degree $n$ Denniston maximal arc, $q = 2^e$, $2 < n < q/2$, with additive subgroup $A$. Define the group $G$ acting on $GF(2^e)$ by

$$G = \{ x \mapsto ax^\sigma : a \in GF(2^e)^*, \sigma \in \text{Aut } GF(2^e) \}.$$  

Then the collineation stabiliser of $\mathcal{K}$ is isomorphic to $C_{2^{e+1}} \rtimes G_A$, the semidirect product of a cyclic group of order $(2^e + 1)$ with the stabiliser of $A$ in $G$.

**Proof.** Let $\mathcal{F} = \{ F_\lambda : \lambda \in GF(2^e) \cup \{ \infty \} \}$, with $F_0$ the nucleus and $F_\infty$ the line at infinity, be the pencil of conics associated with $\mathcal{K}$, and $C_{2^{e+1}}$ be a linear cyclic group of order $2^e + 1$ stabilising $\mathcal{K}$. Notice that the orbits of $C_{2^{e+1}}$ are the elements of the pencil $\mathcal{F}$. We show that the group $PGL(3, q)_\mathcal{K}$ of collineations that stabilise $\mathcal{K}$ is a subgroup of the normaliser $N_{PGL(3, q)}(C_{2^{e+1}})$ of $C_{2^{e+1}}$ in the collineation group of the plane.

Let $g \in PGL(3, q)_\mathcal{K}$, and let $F_\mu, \mu \in A - \{ 0 \}$ be any conic of $\mathcal{F}$ contained in $\mathcal{K}$. Then $F_\infty$ is also a conic contained in $\mathcal{K}$. There are $n-1$ conics in the set $\{ F_\lambda : \lambda \in A - \{ 0 \} \}$. Hence since $n < q/4$ and there are at least $q$ points of $F_\infty$ not on the nucleus of the pencil, it follows that

$$\max \{ |F_\mu \cap F_\lambda| : \lambda \in A - \{ 0 \} \} \geq \frac{q}{n-1} \geq \frac{q}{q/4 - 1} > 4$$

and so $F_\mu$ meets some (non-degenerate) conic of $\mathcal{F}$ in at least 5 points. But 5 points of a non-degenerate conic determine that conic, and so $F_\mu = F_\lambda$ for some $\lambda \in A$. Hence $g$ permutes the conics of $\mathcal{F}$ in $\mathcal{K}$, and $g$ also fixes the nucleus $F_0$. Dually, $g$ fixes the nucleus of the dual (Denniston) maximal arc, and so fixes the line at infinity $F_\infty$ of $\mathcal{F}$.

Notice that a non-degenerate conic $F_\mu$ of $\mathcal{F}$ together with the line at infinity $F_\infty$ uniquely determine a pencil of the form required for the Denniston construction. One way to see this is to notice that the equation of any conic of the pencil can be written as a linear combination of the equation of $F_\mu$ and the equation of $F_\lambda$, and so any collineation group (i.e., $C_{2^{e+1}}$) that preserves $F_\mu$ and $F_\infty$ must preserve all the conics of the pencil. Another way to see this is to note that there is a unique homology group $H$ of order $2^e - 1$ with centre $F_\mu$ and axis $F_\infty$. The conics of the pencil $\mathcal{F}$ are then (uniquely) given as $\{ F_\mu^h : h \in H \}$.

Now $g$ permutes the set of conics of $\mathcal{F}$ in $\mathcal{K}$, and since $g$ fixes the line at infinity, it follows from the previous paragraph that $g$ also permutes the conics of $\mathcal{F}$. Hence $g$ is an element of the collineation stabiliser of the
pencil. We show that the collineation stabiliser of the pencil is the normaliser \( N_{PFL(3, q)}(C_{2^e+1}) \), and so \( g \in N_{PFL(3, q)}(C_{2^e+1}) \).

First note that any \( n \in N_{PFL(3, q)}(C_{2^e+1}) \) permutes the orbits of \( C_{2^e+1} \), and so is an element of the collineation stabiliser of the pencil.

The stabiliser of the pencil is transitive on the \( 2^e - 1 \) non-degenerate conics of the pencil. From the comments in the previous paragraphs, the order of the collineation stabiliser of the pencil is then \( 2^e - 1 \) times the order of the collineation stabiliser of a non-degenerate conic \( F_\infty \) of the pencil and the line \( F_\infty \). But the collineation stabiliser of a non-degenerate conic is isomorphic to \( PFL(2, q) \), which is transitive on lines external to the conic. There are \( \frac{1}{2}(2^e - 2^2) \) such lines (the number of points of the dual of a hyperoval), and so the order of the collineation stabiliser of \( F_\infty \) and \( F_\infty \) is \( |PFL(2, 2^e)|/\left(\frac{1}{2}(2^e - 2^2)\right) = 2e(2^e+1) \). Hence the order of the collineation stabiliser of the pencil is \( 2e(2^e-1) \).

The centraliser \( C_{PFL(3, q)}(C_{2^e+1}) \) is normal in \( N_{PFL(3, q)}(C_{2^e+1}) \), and it follows by [21, Theorem II.7.3] that

\[
N_{PFL(3, q)}(C_{2^e+1}) \cong GF(2^{2e}) \times \Aut GF(2^{2e}).
\]

So the order of \( N_{PFL(3, q)}(C_{2^e+1}) \) is also \( 2e(2^e - 1) \). And hence \( N_{PFL(3, q)}(C_{2^e+1}) \) is exactly the collineation stabiliser of the pencil, and so \( g \in N_{PFL(3, q)}(C_{2^e+1}) \).

So we have shown that \( PFL(3, q) \leq N_{PFL(3, q)}(C_{2^e+1}) \). We now examine how this normaliser acts on the plane.

The centraliser \( C_{PFL(3, q)}(C_{2^e+1}) \cong GF(2^{2e}) \) acts regularly on the points of the plane not on \( F_\infty \), and not equal to \( F_0 \), and so we may identify such points with elements of \( GF(2^{2e}) \) by taking an arbitrary point \( P \) not on \( F_\infty \), and not equal to \( F_0 \), and identifying it with \( 1 \in GF(2^{2e}) \). The group \( N_{PFL(3, q)}(C_{2^e+1}) \) then acts on \( GF(2^{2e}) \) by

\[
x \mapsto ax^\sigma, \quad a \in GF(2^{2e}), \quad \sigma \in \Aut GF(2^{2e}).
\]

Let \( \eta \) be primitive in \( GF(2^{2e}) \). Define \( r = (2^{2e} - 1)/(2^e - 1) = 2^e + 1 \) and \( s = (2^e - 1)/(2^e + 1) = 2^e - 1 \), then \( \eta' \) has order \( 2^e - 1 \) and \( \eta'' \) has order \( 2^e + 1 \) in \( GF(2^{2e}) \). The orbits of \( \langle \eta'^* \rangle \) on the plane are then the conics of the pencil. The conics can then be described by the sets

\[
C_{\eta'^*} = \{ \eta'^*, \eta'^{s+1}, \eta'^{2s+1}, ..., \eta'^{r}\}
\]

\[
C_{\eta''^*} = \{ \eta'', \eta''^{s+1}, \eta''^{2s+1}, ..., \eta''^{r}\}
\]

\[
C_{\eta'''^*} = \{ 1, \eta'^*, \eta'^{2}, ..., \eta'^{r}\}.
\]
Notice that \( \eta \) is a homology that cyclicly permutes the conics. Also \( \langle \eta' \rangle = GF(2^e) \times GF(2^{e+1}) \), and as long as we chose the point of the plane corresponding to 1 to be a point of the conic \( F_1 \), then \( F_{\eta'} = C_{\eta'} \) for every \( i = 1, ..., 2^e - 1 \).

