Quadruple fixed point theorems for nonlinear contractions

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ABSTRACT

The notion of coupled fixed point is introduced by Gnana-Bhaskar and Lakshmikantham. Very recently, the concept of tripled fixed point was introduced by Berinde and Borcut. In this manuscript, a quadruple fixed point is considered and some new related fixed point theorems are obtained. We also give some examples to illustrate our results.

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1. Introduction and preliminaries

In 2006, Gnana-Bhaskar and Lakshmikantham [1] introduced the notion of coupled fixed point and proved some fixed point theorems under certain conditions. Later, Lakshmikantham and Ćirić in [2] extended these results by defining the g-monotone property. Many authors focused on coupled fixed point theory and proved remarkable results (see e.g. [3–16]).

Very recently, Berinde and Borcut [17] introduced the concept of tripled fixed point and proved some related theorems. In this manuscript, the quadruple fixed point is introduced and the existence and uniqueness of quadruple fixed points are considered.

Here we recall some basic definitions and list the results that motivated our quadruple fixed point theorems. Let \((X, d)\) be a metric space and \(X^2 := X \times X\). Then the mapping \(\rho : X^2 \times X^2 \to [0, \infty)\) defined by \(\rho((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + d(y_1, y_2)\) forms a metric on \(X^2\). A sequence \((x_n, y_n) \in X^2\) is said to be a double sequence of \(X\).

**Definition 1** (See [1]). Let \((X, \leq)\) be a partially ordered set and \(F : X^2 \to X\) be a mapping. \(F\) is said to have the mixed monotone property if \(F(x, y)\) is monotone nondecreasing in \(x\) and is monotone non-increasing in \(y\), that is, for any \(x, y \in X\),

\[
\begin{align*}
x_1 \leq x_2 & \Rightarrow F(x_1, y) \leq F(x_2, y), \quad \text{for } x_1, x_2 \in X, \text{ and} \\
y_1 \leq y_2 & \Rightarrow F(x, y_1) \leq F(x, y_2), \quad \text{for } y_1, y_2 \in X.
\end{align*}
\]

**Definition 2** (See [1]). An element \((x, y) \in X^2\) is said to be a coupled fixed point of the mapping \(F : X^2 \to X\) if

\[
F(x, y) = x \quad \text{and} \quad F(y, x) = y.
\]
Throughout this paper, \((X, \leq)\) will denote a partially ordered set and \(d\) will be a metric on \(X\) such that \((X, d)\) is a complete metric space. Further, the product space \(X^2\) satisfies the following:

\[
(u, v) \leq (x, y) \Leftrightarrow u \leq x, \quad y \leq v; \text{ for all } (x, y), (u, v) \in X^2.
\]  

(1.1)

The following two results of Gnana-Bhaskar and Lakshmikantham in [1] were extended to the class of cone metric spaces in [6].

**Theorem 3.** Let \(F : X^2 \rightarrow X\) be a continuous mapping having the mixed monotone property on \(X\). Assume that there exists a \(k \in [0, 1)\) with

\[
d(F(x, y), F(u, v)) \leq \frac{k}{2} [d(x, u) + d(y, v)], \quad \text{for all } u \leq x, y \leq v.
\]  

(1.2)

If there exist \(x_0, y_0 \in X\) such that \(x_0 \leq F(x_0, y_0)\) and \(F(y_0, x_0) \leq y_0\), then there exist \(x, y \in X\) such that \(x = F(x, y)\) and \(y = F(y, x)\).

**Theorem 4.** Let \(F : X^2 \rightarrow X\) be a mapping having the mixed monotone property on \(X\). Suppose that \(X\) has the following properties:

(i) if a non-decreasing sequence \([x_n]\) tends to \(x\), then \(x_n \leq x\), \(\forall n\);

(ii) if a non-increasing sequence \([y_n]\) tends to \(y\), then \(y \leq y_n\), \(\forall n\).

Assume that there exists a \(k \in [0, 1)\) with

\[
d(F(x, y), F(u, v)) \leq \frac{k}{2} [d(x, u) + d(y, v)], \quad \text{for all } u \leq x, y \leq v.
\]  

(1.3)

If there exist \(x_0, y_0 \in X\) such that \(x_0 \leq F(x_0, y_0)\) and \(F(y_0, x_0) \leq y_0\), then there exist \(x, y \in X\) such that \(x = F(x, y)\) and \(y = F(y, x)\).

Inspired by Definition 1, Berinde and Borcut [17] introduced the following partial order on the product space \(X^3 := X \times X \times X\) defined as follows

\[(u, v, w) \leq (x, y, z) \quad \text{if and only if} \quad x \geq u, y \leq v, z \geq w,\]

(1.4)

where \((u, v, w), (x, y, z) \in X^3\). Regarding this partial order, we state the definition of the following mapping.