We now show that \( N_{PGL(3,q)}(C_{2^e+1}) \cong \langle \eta' \rangle \rtimes (\langle \eta' \rangle \rtimes \text{Aut} GF(2^{e+1})). \)

First notice that \( \langle \eta' \rangle \) and \( \langle \eta' \rangle \) are normal subgroups of \( N_{PGL(3,q)}(C_{2^e+1}) \). Consider the group given by the product \( \langle \eta' \rangle \rtimes \text{Aut} GF(2^{e+1}) \). The group \( \langle \eta' \rangle \) acts semiregularly on \( GF(2^{2e+1}), \) in particular the stabiliser of 1 is the identity. The stabiliser of 1 in the group \( \text{Aut} GF(2^e) \) is \( \text{Aut} GF(2^e) \), and so \( \text{Aut} GF(2^e) \) is a complement for \( \langle \eta' \rangle \rtimes \text{Aut} GF(2^{e+1}) \). Hence \( \langle \eta' \rangle \rtimes \text{Aut} GF(2^{e+1}) \cong \langle \eta' \rangle \rtimes \text{Aut} GF(2^{e+1}). \)

Since \( 2^e - 1 \) and \( 2^e + 1 \) are coprime, it follows that the group given by the product \( \langle \eta' \rangle \rtimes \text{Aut} GF(2^{e+1}) \) is a complement for \( \langle \eta' \rangle \rtimes \text{Aut} GF(2^{e+1}) \) in \( N_{PGL(3,q)}(C_{2^e+1}) \). Also, arguing as in the previous paragraph, it follows that \( \text{Aut} GF(2^e) \) is a complement for \( \langle \eta' \rangle \rtimes \text{Aut} GF(2^{e+1}) \) in \( \langle \eta' \rangle \rtimes \text{Aut} GF(2^{e+1}). \) Hence \( N_{PGL(3,q)}(C_{2^e+1}) \cong \langle \eta' \rangle \rtimes (\langle \eta' \rangle \rtimes \text{Aut} GF(2^{e+1})). \)

The stabiliser of the maximal arc \( \mathcal{X} \) is then given by
\[
\langle \eta' \rangle \rtimes (\langle \eta' \rangle \rtimes \text{Aut} GF(2^{e+1}))_G.
\]

The result follows.

**Corollary 2.1.** In \( PG(2, 2^e) \) let \( \mathcal{F} \) be a pencil of conics stabilised by a linear cyclic group of order \( 2^e + 1 \), with \( F_0 \) the nucleus, and \( F_\infty \) the line at infinity. Let \( \mathcal{X}_1 \) and \( \mathcal{X}_2 \) be Denniston maximal arcs constructed from \( \mathcal{F} \) and additive subgroups \( A_1 \) and \( A_2 \) respectively. Then \( \mathcal{X}_1 \) is isomorphic to \( \mathcal{X}_2 \) if and only if \( A_1 = A_2^g \) for some \( g \in G \), where
\[
G = \{ x \mapsto \sigma x^\sigma : a \in GF(2^e)^*, \sigma \in \text{Aut}(GF(2^e)) \}.
\]

**Proof.** The Corollary follows from the details of the proof of the Theorem and by considering the action of \( \text{Aut} GF(2^{e+1}) \) on \( GF(2^e)^* \). The automorphism \( x \mapsto x^2 \) acts as the identity on \( GF(2^e)^* \). Hence \( \text{Aut} GF(2^{e+1}) \) acts on \( GF(2^e) \) as \( \text{Aut} GF(2^e) \) with kernel \( \langle x \mapsto x^2 \rangle \).

Notice that since \( x \mapsto x^2 \) acts as the identity on \( GF(2^e) \), it fixes pointwise the line containing the points corresponding to \( GF(2^e) \). It follows that this line is the axis of the collineation, and since the collineation has order 2 it must be an elation. Also, since each conic \( C_{\eta'} \) contains exactly one point of this line, each of the conics is stabilised by this collineation. Direct calculation then show that \( x \mapsto x^2 \) together with the cyclic group \( C_{2^e+1} \) generate a dihedral group which stabilises the set of conics of the pencil pointwise, and so any Denniston maximal arc in \( PG(2, 2^e) \) admits a dihedral group of order \( 2(2^e + 1) \) in its collineation stabiliser.
Using this Corollary it is easy to show by direct calculation in $GF(16)$ that there are exactly 2 degree 4 Denniston maximal arcs in $PG(2, 16)$ up to isomorphism. Let $\omega$ be a primitive element in $GF(16)$ such that $\omega^4 + \omega = 1$. Then representatives for the additive subgroups may be described by

$$A_1 = GF(4) = \{0, 1, \omega^3, \omega^{10}\} \quad \text{and} \quad A_2 = \{0, 1, \omega^1, \omega^4\}.$$ 

They give rise to maximal arcs with full collineation stabilisers of order 408 and 68 respectively. Since the collineation stabilisers have different orders these maximal arcs have the interesting property that each is isomorphic to its dual maximal arc.

Similarly, in $PG(2, 32)$ it can be shown that there is up to isomorphism a unique degree 4 Denniston maximal arc (and so a unique degree 8 Denniston maximal arc.)

As was shown above, the dual of a Denniston maximal arc is also a Denniston maximal arc. In we conclude this section with some calculations to identify the dual maximal arcs.

**Theorem 2.4.** Let $F_{std}$ be the standard pencil of conics. Let $A$ be an additive subgroup of $GF(2^e)$, and $K$ the Denniston maximal arc in $PG(2, 2^e)$ determined by $A$ and $F_{std}$. Then the dual maximal arc $K'$ of $K$ has points determined by the standard pencil and additive subgroup $A' = \{x^2 : s \in GF(2^e)^* \text{ and trace}_{GF(2^e)}(\lambda s) = 0 \forall \lambda \in A \cup \{0\}\}$.

**Proof.** Firstly, suppose we have a matrix $g \in GL(3, q)$ acting on $PG(2, q)$ by

$$P^g = g \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

for any point $P = (a, b, c)$. Then it is easily verified that the action of $g$ on homogenous coordinates for lines is given by

$$P^g = g^{-T} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

for any line $[x, y, z]$ (where $g^{-T}$ is the inverse transpose of $g$).

We consider the pencil of conics $F'$ stabilised by $C_{2^e+1}$ (defined previously) in the dual plane. Clearly there is a unique such pencil in the dual plane.
Also \( g^T = g \) for every \( g \in C_{2^{r+1}} \), so the orbits of \( C_{2^{r+1}} \) on the lines of \( \text{PG}(2, 2^r) \) are exactly the same as that on points. Hence we may set \( \mathcal{F}' = \mathcal{F}_{\text{std}} \).

We now calculate the elements of the additive subgroup \( A' \) which, together with \( \mathcal{F}' = \mathcal{F}_{\text{std}} \), give rise to the dual (Denniston) maximal arc. To attempt to avoid confusion we shall use italics to distinguish between points of \( \text{PG}(2, 2^r) \) and points of the dual plane, similarly for lines. Note that \((a, b, c)\) is a point of \( \text{PG}(2, 2^r) \) and is a line of the dual plane.

First notice that as with \( \mathcal{K} \) the nucleus of \( \mathcal{K} \) is the point \([0, 0, 1]\) and the line at infinity is \((0, 0, 1)\).

Let \( P = (1, 1, 0) \). We find the lines on \( P \) which are external to \( \mathcal{K} \).

The lines on \( P \) are \([1, 1, d] : d \in GF(2^r) \) \( \cup \{[0, 0, 1] \} \). Notice that \([0, 0, 1]\) is external to \( \mathcal{K} \), and the line \([1, 1, 0]\) is the line joining \( P \) to the nucleus of \( \mathcal{K} \), and is secant to \( \mathcal{K} \), so we assume \( d \neq 0 \). The points on a line \([1, 1, d]\) are \((x, x + d, 1) : x \in GF(2^r) \) \( \cup \{P \} \). So \([1, 1, d]\), \( d \neq 0 \), is external to \( \mathcal{K} \) if and only if

\[
x^2 + x(x + d) + (x + d)^2 + \lambda = 0
\]

has no solution for any \( \lambda \in A \). Which is equivalent to

\[
\text{trace}_{GF(2^r)} \left( \frac{d^2 + \lambda}{x^2} \right) = 1, \quad \forall \lambda \in A.
\]

The trace map is additive, and \( \text{trace}_{GF(2^r)}(1/\lambda) = 1 \) (since \( \xi^2 + \xi + 1 \) is irreducible), so we get the condition

\[
\text{trace}_{GF(2^r)} \left( \frac{\lambda}{x^2} \right) = 0, \quad \forall \lambda \in A.
\]

So the lines on \( P \) that are external to \( \mathcal{K} \) are

\[
\left\{[1, 1, d] : d \in GF(2^r)^* \text{ and } \text{trace}_{GF(2^r)} \left( \frac{\lambda}{x^2} \right) = 0, \forall \lambda \in A \right\} \cup \{[0, 0, 1] \}.
\]

Substituting each of these points into \( x^2 + xy + y^2 + \mu z^2 = 0, \mu \in A' \) and solving for \( \mu \) then gives that

\[
A' = \left\{\lambda/d^2 : d \in GF(2^r)^* \text{ and } \text{trace}_{GF(2^r)} \left( \frac{\lambda}{x^2} \right) = 0 \ \forall \lambda \in A \right\} \cup \{0\}.
\]

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Putting $s = 1/3d^2$ gives the required result. Note that we have implicitly used the fact that $P = (1, 1, 0)$ is a line through the nucleus of $\mathcal{X}'$ so that the elements of $A'$ are in one to one correspondence with the points of $\mathcal{X}'$ on the line $P = (1, 1, 0)$.

The following corollary will be of interest when Thas 1980 maximal arcs are considered in Section 3.

**Corollary 2.2.** In $\text{PG}(2, 2^d)$, the dual of a Denniston maximal arc with additive subgroup $A = GF(2^e)$ is isomorphic to a Denniston maximal arc described by the standard pencil and additive subgroup $A' = \text{Kernel}(\text{trace}_{GF(2^d)/GF(2^e)})$.

**Proof.** By the theorem

$A' = \{x^2s : s \in GF(2^e)^* \text{ and } \text{trace}_{GF(2^d)/GF(2^e)}(\lambda s) = 0 \forall \lambda \in GF(2^e) \cup \{0\} \}$. First notice that by the corollary to Theorem 2.3 that $A'' = x^{-2}A'$ determines an equivalent Denniston maximal arc. We now use some elementary properties of trace to get the required result.

Let $\lambda \in GF(2^e)$. Then $\lambda^2 = \lambda$, and let $s \in \text{Kernel}(\text{trace}_{GF(2^d)/GF(2^e)})$. Then

\[
\text{trace}_{GF(2^d)/GF(2^e)}(\lambda s) = \text{trace}_{GF(2^d)/GF(2^e)}(\text{trace}_{GF(2^d)/GF(2^e)}(\lambda s)) \\
= \text{trace}_{GF(2^d)/GF(2^e)}(\lambda s + (\lambda s)^2 + \cdots + (\lambda s)^{(2^e)d-1}) \\
= \text{trace}_{GF(2^d)/GF(2^e)}(\lambda \text{trace}_{GF(2^d)/GF(2^e)}(s)) \\
= \text{trace}_{GF(2^d)/GF(2^e)}(\lambda, 0) \\
= 0.
\]

Hence every $s \in \text{Kernel}(\text{trace}_{GF(2^d)/GF(2^e)})$ is an element of $A''$. Also $|\text{Kernel}(\text{trace}_{GF(2^d)/GF(2^e)})| = (2^e)^{d-1}$ which is the degree of the maximal arc dual to $\mathcal{X}$. Hence result.

### 3. STABILISERS OF THAS MAXIMAL ARCS

#### 3.1. Thas 1974 Maximal Arcs in Desarguesian Planes

In 1974, J. A. Thas constructed a class of maximal arcs in certain translation planes of order $q^2$ [29]. We shall refer to these as Thas 1974 maximal
arcs to distinguish them from another construction due to Thas (see next subsection). Thas proved the following.

**Theorem 3.1 (Thas [29]).** Let $PG(3, q)$ be embedded as a hyperplane $H$ in $PG(4, q)$. Let $\mathcal{O}$ be an ovoid and $\mathcal{S}$ a spread of $H$, such that each line of $\mathcal{O}$ has exactly one point in common with $\mathcal{O}$. Let $x \in PG(4, q) \setminus H$, define $\mathcal{K}$ to be the union of the points of $PG(4, q) \setminus H$ that are on the lines of $PG(4, q)$ given by the span of $x$ and each of the points of $\mathcal{O}$ (including the point $x$ itself). Then $\mathcal{K}$ is a maximal $\{q^3 - q^2 + q; q\}$-arc in the translation plane $\pi$ defined by the spread $\mathcal{S}$ (via the construction of André [2]).

These were characterised by the authors with the following theorem.

**Theorem 3.2 (Hamilton and Penttila [16]).** Let $\pi$ be a translation plane of order $q^2$. Then a non-trivial maximal arc $\mathcal{K}$ in $\pi$ is a Thas 1974 maximal arc if and only if it is stabilised by a homology of order $q - 1$ with axis a translation line of $\pi$.

For $q$ odd, ovoids of $PG(3, q)$ are classified and are known to be elliptic quadrics [19, 16.1.7]. In [8], it is shown that the configuration required for a Thas 1974 maximal arc does not exist for $q$ odd.

The known ovoids of $PG(3, q)$, $q$ even, are the elliptic quadrics and the Tits ovoids (Tits ovoids are only defined for $q = 2^{2m} + 1, m \geq 1$). It is well known that any ovoid of $PG(3, q)$, $q$ even, gives rise to a symplectic polarity of $PG(3, q)$ [11]. The totally isotropic lines with respect to this polarity are the tangent lines to the ovoid. The points of $PG(3, q)$ together with the totally isotropic lines form a generalised quadrangle $W(q)$ [27, Chapter 3]. In this setting, the ovoid of $PG(3, q)$ is an ovoid of $W(q)$, and a spread of tangent lines is a spread of $W(q)$. If $q$ is even, a spread of $W(q)$ can easily be constructed by taking the spread formed from the image of any ovoid of $W(q)$ under a duality of the associated generalised quadrangle. Other methods of construction exist, such as exploiting certain actions of the Suzuki group on $PG(3, q)$, see for instance [24, Theorem 27.3].

Given either a Tits ovoid or an elliptic quadric, the known spreads of $PG(3, q)$ of the form required for the Thas construction are the Desarguesian (regular) spreads and the Lüneburg spreads (Lüneburg spreads are only defined for $q = 2^{2m} + 1$, $m \geq 1$, and are the images of the Tits ovoids under a duality of $W(q)$) [30]. Note that given an elliptic quadric both Desarguesian and Lüneburg spreads of the required form exist; similarly, given a Tits ovoid both Desarguesian and Lüneburg spreads of the required form exist [30]. Hence the Thas construction is
known to give maximal arcs in the Desarguesian and Lüneburg planes of even order.

In the next subsection it will be shown that an elliptic quadric-Desarguesian spread pair of the form required for the Thas 1974 construction gives rise to Denniston maximal arcs (an observation that was first made in [29]). It is also shown that up to isomorphism there is one such elliptic quadric-Desarguesian spread pair. Hence the remaining case to consider for Thas 1974 maximal arcs in Desarguesian projective planes is that of a Desarguesian spread—Tits ovoid pair. We conclude this section with a series of Lemmas to calculate the number of such pairs up to isomorphism and the collineation stabilisers of the associated maximal arcs.