**Definition 5** [See [17]]. Let \((X, \leq)\) be a partially ordered set and \(F : X^3 \rightarrow X\) be a mapping. We say that \(F\) has the mixed monotone property if \(F(x, y, z)\) is monotone non-decreasing in \(x\) and \(z\), and it is monotone non-increasing in \(y\), that is, for any \(x, y, z \in X\) the implications below hold

\[
x_1, x_2 \in X, \quad x_1 \leq x_2 \Rightarrow F(x_1, y, z) \leq F(x_2, y, z),
\]

\[
y_1, y_2 \in X, \quad y_1 \leq y_2 \Rightarrow F(x, y_1, z) \geq F(x, y_2, z),
\]

\[
z_1, z_2 \in X, \quad z_1 \leq z_2 \Rightarrow F(x, y, z_1) \leq F(x, y, z_2).
\]  

(1.5)

**Definition 6** [See [17]]. An element \((x, y, z) \in X^3\) is called a tripled fixed point of \(F : X^3 \rightarrow X\) if

\[
F(x, y, z) = x \quad \text{and} \quad F(y, x, y) = y \quad \text{and} \quad F(z, y, x) = z.
\]  

(1.6)

For a metric space \((X, d)\), the function \(\rho : X^3 \times X^3 \rightarrow [0, \infty)\), given by,

\[
\rho((x, y, z), (u, v, w)) := d(x, u) + d(y, v) + d(z, w)
\]

is a metric on \(X^3\), that is, the pair \((X^3, \rho)\) is a metric space induced by \(d\).

**Theorem 7** [17]. Let \((X, \leq)\) be a partially ordered set and \((X, d)\) a complete metric space. Let \(F : X^3 \rightarrow X\) be a continuous mapping having the mixed monotone property on \(X\). Assume that there exist constants \(a, b, c \in [0, 1)\) such that \(a + b + c < 1\) for which

\[
d(F(x, y, z), F(u, v, w)) \leq ad(x, u) + bd(y, v) + cd(z, w)
\]  

(1.7)

for all \(x \geq u, y \leq v, z \geq w\). If there exist \(x_0, y_0, z_0 \in X\) such that

\[
x_0 \leq F(x_0, y_0, z_0), \quad y_0 \geq F(y_0, x_0, y_0), \quad z_0 \leq F(x_0, y_0, z_0),
\]

then there exist \(x, y, z \in X\) such that

\[
F(x, y, z) = x \quad \text{and} \quad F(y, x, y) = y \quad \text{and} \quad F(z, y, x) = z.
\]
Theorem 11. Let $(X, \leq)$ be a partially ordered set and $(X, d)$ be a complete metric space. Let $F : X^4 \to X$ be a mapping having the mixed monotone property on $X$. Assume that there exist constants $a, b, c \in [0, 1)$ such that $a + b + c < 1$ for which

$$d(F(x, y, z), F(u, v, w)) \leq ad(x, u) + bd(y, v) + cd(z, w)$$

(1.8)

for all $x \geq u$, $y \leq v$, $z \geq w$. Assume that $X$ has the following properties:

(i) if non-decreasing sequence $x_n$ tends to $x$, then $x_n \leq x$ for all $n$,

(ii) if non-increasing sequence $y_n$ tends to $y$, then $y_n \geq y$ for all $n$.

If there exist $x_0, y_0, z_0 \in X$ such that

$$x_0 \leq F(x_0, y_0, z_0), \quad y_0 \geq F(y_0, x_0, y_0), \quad z_0 \leq F(x_0, y_0, z_0),$$

then there exist $x, y, z \in X$ such that

$$F(x, y, z) = x \quad \text{and} \quad F(y, x, y) = y \quad \text{and} \quad F(z, y, x) = z.$$  

Inspired by the results on coupled fixed points and tripled fixed points, Karapınar [18] introduced the notion of quadruple fixed point and proved some related fixed point theorems in partially ordered metric spaces (see also [19–21]). The aim of this paper is to introduce the concept of quadruple fixed point and prove the related fixed point theorems.

2. Quadruple fixed point theorems

Let $(X, \leq)$ be a partially ordered set and $(X, d)$ be a complete metric space. We state the definition of the following mapping.

Definition 9. Let $F : X^4 \to X$ be a mapping. We say that $F$ has the mixed monotone property if $F(x, y, z, w)$ is monotone non-decreasing in $x$ and $z$, and it is monotone non-increasing in $y$ and $w$, that is, for any $x, y, z, w \in X$

$$x_1, x_2 \in X, \quad x_1 \leq x_2 \Rightarrow F(x_1, y, z, w) \leq F(x_2, y, z, w),$$

$$y_1, y_2 \in X, \quad y_1 \leq y_2 \Rightarrow F(x, y_1, z, w) \geq F(x, y_2, z, w),$$

$$z_1, z_2 \in X, \quad z_1 \leq z_2 \Rightarrow F(x, y, z_1, w) \leq F(x, y, z_2, w),$$

$$w_1, w_2 \in X, \quad w_1 \leq w_2 \Rightarrow F(x, y, z, w_1) \geq F(x, y, z, w_2).$$

(2.1)

Definition 10. An element $(x, y, z, w) \in X^4$ is called a quadruple fixed point of $F : X^4 \to X$ if

$$F(x, y, z, w) = x, \quad F(y, z, w, x) = y, \quad F(z, w, x, y) = z \quad \text{and} \quad F(w, x, y, z) = w.$$  

(2.2)

For a metric space $(X, d)$, the function $\rho : X^4 \times X^4 \to [0, \infty)$, given by,

$$\rho((x, y, z, w), (u, v, r, t)) := d(x, u) + d(y, v) + d(z, r) + d(w, t)$$

is a metric on $X^4$, that is, $(X^4, \rho)$ is a metric space induced by $d$.