**Lemma 3.1.** There is a unique conjugacy class of cyclic subgroups of $\text{PGL}(4, q)$, $q = 2^h$, $h$ odd, of order (i) $q + \sqrt{2q} + 1$ (ii) $q - \sqrt{2q} + 1$.

**Proof.** Let $p$ be a prime divisor of $q + \sqrt{2q} + 1$. Then $p$ does not divide $|\text{PGL}(4, q)|/(q + \sqrt{2q} + 1)$, since the greatest common divisor of the order of $\text{PGL}(4, q)$ and $q + \sqrt{2q} + 1$ is $q - \sqrt{2q} + 1$. Let $x$ be a Singer cycle in $\text{PGL}(4, q)$. Then $x^{q + \sqrt{2q} + 1}$ has order $q + \sqrt{2q} + 1$. Thus a Sylow $p$-subgroup $P$ of $\text{PGL}(4, q)$ is cyclic. By [21, II.7.3], the normaliser of $P$ is the normaliser of the subgroup generated by a Singer cycle. Since all Sylow $p$-subgroups of $\text{PGL}(4, q)$ are conjugate, so are their normalisers. Since a cyclic subgroup of order $q + \sqrt{2q} + 1$ is contained in the normaliser of a Sylow $p$-subgroup as a characteristic subgroup, it follows that any two such are conjugate.

**Lemma 3.2.** There is a unique conjugacy class of cyclic subgroups of $\text{PSp}(4, q)$, $q = 2^h$, $h$ odd, of order (i) $q + \sqrt{2q} + 1$ (ii) $q - \sqrt{2q} + 1$.

**Proof.** Since $\text{PO}^{-}(4, q)$ is a subgroup of $\text{PSp}(4, q)$, and $\text{PO}^{-}(4, q)$ contains a cyclic subgroups of order $q^2 + 1$, the argument of the previous lemma applies.

**Theorem 3.3 (Suzuki [28]).** The Suzuki groups $Sz(q)$ have a unique class of maximal subgroups of order divisible by $q + \sqrt{2q} + 1$. It is a class of dihedral subgroups of order $4(q + \sqrt{2q} + 1)$.

**Theorem 3.4.** The stabiliser $G$ in $\text{PGL}(4, q)$, $q = 2^h$, $h$ odd, $h \geq 3$, of a Tits ovoid $O$ has two orbits on regular spreads of tangent lines to the ovoid, with stabilisers of orders $4(q + \sqrt{2q} + 1)h$. 
Proof. Let $S$ be the generalised quadrangle of points of $\text{PG}(3, q)$ and tangent lines to $O$. Since $G$ centralises a polarity of $O$, it is sufficient to show that $G$ has two orbits on elliptic quadric ovoids of $S$, with stabilisers of orders $4(q \pm \sqrt{2q} + 1)$.

By the last theorem, $G$ contains cyclic subgroup of orders $q \pm \sqrt{2q} + 1$. By the previous two lemmas, this is a subgroup of a cyclic subgroup $C$ of $\text{PSp}(4, q)$ of order $q^2 + 1$. It is well known that all orbits of $C$ on points of $\text{PG}(3, q)$ are elliptic quadrics. Since, by the proof of Lemma 3.2, all cyclic subgroups of $\text{PSp}(4, q)$ of order $q^2 + 1$ are conjugate, and elliptic quadric ovoids of $S$ exist, at least one of these orbits (in fact exactly one, but we don’t need this—$H(q)$ can’t have disjoint ovoids) is an ovoid of $S$. Thus we have two elliptic quadric ovoids of $S$, $O^+$ stabilised by a subgroup of $\text{Sz}(q)$ of order divisible by $q + \sqrt{2q} + 1$ and $O^-$ stabilised by a subgroup of $\text{Sz}(q)$ of order divisible by $q - \sqrt{2q} + 1$. By the Theorem of Suzuki it follows that $\text{Sz}(q)_{q^2}$ has order at most $4(q + \sqrt{2q} + 1)$ and that $\text{Sz}(q)_{q^2}$ has order at most $4(q - \sqrt{2q} + 1)$. Thus we have at least $(q - \sqrt{2q} + 1)q^2(q - 1)/4$ elliptic quadric ovoids of $S$ in the orbit of $O^+$ and at least $(q + \sqrt{2q} + 1)q^2(q - 1)/4$ elliptic quadric ovoids of $S$ in the orbit of $O^-$. The sum of these two numbers is $q^2(q^2 - 1)/2$, which is the total number of elliptic quadric ovoids of $S$, being the index of $PO^-(4, q)$ in $\text{PSp}(4, q)$. The result follows.

Corollary 3.1. There are, up to equivalence under $\text{PGL}(3, q^2)$, two 1974 maximal arcs in $\text{PG}(2, q^2)$ arising from Tits ovoids. They have stabilisers in $\text{PGL}(3, q^2)$ given by the semidirect product of a dihedral group of order $4(q \pm (2q)^{1/2} + 1)(q - 1)$ by a cyclic group of order $\omega q + 1$, where $q = 2^{\omega + 1}$.

3.2. That 1980 Maximal Arcs in Desarguesian Planes

In 1980 J. A. Thas gave the following construction of maximal arcs in certain translation planes of order $q^d$ whose kernel contains $GF(q)$.

Theorem 3.5 (Thas [31]). Let $Q^- = Q^-((2t - 1, q))$ be a non-singular elliptic quadric in $\text{PG}(2t - 1, q)$, $t > 1$, and let $\mathcal{F}^-$ be a $(t - 2)$-spread of $Q^-$. Suppose there exists a $(t - 1)$-spread $\mathcal{F} = \{s_1, s_2, \ldots, s_{t - 1}\}$ of $\text{PG}(2t - 1, q)$ such that $\mathcal{F}^- = \{Q^- \cap s_1, Q^- \cap s_2, \ldots, Q^- \cap s_{t - 1}\}$. Embed $\text{PG}(2t - 1, q)$ as a hyperplane $H$ in $\text{PG}(2t, q)$ and choose any point $x \in \text{PG}(2t, q) \setminus H$. Let $\mathcal{K}$ be the union of the points of $\text{PG}(2t, q) \setminus H$ that are on the lines of $\text{PG}(2t, q)$ given by the span of $x$ and each of the points of $Q^-$(including the point $x$ itself), then $\mathcal{K}$ is the set of points of a maximal $\{q^{d - 1} - q^d + q^{d - 1}; q^{d - 1}\}$-arc in the translation plane $\pi(\mathcal{F})$ of order $q^d$ determined by the spread $\mathcal{F}$. 


In the above, if \( t = 2 \) the Thas 1980 construction is just the Thas 1974 construction with the restriction that the ovoid be an elliptic quadric. For \( q \) is odd it is shown in [8] that Thas 1980 maximal arcs do not exist, that is, that the hypotheses of Theorem 3.5 are never satisfied.

When \( q \) is even we first note that the spread \( \mathcal{S} \) of \( H \) must be symplectic. The quadric \( Q^-(2t-1, q) \) induces a symplectic polarity \( \theta \) of \( H \) [20, Lemma 22.3.3]. Given a \((t-2)\)-spread \( \mathcal{S}' \) of \( Q^-(2t-1, q) \), any spread \( \mathcal{S} \) of \( H \) that “fits” \( \mathcal{S}' \) has elements which are (maximal) totally isotropic with respect to \( \theta \) \((s^\theta = s \text{ for every } s \in \mathcal{S})\), and so is a symplectic spread. This follows since any \( s \in \mathcal{S} \) can be written as a span \( s = \langle p, s^- \rangle \) for some \( s^- \in \mathcal{S}' \) and \( p \in (s^-)^\theta \setminus Q^-(2t-1, q) \). Then \( s^\theta = \langle p, s^- \rangle^\theta = p^\theta \cap (s^-)^\theta \), which contains both \( p \) and \( s^- \) \((\theta \text{ is symplectic so } p < p^\theta)\), and so \( s^\theta = s \).