Let $\Phi$ denote all the functions $\phi : [0, \infty) \to [0, \infty)$ which satisfy that $\lim_{t \to 0^+} \phi(t) > 0$ for all $r > 0$ and $\lim_{t \to 0^+} \phi(t) = 0$.

Let $\Psi$ denote all the functions $\psi : [0, \infty) \to [0, \infty)$ which satisfy

(i) $\psi(t) = 0$ if and only if $t = 0$,

(ii) $\psi$ is continuous and non-decreasing,

(iii) $\psi(s + t) \leq \psi(s) + \psi(t), \forall s, t \in [0, \infty)$.

Examples of typical functions $\phi$ and $\psi$ are given in [3].

The aim of this paper is to prove the following theorem.

Theorem 11. Let $(X, \leq)$ be a partially ordered set and $(X, d)$ be a complete metric space. Let $F : X^4 \to X$ be a mapping having the mixed monotone property on $X$. Assume that for all $x \geq u$, $y \leq v$, $z \geq r$, $w \leq t$,

$$\psi(d(F(x, y, z, w), F(u, v, r, t))) \leq \frac{1}{4} \psi(d(x, u) + d(y, v) + d(z, r) + d(w, t)) - \phi(d(x, u) + d(y, v) + d(z, r) + d(w, t))$$

(2.3)

where $\phi \in \Phi$ and $\psi \in \Psi$. Suppose that there exist $x_0, y_0, z_0, w_0 \in X$ such that

$$x_0 \leq F(x_0, y_0, z_0, w_0), \quad y_0 \geq F(y_0, z_0, w_0, x_0),$$

$$z_0 \leq F(z_0, w_0, x_0, y_0), \quad w_0 \geq F(w_0, x_0, y_0, z_0).$$
Suppose that either
(a) $F$ is continuous, or
(b) $X$ has the following property:
(i) if non-decreasing sequence $x_n$ tends to $x$, then $x_n \leq x$ for all $n$,
(ii) if non-increasing sequence $y_n$ tends to $y$, then $y_n \geq y$ for all $n$.

then there exist $x, y, z, w \in X$ such that

$$F(x, y, z, w) = x, \quad F(y, z, w, x) = y,$$
$$F(z, w, x, y) = z, \quad F(w, x, y, z) = w.$$  

Proof. Let $x_0, y_0, z_0, w_0 \in X$ be such that

$$x_0 \leq F(x_0, y_0, z_0, w_0), \quad y_0 \geq F(y_0, z_0, w_0, x_0),$$
$$z_0 \leq F(z_0, w_0, x_0, y_0), \quad w_0 \geq F(w_0, x_0, y_0, z_0).$$

We construct the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ and $\{w_n\}$ as follows

$$x_n = F(x_{n-1}, y_{n-1}, z_{n-1}, w_{n-1}),$$
$$y_n = F(y_{n-1}, z_{n-1}, w_{n-1}, x_{n-1}),$$
$$z_n = F(z_{n-1}, w_{n-1}, x_{n-1}, y_{n-1}),$$
$$w_n = F(w_{n-1}, x_{n-1}, y_{n-1}, z_{n-1}),$$  \hspace{1cm} (2.4)

for $n = 1, 2, 3, \ldots$

By the mixed monotone property of $F$, it is easy to show that

$$x_0 \leq x_1 \leq \cdots \leq x_n \leq \cdots$$
$$y_0 \geq y_1 \geq \cdots \geq y_n \geq \cdots$$
$$z_0 \leq z_1 \leq \cdots \leq z_n \leq \cdots$$
$$w_0 \geq w_1 \geq \cdots \geq w_n \geq \cdots.$$  \hspace{1cm} (2.5)

Due to (2.3)–(2.5), we have

$$
\psi \left( d(x_{n+1}, x_{n+2}) \right) = \psi \left( d(F(x_n, y_n, z_n, w_n), F(x_{n+1}, y_{n+1}, z_{n+1}, w_{n+1})) \right)
\leq \frac{1}{4} \psi \left( d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(z_n, z_{n+1}) + d(w_n, w_{n+1}) \right)
- \phi \left( d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(z_n, z_{n+1}) + d(w_n, w_{n+1}) \right),
\tag{2.6}
$$

$$
\psi \left( d(y_{n+1}, y_{n+2}) \right) = \psi \left( d(F(y_n, z_n, w_n, x_n), F(y_{n+1}, z_{n+1}, w_{n+1}, x_{n+1})) \right)
\leq \frac{1}{4} \psi \left( d(y_n, y_{n+1}) + d(z_n, z_{n+1}) + d(w_n, w_{n+1}) + d(x_n, x_{n+1}) \right)
- \phi \left( d(y_n, y_{n+1}) + d(z_n, z_{n+1}) + d(w_n, w_{n+1}) + d(x_n, x_{n+1}) \right),
\tag{2.7}
$$