In fact, for \( q \) even, as is noted in the original paper, the Thas 1980 construction gives maximal arcs in every translation plane whose spread is symplectic. It is well known that any symplectic spread of \( PG(2t-1, q) \) gives rise to a spread of a non-singular quadric \( Q(2t, q) \) in \( PG(2t, q) \) [15, Sect. 3.2] (and conversely). Further, such a spread of \( Q(2t, q) \) induces a spread of a quadric \( Q^-(2t-1, q) \) in the \( PG(2t-1, q) \) which is of the right form with respect to the symplectic spread for the Thas 1980 construction [15, Sect. 3.2]. Another way to say this is that a symplectic spread of \( PG(2t-1, q) \) will induce (by intersection) a spread of any quadric \( Q^-(2t-1, q) \) with the same polarity.

When \( q \) is even, many examples of symplectic spreads are known for all \( t > 1 \). In particular spreads which give rise to Desarguesian projective planes can be viewed as symplectic spreads. We now consider the Thas 1980 maximal arcs that arise in Desarguesian projective planes.

**Theorem 3.6.** The Thas 1980 maximal arcs that occur in Desarguesian planes of even order are of Denniston type.

**Proof.** First note that the collineation group of a translation plane is transitive on the affine points and so the different choices of the affine point \( x \) in the Thas construction give rise to isomorphic maximal arcs (for a given spreads of \( PG(2t-1, q) \) and \( Q^-(2t-1, q) \)).

Let \( \mathcal{S} = \{s_1, \ldots, s_{q^t+1} \} \) be a Desarguesian spread of \( PG(2t-1, q) \), \( t > 1 \), \( q \) even. We show that any two non-singular elliptic quadrics \( Q^-(2t-1, q) \) and \( Q^-(2t-1, q') \) of the form required for the Thas construction with respect to \( \mathcal{S} \) can be mapped to one another by elements of the stabiliser \( GL(2t, q) \), and hence give rise to isomorphic maximal arcs. An example is then given of a Desarguesian spread—quadric spread pair that is stabilised by a linear cyclic group of order \( q^t+1 \). It then follows that any Desarguesian spread—quadric spread pair of the form required for the
Thus construction gives rise to a maximal arc stabilised by a linear cyclic group of order $q^t+1$, and so the maximal arc is of Denniston type (by Theorem 2.2).

Let $Q^- = Q^-(2t-1, q)$ and $Q'^- = Q^-(2t-1, q')$ be non-singular elliptic quadrics, with $(t-2)$-spreads $\mathcal{S}^-$ and $\mathcal{S}'^-$, respectively, and such that

$$\mathcal{S}^- = \{Q^- \cap s_1, \ldots, Q^- \cap s_{q^t+1}\}$$

and

$$\mathcal{S}'^- = \{Q'^- \cap s_1, \ldots, Q'^- \cap s_{q^t+1}\}.$$  

Since $q$ is even, $Q^-$ and $Q'^-$ induce symplectic polarities $\theta$ and $\theta'$, respectively, of $PG(2t-1, q)$. As was shown in the last section note that $s^\theta = s = s'^\theta$ for every $s \in \mathcal{S}$.

The following is most easily understood in the (equivalent) André construction of translation planes [2]. In the André construction our spread $\mathcal{S}$ is considered as a collection of $t$ dimensional subspaces of the vector space $V(2t, q)$ which partition the points of $V(2t, q) - \{0\}$. The points of the (affine) translation plane are then the points of $V(2t, q)$, and the lines are the cosets of the spread $\mathcal{S}$ (i.e., lines are of the form $v + s$ for $v \in V(2t, q), s \in \mathcal{S}$).

We employ an argument used in [22, Theorem 3.5] to show $\theta$ and $\theta'$ are conjugate in the stabiliser of $\mathcal{S}$ in $\Gamma L(2t, q)$. The product $\theta\theta'$ is a collineation of $PG(2t-1, q)$ and hence is induced by a (invertible) semi-linear transformation on the underlying vector space $V(2t, q)$. By the previous paragraphs, $\theta\theta'$ acts as the identity on $\mathcal{S}$, and so is a homology of the affine translation plane determined by $\mathcal{S}$. Since $q$ is even the order of $\theta\theta'$ must be odd.

Consider the group $D_{2|\theta\theta'} = \langle \theta, \theta\theta' \rangle$ acting on $V(2t, q)$. It is easily shown to be dihedral of (odd) order $(\theta^2 = 1 = \theta^2, \theta(\theta') = (\theta\theta')^{-1} \theta$. Then $\theta$ and $\theta'$ are elements of order 2 in $D_{2|\theta\theta'}$ and so are conjugate under some power of $\theta\theta'$ (since $\theta\theta'$ has odd order). The group generated by $\theta\theta'$ stabilises $\mathcal{S}$, and so $\theta$ and $\theta'$ are conjugate in the $\Gamma L(2t, q)$ stabiliser of $\mathcal{S}$.

It follows that the symplectic spaces arising from $\theta$ and $\theta'$ can be mapped to one another by some element of the stabiliser of $\mathcal{S}$ in $\Gamma L(2t, q)$. So up to equivalence in $\Gamma L(2t, q)$ we may assume $\theta = \theta'$.

In [15], R. H. Dye constructs a spread denoted $\mathcal{P}$ of the symplectic space $Sp(2t, q)$, $q$ even, which is a Desarguesian spread of $PG(2t-1, q)$. Though not explicitly said to be Desarguesian, $\mathcal{P}$ admits a group triply transitive on the elements of the spread [15, Theorem 7], and so is Desarguesian [24, 29.3]. Dye proves by some lengthy calculations:

**Theorem 3.7 (Dye [15, Theorem 14]).** The stabiliser in the group $Sp(2t, q)$, $q$ even, of $\mathcal{P}$ acts triply transitive on the elliptic quadrics with the same polarity as the space $Sp(2t, q)$. **
Hence, since we have shown up to equivalence in $\mathcal{L}(2t, q_{\mathcal{L}})$ that $\theta = \theta'$, it follows from Dye's result that the quadrics $Q^-$ and $Q'^-$ can be mapped to one another by elements of the stabiliser $\mathcal{L}(2t, q_{\mathcal{L}})$. It follows that the Thas 1980 maximal arcs of a given degree in a Desarguesian plane are all isomorphic.

In only remains to give an example (for each $t$) of a Thas 1980 maximal arc in a Desarguesian plane that admits a linear cyclic group of order $q^t + 1$ to complete the proof. We construct a quadric - Desarguesian spread pair of the form required for the Thas construction that admits a linear cyclic group of order $q^t + 1$. There are a number of ways the existence of such a quadric-spread pair could be proved, but here is a fairly concrete one.

Let $\xi^2 + x^2 + 1$ be an irreducible polynomial over $GF(q^t)$, $q$ even. Define a quadratic form on $GF(q^t)$ by $f(x, y) = x^2 + xy + y^2$.

The field $GF(q^t)$ is a vector space over $GF(q)$, so for $a \in GF(q^t)$ denote the corresponding vector in $GF(q)$ by $a$. Define a quadratic form on the 2t-dimensional vector space $V(2t, q)$ by $Q(x, y) = \text{trace}_{GF(q^t)/GF(q)}(f(x, y))$. Then it is well known (see for instance [22, 32]) and not too difficult to verify that $Q$ is a non-degenerate quadratic form on $V(2t, q)$ and is of elliptic type.