$$
\psi \left( d(z_{n+1}, z_{n+2}) \right) = \psi \left( d(F(z_n, w_n, x_n, y_n), F(z_{n+1}, w_{n+1}, x_{n+1}, y_{n+1})) \right)
\leq \frac{1}{4} \psi \left( d(z_n, z_{n+1}) + d(w_n, w_{n+1}) + d(x_n, x_{n+1}) + d(y_n, y_{n+1}) \right)
- \phi \left( d(z_n, z_{n+1}) + d(w_n, w_{n+1}) + d(x_n, x_{n+1}) + d(y_n, y_{n+1}) \right),
\tag{2.8}
$$

$$
\psi \left( d(w_{n+1}, w_{n+2}) \right) = \psi \left( d(F(w_n, x_n, y_n, z_n), F(w_{n+1}, x_{n+1}, y_{n+1}, z_{n+1})) \right)
\leq \frac{1}{4} \psi \left( d(w_n, w_{n+1}) + d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(z_n, z_{n+1}) \right)
- \phi \left( d(w_n, w_{n+1}) + d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(z_n, z_{n+1}) \right).
\tag{2.9}
$$

Due to (2.6)–(2.9), we conclude that

$$
\psi \left( d(x_{n+1}, x_{n+2}) \right) + \psi \left( d(y_{n+1}, y_{n+2}) \right) + \psi \left( d(z_{n+1}, z_{n+2}) \right) + \psi \left( d(w_{n+1}, w_{n+2}) \right)
\leq \psi \left( d(z_n, z_{n+1}) + d(w_n, w_{n+1}) + d(x_n, x_{n+1}) + d(y_n, y_{n+1}) \right)
- 4\phi \left( d(z_n, z_{n+1}) + d(w_n, w_{n+1}) + d(x_n, x_{n+1}) + d(y_n, y_{n+1}) \right).
\tag{2.10}
$$

From the property (iii) of $\psi$, we have

$$
\psi \left( d(x_{n+1}, x_{n+2}) + d(y_{n+1}, y_{n+2}) + d(z_{n+1}, z_{n+2}) + d(w_{n+1}, w_{n+2}) \right)
\leq \psi \left( d(x_{n+1}, x_{n+2}) \right) + \psi \left( d(y_{n+1}, y_{n+2}) \right) + \psi \left( d(z_{n+1}, z_{n+2}) \right) + \psi \left( d(w_{n+1}, w_{n+2}) \right).
\tag{2.11}
$$
Combining (2.10) and (2.11), we get that
\[
\psi(d(x_{n+1}, x_{n+2}) + d(y_{n+1}, y_{n+2}) + d(z_{n+1}, z_{n+2}) + d(w_{n+1}, w_{n+2})) \\
\leq \psi(d(z_n, z_{n+1}) + d(w_n, w_{n+1}) + d(x_n, x_{n+1}) + d(y_n, y_{n+1})) \\
- 4\phi(d(z_n, z_{n+1}) + d(w_n, w_{n+1}) + d(x_n, x_{n+1}) + d(y_n, y_{n+1})).
\]  
(2.12)

Set \(\delta_n = d(x_n, x_{n-1}) + d(y_n, y_{n-1}) + d(z_n, z_{n-1}) + d(w_n, w_{n-1})\). Then we have
\[
\psi(\delta_{n+2}) \leq \psi(\delta_{n+1}) - 4\phi(\delta_{n+1}) \quad \text{for all } n,
\]  
(2.13)

which yields that
\[
\psi(\delta_{n+2}) \leq \psi(\delta_{n+1}) \quad \text{for all } n.
\]  
(2.14)

Since \(\psi\) is non-decreasing, we get that \(\delta_{n+2} \leq \delta_{n+1}\) for all \(n\). Hence \(\{\delta_n\}\) is a non-increasing sequence. Since it is bounded below, there is some \(\delta \geq 0\) such that
\[
\lim_{n \to \infty} \delta_n = \delta.
\]  
(2.15)

We shall show that \(\delta = 0\). Suppose, on the contrary, that \(\delta > 0\).

Letting \(n \to \infty\) in (2.13) and having in mind that we suppose \(\lim_{t \to r} \phi(t) > 0\) for all \(r > 0\) and \(\lim_{t \to 0^+} \phi(t) = 0\), we have
\[
\delta \leq \delta - 4\phi(\delta) < \delta
\]  
(2.16)

which is a contradiction. Thus, \(\delta = 0\), that is,
\[
\lim_{n \to \infty} \delta_n = \lim_{n \to \infty} [d(x_n, x_{n-1}) + d(y_n, y_{n-1}) + d(z_n, z_{n-1}) + d(w_n, w_{n-1})] = 0.
\]  
(2.17)