Define

$$C_{q^t + 1} = \left\{ \begin{pmatrix} a + xb & b \\ b & a \end{pmatrix} : a^2 + xab + b^2 = 1 \right\}.$$ 

Then $C_{q^t + 1}$ is cyclic of order $q^t + 1$ and $f(g(x, y)) = f(x, y)$ for every $g \in C_{q^t + 1}$. So $C_{q^t + 1}$ preserves the quadratic form $f$, and so preserves $Q$ when made to act on $V(2t, q)$. It also preserves the (vector space) spread

$$\{ \{ (a, ab) : a \in GF(q^t) \} : b \in GF(q^t) \} \cup \{ (0, b) : b \in GF(q^t) \}.$$ 

Further the points of the quadric determined by $Q$ meet any element of the spread in a subspace, and the spread is maximal totally isotropic with respect to the polar form of the quadratic form. 

In the proof of the theorem it is hardly a coincidence that the group $C_{q^t + 1}$ is the same as the group given by Abatangelo and Larato in their characterisation of Denniston maximal arcs.

We can identify which Denniston maximal arcs the Thas 1980 construction gives rise to using the following lemma.

**Lemma 3.3.** In $PG(2, 2^e)$, $t > e > 1$, there is up to isomorphism a unique degree $2^{e(t-1)}$ Denniston maximal arc $\mathcal{X}$ stabilised by a homology of order $2^e - 1$.
Proof. Since in $PG(2, 2^e)$ a degree $2^{(t-1)}$ maximal arc is dual to a degree 2 maximal arc in $PG(2, 2^e)$ it is sufficient to show that there is a up to isomorphism a unique degree 2 Denniston maximal arc $\mathcal{X}'$ stabilised by a homology of order $2^e - 1$.

First note $\mathcal{X}'$ is isomorphic to a maximal arc $\mathcal{X}''$ described by the following standard pencil of conics, and some additive subgroup $A''$ of $GF(2^e)$, $|A''| = 2^e$.

Let $z^2 + xz + 1$ be an irreducible polynomial over $GF(2^e)$, and chose an additive subgroup $A''$ of $GF(2^e)$ such that the maximal arc $\mathcal{X}''$ described by the set of points of $F_z : x^2 + xz + y^2 + z^2 = 0$, $\lambda \in A''$, is isomorphic to $\mathcal{X}'$.

The nucleus of $\mathcal{X}''$ is the point $(0, 0, 1)$, and the line at infinity is $[0, 0, 1]$. Let $h$ be the homology of order $2^e - 1$ stabilising $\mathcal{X}''$. Then arguing as in Theorem 2.3, $h$ fixes the nucleus and the line at infinity, and cyclically permutes the conics of $\mathcal{X}''$.

The group of all homologies with centre $(0, 0, 1)$ and axis $[0, 0, 1]$ can be described by the following set of matrices:

$$H_{2^e-1} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & b \end{pmatrix} : b \in GF(2^e)^* \right\}. $$

Note that the order of $H_{2^e-1}$ is $2^e - 1$, and since the kernel of $PG(2, 2^e)$ is $GF(2^e)^*$ [24], these are all the homologies with this centre and axis. So $h \in H_{2^e-1}$.

Since $H_{2^e-1}$ acts regularly on the non-fixed points of any line through the centre, it transitivity permutes the orbits of $h$ of length $2^e - 1$ on any line through the center of $h$. Hence all degree $2^e$ Denniston maximal arcs that are a subset of the pencil $F_z$, $\lambda \in GF(2^e)$, and that are stabilised by $h$ are isomorphic.

In the lemma if $e = 1$ then $\mathcal{X}$ is the dual of a conic, and so uniqueness follows (though $2^e - 1 = 1$ and so there is no homology of that order).

Corollary 3.2. In $PG(2, 2^e)$ there is up to isomorphism a unique degree $2^{(t-1)}$ Thas 1980 maximal arc. It is isomorphic to a Denniston maximal arc with the standard pencil and additive subgroup $Kernel(trace_{GF(2^e)}GF(2^e))$.

Proof. Uniqueness was shown in the proof of Theorem 3.6. Note that such a Thas 1980 maximal arc is stabilised by a homology of order $2^e - 1$. Then note that a Denniston maximal arc with standard pencil and additive
subgroup $GF(2^e)$ is clearly stabilised by a homology of order $2^e - 1$. The corollary to Theorem 2.4 gave the calculation of the dual parameters of such a Denniston maximal arc.

4. POINT TRANSITIVE MAXIMAL ARCS IN DESARGUESIAN PLANES

In this section maximal arcs in Desarguesian planes whose collineation stabiliser is transitive on the points of the maximal arc are classified. Since non-trivial maximal arcs do not exist in odd order Desarguesian planes [4] we need only consider the even order case.

In 1990, Delandtsheer and Doyen classified the maximal arcs in finite projective planes whose collineation stabiliser is transitive on the set of lines secant to the maximal arc. They show the following.

**Theorem 4.1** (Delandtsheer and Doyen [10]). Let $\pi$ be a projective plane of order $q$, $q$ not necessarily a prime power. If $G \leq \text{Aut}(\pi)$ stabilises a maximal arc $\mathcal{K}$ of degree $n$, and acts transitively on the set of lines secant to $\mathcal{K}$, then one of the following holds:

(i) $q = p^e, \ c \geq 1, p$ prime, $n = q, \mathcal{K}$ is a line transitive affine translation plane $A$, and $G$ contains the translation group of $A$.

(ii) $q = 2^e, \ e \geq 3, n = q/2, \mathcal{K}$ is the dual of a regular hyperoval in $PG(2, 2^e)$, and $PSL(2, 2^e) \triangleleft G \leq PFL(2, 2^e)$.

(iii) $q = 4, n = 2, \mathcal{K}$ is a (regular) hyperoval in $PG(2, 4)$ and $G = Alt(6), Sym(6), PSL(2, 5)$ or $PGL(2, 5)$.

Further, in each case except (iii), $G$ stabilises a line of $\pi$ disjoint from $\mathcal{K}$.

Considering a degree $n$ maximal arc in a plane of order $q$ as a $(q(n-1)+n, n, 1)$-BIBD, it follows from the results of [9] that transitivity on secant lines to $\mathcal{K}$ implies transitivity on points of $\mathcal{K}$ (but not conversely). So the general problem of classifying maximal arcs whose collineation stabiliser is transitive on points remains open. We shall call such maximal arcs **point transitive**.

In [23], Korchmaros classified the point transitive hyperovals in Desarguesian projective planes. Partial results have also been obtained by various authors for hyperovals in non-Desarguesian projective planes [7].

A necessary condition for a maximal arc in a projective plane to be point transitive is that the size of the maximal arc divides the order of the collineation group of the plane. Using elementary number theory this
condition can be used to show the non-existence of transitive maximal arcs of a given degree for a large number of cases. The remaining cases can then be dealt with using knowledge of the types of group elements that can stabilise a maximal arc and some elementary group theory.

Denote the greatest common factor of two integers \(x\) and \(y\) by \(\gcd(x, y)\).

**Lemma 4.1.** Let \(\mathcal{X}\) be a maximal arc of degree \(n\) in \(PG(2, q)\), \(2^r = n < q = 2^e\). If \(\mathcal{X}\) is point transitive then one of the following conditions holds:

(i) \(r = e - 1\).

(ii) \(e\) is even, and (a) \(r = \frac{e}{2}\) or (b) \(r = \frac{e}{2} - 1\).

(iii) \(e\) is either 1, 2, or 4, and \(r = 1\).

**Proof.** For \(\mathcal{X}\) to be point transitive, it is necessary that the size of \(\mathcal{X}\) divides \(|\mathcal{P}L(3, q)| = (q^2 + q + 1)(q + 1)q^3(q - 1)^3\) \(e\). We use elementary properties of greatest common divisors, and that \(n\) and \(q\) are powers of 2, to estimate \(\gcd(|\mathcal{X}|, |\mathcal{P}L(3, q)|)\). Now \(\gcd(e, q(n - 1) + n)\) is at most \(e\).

\[
\gcd(q + 1, q(n - 1) + n) = \gcd(q + 1, (q + 1)(n - 1) + 1) = 1.
\]
\[
\gcd(q^2, q(n - 1) + n) = \gcd(q^2, q(n - 1) + 1) = n = n.
\]
\[
\gcd(q - 1, q(n - 1) + n) = \gcd(q - 1, (q - 1)(n - 1) + 2n - 1)
\]
\[= \gcd(q - 1, 2n - 1).\]

Hence the \(\gcd((q - 1)^2, q(n - 1) + 1)\) is at most \((2^{\gcd(e, r + 1)} - 1)^2\).

\[
\gcd(q^2 + q + 1, q(n - 1) + n)
\]
\[= \gcd(q^2 + q + 1, q - q/n + 1)
\]
\[= \gcd(q(q - q/n + 1) + q^2/n + 1, q - q/n + 1)
\]
\[= \gcd(q^2/n + 1, q - q/n + 1)
\]
\[= \gcd(q^2/n - q + q/n, q - q/n + 1)
\]
\[= \gcd(q/n(q - n + 1), q - q/n + 1)
\]
\[= \gcd(q - n + 1, q - q/n + 1)
\]
\[= \gcd(q/n - n, q - q/n + 1)
\]
\[= \gcd(q/n - n, (n - 1)(q/n - n) + n(n - 1) + 1)
\]
\[= \gcd(q/n - n, n(n - 1) + 1)
\]
\[= \gcd(q - n^2, n(n - 1) + 1).
\]
Hence the $\gcd(q^2 + q + 1, |K|)$ is at most $(q^2 + q + 1, q - n^2)$. Now $q^3 - 1 = (q - 1)(q^2 + q + 1)$. And so $\gcd(q^2 + q + 1, |K|)$ is at most $\gcd(q^3 - 1, q - n^2) = 2^{\gcd(3e, e - 2r)} - 1$.

For $\mathcal{X}$ to be transitive it is then required that the product of these terms is greater than or equal to the size of $\mathcal{X}$, and so

$$e \cdot 1 \cdot 2^r (2^{\gcd(e, r + 1)} - 1)^2 \cdot (2^{\gcd(3e, e - 2r)} - 1) \geq 2^r (2^r - 1) + 2^r.$$  

Dividing by $2^r$ then gives

$$e (2^{\gcd(e, r + 1)} - 1)^2 \cdot (2^{\gcd(3e, e - 2r)} - 1) \geq 2^r - 2^{e - r} + 1. \quad (1)$$

Suppose some integer $p$ divides $\gcd(e, r + 1)$. Then $p$ does not divide $e$, and so does not divide $e - 2r$ with the possible exception of $p = 2$. Put $x = \gcd(e, r + 1)$, then $x$ divides $e$ and $\gcd(x, e - 2r)$ is at most 2. It follows $\gcd(3e, e - 2r)$ is at most $\gcd(\frac{6e}{x}, e - 2r)$, which is at most $\frac{6e}{x}$. To be transitive we then require that

$$e(2^x - 1)^2 \cdot (2^{\frac{6e}{x}} - 1) \geq 2^x - 2^{x - r} + 1 \Rightarrow e^{2^{\frac{6e}{x}} + 6e/x} \geq 2^x - 2^{x - r} + 1$$

$$\Rightarrow 2^{(\log_e e) + 2x + 6e/x} \geq 2^x$$

$$\Rightarrow (\log_e e) + 2x + \frac{6e}{x} \geq e - 1$$

$$\Rightarrow 2x^2 + (1 + (\log_e e) - e) \cdot x + 6e \geq 0.$$

This quadratic in $x$ has zeros for $e \geq 62$ at

$$x = \frac{e - (\log_e e) - 1 \pm \sqrt{(1 + (\log_e e) - e)^2 - 48e}}{4}.$$

Hence for $e \geq 62$ we require that either

$$x \leq \frac{e - (\log_e e) - 1 - \sqrt{(1 + (\log_e e) - e)^2 - 48e}}{4}$$

or

$$x \geq \frac{e - (\log_e e) - 1 + \sqrt{(1 + (\log_e e) - e)^2 - 48e}}{4}.$$

We consider the second case first. Then for $e \geq 62$, direct calculation shows that $x > \frac{e}{3}$. But since $x$ divides $e$, $x$ must be one of $e$, $\frac{e}{3}$, or $\frac{e}{6}$. If $x = e$ then $r + 1 = e$ and we are in case (i) of the statement of the theorem. Also $x = \frac{e}{3}$. 


gives $r + 1 = \frac{93}{64}$ which is included in case (ii). If $x = \frac{93}{64}$, then $r + 1 = \frac{93}{64}$ or $\frac{93}{64}$. When $r + 1 = \frac{93}{64}$, $\gcd(3e, e - 2r) = (18, \frac{93}{64} + 2)$, which is at most 18. Using this in (1) gives

$$e(2e^{\log 2 - 1})^2 (2^{18} - 1) \geq 2e^{2e^{\log 2 - 1}} + 1 \Rightarrow 2^{(\log 2 + 1)(2^{\log 2 - 1})} \geq 2^{e - 1}$$

$$\Rightarrow (\log 2) + \frac{2e}{3} + 18 \geq e - 1$$

$$\Rightarrow 3(\log 2) + 57 \geq e$$

$$\Rightarrow 62 \geq e$$

Hence for $e > 62$, transitive maximal arcs with $r + 1 = \frac{93}{64}$ cannot occur.

When $r + 1 = \frac{93}{64}$ similar calculations show that transitive maximal arcs cannot occur for $e > 62$.

We now consider the first case for the range of $x$. Direct calculation shows that for $e \geq 95$,

$$x \leq \frac{e - (\log 2 + 1 - \sqrt{(1 + (\log 2 + 1)(2^{\log 2 - 1})^2 - 48e)}}}{4} \leq 8.$$ 

Now suppose that $e \neq 2r$, then $\gcd(3e, e - 2r)$ is at most $|e - 2r|$. Then, to be transitive for $e \geq 95$, we require that

$$e(2^{e^{\log 2 - 1}} - 1)(2^8 - 1)^2 \geq 2^e - 2e^{e - r} + 1.$$ 

Arguing as before this gives that for $e > 2r$ we must have $(\log 2 + 17 \geq 2r$, and for $e < 2r$ that $(\log 2 + 1) \geq 2(e - r)$.

So suppose for $e > 2r$ we have $(\log 2 + 17 \geq 2r$. Then $\gcd(3e, e - 2r) = \gcd(3e - 3(e - 2r), e - 2r) = (6r, e - 2r)$, which is then at most $3((\log 2 + 1) + 17)$. Similarly for $e < 2r$ we have $(\log 2 + 1) \geq 2(e - r)$. So $\gcd(3e, e - 2r) = \gcd(3e + 3(e - 2r), e - 2r) = \gcd(6(e - r), e - 2r)$, which is then also at most $3((\log 2 + 1) + 17)$. We have the upper bound of 8 for $\gcd(|\mathcal{A}|, (q - 1)^2)$. Hence to be transitive requires that

$$e(2^{e^{\log 2 - 1}} - 1)(2^8 - 1)^2 \geq 2^e - 2e^{e - r} + 1$$

$$\Rightarrow (\log 2) + 16 + 3((\log 2 + 17) \geq e - 1$$

$$\Rightarrow 92 \geq e.$$

Hence the case of the lower bound for $x$ does not give transitive maximal arcs, except those listed in the statement of the theorem, for $e > 95$ (so that $x < 8$). Similarly other case for the range of $x$ did not give transitive maximal arcs for $e \geq 62$. 
Simple calculations for $e \leq 95$ show that the size of $K$ divides the order of $PIL(3, q)$ only for the cases given in the statement of the lemma. Straightforward calculations show that the size of $K$ does divide the order of $PIL(3, q)$ in the cases listed in the statement of the lemma. The result follows.