Now, we shall prove that \([x_n],[y_n],[z_n]\) and \([w_n]\) are Cauchy sequences. Suppose to the contrary that at least one of \([x_n],[y_n],[z_n]\) and \([w_n]\) is not Cauchy. So, there exists \(\varepsilon > 0\) for which we can find subsequences \([x_{n(k)}],[x_{m(k)}]\) of \([x_n]\) and \([y_{n(k)}],[y_{m(k)}]\) of \([y_n]\) and \([z_{n(k)}],[z_{m(k)}]\) of \([z_n]\) and \([w_{n(k)}],[w_{m(k)}]\) of \([w_n]\) with \(n(k) > m(k) \geq k\) such that
\[
d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)}) + d(z_{n(k)}, z_{m(k)}) + d(w_{n(k)}, w_{m(k)}) \geq \varepsilon.
\]  
(2.18)

Additionally, corresponding to \(m(k)\), we may choose \(n(k)\) such that it is the smallest integer satisfying (2.18) and \(n(k) > m(k) \geq k\). Thus,
\[
d(x_{n(k)-1}, x_{m(k)}) + d(y_{n(k)-1}, y_{m(k)}) + d(z_{n(k)-1}, z_{m(k)}) + d(w_{n(k)-1}, w_{m(k)}) < \varepsilon.
\]  
(2.19)

By using the triangle inequality and having (2.18) and (2.19) in mind we obtain
\[
\varepsilon \leq t_k = d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)}) + d(z_{n(k)}, z_{m(k)}) + d(w_{n(k)}, w_{m(k)}) \\
\leq d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)}) + d(y_{n(k)}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{m(k)}) \\
+ d(z_{n(k)}, z_{n(k)-1}) + d(z_{n(k)-1}, z_{m(k)}) + d(w_{n(k)}, w_{n(k)-1}) + d(w_{n(k)-1}, w_{m(k)}) \\
< d(x_{n(k)}, x_{n(k)-1}) + d(y_{n(k)}, y_{n(k)-1}) + d(z_{n(k)}, z_{n(k)-1}) + d(w_{n(k)}, w_{n(k)-1}) + \varepsilon.
\]  
(2.20)

Letting \(k \to \infty\) in (2.20) and using (2.17) we get
\[
\lim_{k \to \infty} t_k = \lim_{k \to \infty} d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)}) + d(z_{n(k)}, z_{m(k)}) + d(w_{n(k)}, w_{m(k)}) = \varepsilon.
\]  
(2.21)

Again by the triangle inequality,
\[
t_k = d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)}) + d(z_{n(k)}, z_{m(k)}) + d(w_{n(k)}, w_{m(k)}) \\
\leq d(x_{n(k)}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{m(k)+1}) + d(x_{n(k)+1}, x_{m(k)}) + d(y_{n(k)}, y_{n(k)+1}) + d(y_{n(k)+1}, y_{m(k)}) \\
+ d(y_{m(k)}, y_{m(k)+1}) + d(y_{m(k)+1}, z_{m(k)+1}) + d(z_{m(k)+1}, z_{n(k)+1}) + d(z_{n(k)+1}, z_{m(k)}) \\
+ d(w_{n(k)}, w_{n(k)+1}) + d(w_{n(k)+1}, w_{m(k)+1}) + d(w_{m(k)+1}, w_{m(k)}) \\
\leq \delta_{m(k)+1} + \delta_{n(k)+1} + d(x_{n(k)+1}, x_{m(k)+1}) + d(y_{n(k)+1}, y_{m(k)+1}) \\
+ d(z_{n(k)+1}, z_{m(k)+1}) + d(w_{n(k)+1}, w_{m(k)+1}).
\]  
(2.22)

Since \(n(k) > m(k)\), then
\[
x_{n(k)} \geq x_{m(k)} \quad \text{and} \quad y_{n(k)} \leq y_{m(k)},
\]  
(2.23)
Hence from (2.3), (2.4) and (2.23), we have,

\[
\psi (d(x_{n(k)} + 1, x_{m(k) + 1})) = \psi \left( d(F(x_{n(k)}, y_{n(k)}, z_{n(k)}, w_{n(k)}), F(x_{m(k)}, y_{m(k)}, z_{m(k)}, w_{m(k)})) \right) \\
\leq \frac{1}{4} \psi \left( d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)}) + d(z_{n(k)}, z_{m(k)}) + d(w_{n(k)}, w_{m(k)}) \right) \\
- \phi \left( d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)}) + d(z_{n(k)}, z_{m(k)}) + d(w_{n(k)}, w_{m(k)}) \right).
\]

(2.24)

\[
\psi (d(y_{n(k)} + 1, y_{m(k) + 1})) = \psi \left( d(F(y_{n(k)}, z_{n(k)}, w_{n(k)}, x_{n(k)}), F(y_{m(k)}, z_{m(k)}, w_{m(k)}, x_{m(k)})) \right) \\
\leq \frac{1}{4} \psi \left( d(y_{n(k)}, y_{m(k)}) + d(z_{n(k)}, z_{m(k)}) + d(w_{n(k)}, w_{m(k)}) + d(x_{n(k)}, x_{m(k)}) \right) \\
- \phi \left( d(y_{n(k)}, y_{m(k)}) + d(z_{n(k)}, z_{m(k)}) + d(w_{n(k)}, w_{m(k)}) + d(x_{n(k)}, x_{m(k)}) \right).
\]

(2.25)

\[
\psi (d(z_{n(k)} + 1, z_{m(k) + 1})) = \psi \left( d(F(z_{n(k)}, w_{n(k)}, x_{n(k)}, y_{n(k)}), F(z_{m(k)}, w_{m(k)}, x_{m(k)}, y_{m(k)})) \right) \\
\leq \frac{1}{4} \psi \left( d(z_{n(k)}, z_{m(k)}) + d(w_{n(k)}, w_{m(k)}) + d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)}) \right) \\
- \phi \left( d(z_{n(k)}, z_{m(k)}) + d(w_{n(k)}, w_{m(k)}) + d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)}) \right).
\]

(2.26)

\[
\psi (d(w_{n(k)} + 1, w_{m(k) + 1})) = \psi \left( d(F(w_{n(k)}, x_{n(k)}, y_{n(k)}, z_{n(k)}), F(w_{m(k)}, x_{m(k)}, y_{m(k)}, z_{m(k)})) \right) \\
\leq \frac{1}{4} \psi \left( d(w_{n(k)}, w_{m(k)}) + d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)}) + d(z_{n(k)}, z_{m(k)}) \right) \\
- \phi \left( d(w_{n(k)}, w_{m(k)}) + d(x_{n(k)}, x_{m(k)}) + d(y_{n(k)}, y_{m(k)}) + d(z_{n(k)}, z_{m(k)}) \right).
\]

(2.27)

Combining (2.22) with (2.24)–(2.27), we obtain that

\[
\psi (t_k) \leq \psi (\delta_{n(k)} + 1 + \delta_{m(k)} + 1 + d(x_{n(k)} + 1, x_{m(k)}) + d(y_{n(k)} + 1, y_{m(k)} + 1) + d(z_{n(k)} + 1, z_{m(k)} + 1) + d(w_{n(k)} + 1, w_{m(k)} + 1)) \\
\leq \psi (\delta_{n(k)} + 1 + \delta_{m(k)} + 1) + \psi (t_k) - 4 \phi (t_k).
\]

(2.28)

Letting \( k \to \infty \), we get a contradiction. This shows that \( \{x_n\}, \{y_n\}, \{z_n\} \) and \( \{w_n\} \) are Cauchy sequences. Since \( X \) is a complete metric space, there exist \( x, y, z, w \in X \) such that

\[
\lim_{n \to \infty} x_n = x \quad \text{and} \quad \lim_{n \to \infty} y_n = y, \\
\lim_{n \to \infty} z_n = z \quad \text{and} \quad \lim_{n \to \infty} w_n = w.
\]

(2.29)

Suppose that the assumption (a) holds. Then by (2.4) and (2.29), we have

\[
x = \lim_{n \to \infty} x_n = \lim_{n \to \infty} F(x_{n-1}, y_{n-1}, z_{n-1}, w_{n-1}) \\
= F \left( \lim_{n \to \infty} x_{n-1}, \lim_{n \to \infty} y_{n-1}, \lim_{n \to \infty} z_{n-1}, \lim_{n \to \infty} w_{n-1} \right) \\
= F(x, y, z, w).
\]

(2.30)

Analogously, we also observe that

\[
y = \lim_{n \to \infty} y_n = \lim_{n \to \infty} F(y_{n-1}, z_{n-1}, w_{n-1}, x_{n-1}) = F(y, z, w, x) \\
z = \lim_{n \to \infty} z_n = \lim_{n \to \infty} F(z_{n-1}, w_{n-1}, x_{n-1}, y_{n-1}) = F(z, w, x, y) \\
w = \lim_{n \to \infty} w_n = \lim_{n \to \infty} F(w_{n-1}, x_{n-1}, y_{n-1}, z_{n-1}) = F(w, x, y, z).
\]

(2.31)

Thus, we have

\[
F(x, y, z, w) = x, \quad F(y, z, w, x) = y, \\
F(z, w, x, y) = z, \quad F(w, x, y, z) = w.
\]

Let us assume that the assumption (b) holds. Since \( \{x_n\} \) and \( \{z_n\} \) are non-decreasing and \( x_n \) tends to \( x \) and \( z_n \) tends to \( z \) and also \( \{y_n\} \) and \( \{w_n\} \) are non-increasing and \( y_n \) tends to \( y \) and \( w_n \) tends to \( w \), we have

\[
x_n \geq x, \quad y_n \leq y, \quad z_n \geq z, \quad w_n \leq w
\]

for all \( n \), by the assumption (b). Consider now

\[
d(x, F(x, y, z, w)) \leq d(x, x_{n+1}) + d(x_{n+1}, F(x, y, z, w)) \\
= d(x, x_{n+1}) + d(F(x_n, y_n, z_n, w_n), F(x, y, z, w))
\]
As $n$ tends to $\infty$ in (2.32) and using (2.29), we get that $d(x,F(x,y,z,w)) = 0$. Thus, $x = F(x,y,z,w)$. Analogously, we get that

$$F(y,z,w,x) = y, \quad F(z,w,x,y) = z, \quad F(w,x,y,z) = w.$$  

Thus, we proved that $F$ has a quadruple fixed point. \(\Box\)