**Theorem 4.2.** Let $K$ be a non-trivial maximal arc in $PG(2, q)$, $q > 2$, such that the collineation stabiliser of $K$ is transitive on the points of $K$, then $K$ is isomorphic to one of the following:

(i) a regular hyperoval in $PG(2, 2)$ or $PG(2, 4)$, or a Lunelli–Sce hyperoval in $PG(2, 16)$.

(ii) the dual of a translation oval in $PG(2, q)$ for any even $q$.

**Proof.** We first note that non-trivial maximal arcs are known not to exist in $PG(2, q)$ for $q$ odd [4]. In considering existence in $PG(2, q)$, $q = 2^e$, we rely on using divisibility conditions and Hartley’s classification of maximal subgroups of $PSL(3, 2^e)$ [17]. We consider each of the cases of the previous lemma.

Case (iii) of Lemma 4.1 is that of a hyperoval in $PG(2, 2)$, $PG(2, 4)$ or $PG(2, 16)$. For each of these planes hyperovals have been classified, and those that are transitive are exactly those listed in (i) of the statement of the theorem.

In case (ii)(a) of Lemma 4.1, the size of $K$ is divisible by $q - \sqrt{q} + 1$, so the collineation group stabilising $K$ has order divisible by primitive prime divisors of $q^3 - 1$ (for $q > 4$). By Hartley’s results, such groups are subgroups of the normaliser of a Singer cycle or of $PIL(3, q)$, and, if in the last case but not the former, they contain $PSU(3, q)$. Groups containing $PSU(3, q)$ stabilise no maximal arc, as they have just two orbits on points of $PG(2, q)$: the classical unital and its complement. If the group is a subgroup of the normaliser of a Singer cycle, then as $\sqrt{q}$ divides $|K|$, it divides the order of the group, and so the order $3e(q^3 + q + 1)$ of the normaliser of a Singer cycle. Hence $\sqrt{q}$ divides $e$ (as $3(q^3 + q + 1)$ is odd), giving $q \leq 16$. Easy calculations show that a degree 4 maximal arc in $PG(2, 16)$ cannot be stabilised by a group of order 13, and so this case cannot occur. The case $q = 4$ is covered by case (iii) of the theorem. Hence case (ii)(a) of Lemma 4.1 only gives rise to transitive maximal arcs when $q = 4$.

In case (ii)(b) of Lemma 4.1, Hartley’s results imply that, if $q > 4$, the group fixes a point or a line or a Baer subplane, and if in the latter case but not in the former two, it must contain $PSL(3, \sqrt{q})$, and so cannot stabilise a maximal arc ($PSL(3, \sqrt{q})$ has two orbits on $PG(2, q)$: a Baer subplane and its complement). Now the action on the fixed line or the quotient space of the fixed point has to have a kernel of order divisible by
\[ \sqrt{q} - 1 \]. In other words, the maximal arc is stabilised by a non-trivial homology of order \( \sqrt{q} - 1 \), contrary to having degree \( \sqrt{q}/2 \). So \( q = 4 \), and the maximal arc is a point (trivial).

In case (i) of Lemma 4.1, \( \mathcal{K} \) is a dual hyperoval. Thus we must classify hyperovals \( \mathcal{K} \) whose collineation stabilisers \( G \) are transitive on the external lines to \( \mathcal{K} \). The remainder of the proof is devoted to showing that such a hyperoval is a translation hyperoval.

The number of points not on \( \mathcal{K} \) is odd, so there must be an orbit \( O \) on such points of odd length. Hence any point \( Q \) of \( O \) has stabiliser containing a Sylow 2-subgroup. If \( q > 4 \), then \( q/2 \) does not divide \( e \), and so there are homographies of order 2 in the group; these are elations. Since they fix a hyperoval, they have centre a point not on the hyperoval and axis a secant line to the hyperoval (as \( q > 2 \) [26]). Since every point \( Q \) of \( O \) has stabiliser containing a Sylow 2-subgroup, it follows that every point \( Q \) of \( O \) is fixed by every elation in some Sylow 2-subgroup \( S \). Two non-trivial elations in a 2-subgroup fixing a hyperoval have common axis [26]. Moreover, every line on the centre of a non-trivial elation is fixed by that elation, so some external line is fixed by a non-trivial elation, so \( q \) divides the order of the group \( S \).

Hartley's results imply that, if \( q > 4 \), the group \( G \) fixes a point, a line, a subplane of order \( \sqrt{q} \) or \( \sqrt[3]{q} \) or a classical unital. In the subplane cases, but not in the first two, either \( PSL(3, \sqrt{q}) \) or \( PSL(3, \sqrt[3]{q}) \) is contained in the collineation stabiliser, these can then be excluded by an homology argument as before. If in the unital case but not the first two, the group contains \( PSU(3, q) \) and so fixes no hyperoval. Any fixed point must be on the hyperoval. Any fixed line must be secant to the hyperoval. In the following let \( c \) be the size of the intersection of \( O \) with an external line (a constant, by transitivity on external lines). Counting incident (point of \( O \), external line) pairs gives \( |O| = c(q - 1) \).

For \( q > 16 \), the homography stabiliser of \( \mathcal{K} \) has order divisible by 4. Moreover, the Sylow 2-subgroups of \( G \) intersect \( PGL(3, q) \) are elementary abelian, for no homography \( g \) of order 4 can fix a hyperoval. (\( g^2 \) must be an elation with axis a secant line to the hyperoval, and centre a point not on the hyperoval. Considering a secant line on the centre of this elation other than the axis of this elation, we see that the points of secancy are interchanged by \( g^2 \), but must be interchanged or fixed by \( g \), a contradiction.) Hence if a line is fixed by a Sylow 2-subgroup \( S \), it must be the common axis of all elations in \( S \). So, for \( q > 16 \), any fixed line must be the common axis of all elations in each Sylow 2-subgroup, as its stabiliser contains a Sylow 2-subgroup and these are all conjugate. So, there is at most one fixed line \( l \). In this case, \( |O| = q - 1 \), so \( c = 1 \) and \( O \) is \( l - \mathcal{K} \). Hence there are at least \( q - 1 \) elations with axis \( l \) fixing \( \mathcal{K} \) (there exists one
with axis \( l \) and centre \( C \) on \( l \) and hence \( C \) in \( O \); conjugate) and these elations commute (having common axis) and so generate an elementary abelian elation group of order \( q \), thus \( \mathcal{X} \) is a translation oval.

Suppose \( G \) fixes no line. Then there is a fixed point \( P \). Let \( Q \) be a point of \( O \). Since \( |O| \) is odd, it follows that the length of the orbit of \( l \) is odd (indeed, it divides \( c(q - 1) \)). If \( q > 16 \), then the size of a Sylow 2-subgroup of \( G \) intersect \( PGL(3, q) \) is bigger than \( \sqrt{q} \), and the greatest common divisor of \( q - 1 \) and \( |G \cap PGL(3, q)| \) is not 1, so the group induced by \( G \cap PGL(3, q) \) on \( PG(2, q)/P \) must contain \( AGL(1, q) \), by Dickson's list of subgroups of \( PSL(2, q) \) \[14\]. Hence \( q \) divides \( |G \cap PGL(3, q)| \), and the result now follows from \[25\, \text{Theorem 3.6}\].

Thus it remains to deal with \( q = 16 \). Here the hyperovals are classified, and the only non-translation hyperoval, the Lunelli–Sce hyperoval, has a collineation stabiliser of order 144, which is not divisible by 120, and so is not transitive on the external lines to the hyperoval.

REFERENCES

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