**Corollary 12.** Let $(X, \leq)$ be a partially ordered set and $(X, d)$ be a complete metric space. Let $F : X^4 \to X$ be a mapping having the mixed monotone property on $X$. Assume that there exists $k \in [0, 1)$ such that

$$d(F(x,y,z,w), F(u,v,r,t)) \leq \frac{k}{4} [d(x,u) + d(y,v) + d(z,r) + d(w,t)]$$

for all $x \geq u, y \leq v, z \geq r$ and $w \leq t$. Suppose that there exist $x_0, y_0, z_0, w_0 \in X$ such that

$$x_0 \leq F(x_0, y_0, z_0, w_0), \quad y_0 \geq F(y_0, z_0, w_0, x_0),$$

$$z_0 \leq F(z_0, w_0, x_0, y_0), \quad w_0 \geq F(w_0, x_0, y_0, z_0).$$

Suppose that either

(a) $F$ is continuous, or

(b) $X$ has the following property:

(i) if non-decreasing sequence $x_n \to x$, then $x_n \leq x$ for all $n$,

(ii) if non-increasing sequence $y_n \to y$, then $y_n \geq y$ for all $n$,

then there exist $x, y, z, w \in X$ such that

$$F(x,y,z,w) = x, \quad F(y,z,w,x) = y,$$

$$F(z,w,x,y) = z, \quad F(w,x,y,z) = w.$$  

**Proof.** It is sufficient to take $\psi(t) = t$ and $\phi = \frac{1-k}{4}t$ in the previous theorem. \(\Box\)

3. Uniqueness of quadruple fixed point

In this section we shall prove the uniqueness of the quadruple fixed point. For a product $X^4$ of a partially ordered set $(X, \leq)$ we define a partial ordering in the following way: For all $(x,y,z,w), (u,v,r,t) \in X^4$

$$(x,y,z,w) \leq (u,v,r,t) \iff x \leq u, \quad y \geq v, \quad z \leq r, \quad w \geq t.$$  

(3.1)

We say that $(x,y,z,w)$ is equal to $(u,v,r,t)$ if and only if $x = u, y = v, z = r$ and $w = t$.

**Theorem 13.** In addition to the hypothesis of Theorem 11, suppose that for all $(x,y,z,w), (u,v,r,t) \in X^4$, there exists $(a,b,c,d) \in X^4$ that is comparable to $(x,y,z,w)$ and $(u,v,r,t)$, then $F$ has a unique quadruple fixed point.

**Proof.** The set of quadruple fixed points of $F$ is not empty due to Theorem 11. Assume that $(x,y,z,w)$ and $(u,v,r,t)$ are quadruple fixed points of $F$, that is,

$$F(x,y,z,w) = x, \quad F(u,v,r,t) = u,$$

$$F(y,z,w,x) = y, \quad F(v,r,t,u) = v,$$

$$F(z,w,x,y) = z, \quad F(r,t,u,v) = r,$$

$$F(w,x,y,z) = w, \quad F(t,u,v,r) = t.$$  

We shall show that $(x,y,z,w)$ and $(u,v,r,t)$ are equal. By the assumption of the theorem, there exists $(a,b,c,d) \in X^4$ that is comparable to $(x,y,z,w)$ and $(u,v,r,t)$. Define sequences $\{a_n\}, \{b_n\}, \{c_n\}$ and $\{d_n\}$ such that

$$a = a_0, \quad b = b_0, \quad c = c_0, \quad d = d_0$$

and

$$a_n = F(a_{n-1}, b_{n-1}, c_{n-1}, d_{n-1}),$$

$$b_n = F(b_{n-1}, c_{n-1}, d_{n-1}, a_{n-1}),$$

$$c_n = F(c_{n-1}, d_{n-1}, a_{n-1}, b_{n-1}),$$

$$d_n = F(d_{n-1}, a_{n-1}, b_{n-1}, c_{n-1})$$

(3.2)
by (2.3) and (3.3), we have
\[
\gamma \leq a_n + b_n + c_n + d_n.
\] Consequently, we have
\[
\gamma \leq a_n + b_n + c_n + d_n.
\]
Recursively, we get that
\[
\gamma \leq a_n + b_n + c_n + d_n.
\]

By (2.3) and (3.3), we have
\[
\psi(d(x, a_{n+1})) = \psi(d(F(x, y, z, w), F(a_n, b_n, c_n, d_n)))
\]
\[
\leq \frac{1}{2} \psi(d(x, a_n) + d(y, b_n) + d(z, c_n) + d(w, d_n))
\]
\[
- \phi(d(x, a_n) + d(y, b_n) + d(z, c_n) + d(w, d_n)),
\]
\[
\psi(d(b_{n+1}, y)) = \psi(d(F(b_n, c_n, d_n, a_n), F(y, z, w, x)))
\]
\[
\leq \frac{1}{2} \psi(d(b_n, y) + d(c_n, z) + d(d_n, w) + d(a_n, x))
\]
\[
- \phi(d(b_n, y) + d(c_n, z) + d(d_n, w) + d(a_n, x)),
\]
\[
\psi(d(z, c_{n+1})) = \psi(d(F(z, w, x, y), F(c_n, d_n, a_n, b_n)))
\]
\[
\leq \frac{1}{2} \psi(d(z, c_n) + d(w, d_n) + d(x, a_n) + d(y, b_n))
\]
\[
- \phi(d(z, c_n) + d(w, d_n) + d(x, a_n) + d(y, b_n)),
\]
\[
\psi(d(d_{n+1}, w)) = \psi(d(F(d_n, a_n, b_n, c_n), F(w, x, y, z)))
\]
\[
\leq \frac{1}{2} \psi(d(d_n, w) + d(a_n, x) + d(b_n, y) + d(c_n, z))
\]
\[
- \phi(d(d_n, w) + d(a_n, x) + d(b_n, y) + d(c_n, z)).
\]

Set \(\gamma_n = d(x, a_n) + d(y, b_n) + d(z, c_n) + d(w, d_n)\). Then, due to (3.4)-(3.7), we have
\[
\psi(\gamma_{n+1}) \leq \psi(\gamma_n) - 4\phi(\gamma_n)
\]
for all \(n\), which implies
\[
\gamma_{n+1} \leq \gamma_n.
\]
Hence, the sequence \(\{\gamma_n\}\) is decreasing and bounded below. Thus, there exists \(\gamma \geq 0\) such that
\[
\lim_{n \to \infty} \gamma_n = \gamma.
\]
Now, we shall show that \(\gamma = 0\). Suppose to the contrary that \(\gamma > 0\).

Letting \(n \to \infty\) in
\[
\psi(\gamma_{n+1}) \leq \psi(\gamma_n) - 4\phi(\gamma_n),
\]
we obtain that
\[
\psi(\gamma) \leq \psi(\gamma) - 4 \lim_{n \to \infty} \phi(\gamma_n) < \psi(\gamma)
\]
which is a contradiction. Therefore, \(\gamma = 0\). That is,
\[
\lim_{n \to \infty} \gamma_n = 0.
\]
Consequently, we have
\[
\lim_{n \to \infty} d(x, a_n) = 0, \quad \lim_{n \to \infty} d(y, b_n) = 0,
\]
\[
\lim_{n \to \infty} d(z, c_n) = 0, \quad \lim_{n \to \infty} d(w, d_n) = 0.
\]

Similarly, we show that
\[
\lim_{n \to \infty} d(u, a_n) = 0, \quad \lim_{n \to \infty} d(v, b_n) = 0,
\]
\[
\lim_{n \to \infty} d(r, c_n) = 0, \quad \lim_{n \to \infty} d(s, d_n) = 0.
\]
Combining (3.8) and (3.9) yields that \((x, y, z, w)\) and \((u, v, r, t)\) are equal. \(\square\)

**Remark 14.** We would like to point out that the discussions we presented above can be generalized for the mappings
\(F : X^{2n} \to X, (n = 2, 3, \ldots)\). Notice that the techniques introduced in [17] are required to produce some related results for the mappings \(F : X^{2n+1} \to X, (n = 2, 3, \ldots)\).
4. Examples

In this section we give some examples to show that our results are effective.

Example 15. Let $X = [0, \infty)$ with the metric $d(x, y) = |x - y|$, for all $x, y \in X$ and the following order relation:

$x, y \in X, \quad x \leq y \iff x = y$ or $(x, y \in Z$ and $x \leq y)$,

where $Z$ is the set of integers and $\leq$ is the usual ordering.

Let $F : X^4 \to X$ be given by

$$F(x, y, z, w) = \begin{cases} 1, & \text{if } xyzw \neq 0 \\ 0, & \text{if } xyzw = 0 \end{cases}$$

for all $x, y, z, w \in X$.

Let $\psi, \phi : [0, \infty) \to [0, \infty)$ be given by

$$\psi(t) = t, \quad \text{and} \quad \phi(t) = \frac{t}{10}$$

for all $t \in [0, \infty)$.

It is easy to check that all the conditions of Theorem 11 are satisfied. By applying Theorem 11 we conclude that $F$ has a quadruple fixed point. In fact, $F$ has two quadruple fixed points. They are $(0, 0, 0, 0)$ and $(1, 1, 1, 1)$. Therefore, the conditions of Theorem 11 are not sufficient for the uniqueness of a quadruple fixed point.

Example 16. Let $X = \mathbb{R}$ with the metric $d(x, y) = |x - y|$, for all $x, y \in X$ and the usual ordering.

Let $F : X^4 \to X$ be given by

$$F(x, y, z, w) = \frac{x - y + z - w}{16}, \quad \text{for all } x, y, z, w \in X.$$

Let $\psi, \phi : [0, \infty) \to [0, \infty)$ be given by

$$\psi(t) = t, \quad \text{and} \quad \phi(t) = \frac{3t}{16}$$

for all $t \in [0, \infty)$.

It is easy to check that all the conditions of Theorem 13 are satisfied and $(0, 0, 0, 0)$ is the unique quadruple fixed point of $F$.

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References


