On the isomorphism problem for a family of cubic metacirculant graphs

Ngo Dac Tan

Institute of Mathematics, P.O. Box 631 Bo Ho, 10.000 Hanoi, Viet Nam

Received 15 November 1991; revised 15 August 1992

Abstract

In this paper an isomorphism testing algorithm for graphs in the family of all cubic metacirculant graphs with non-empty first symbol $S_0$ is given. The time complexity of this algorithm is also evaluated.

1. Introduction

Throughout this paper the term graph always means a finite undirected graph without loops and multiple edges. We write $\mathbb{Z}_n$ for the ring of integers modulo $n$ and $\mathbb{Z}_n^*$ for the multiplicative group of units in $\mathbb{Z}_n$, where $n$ is a positive integer.

Let $m$ and $n$ be two positive integers, $\alpha \in \mathbb{Z}_n^*$, $\mu = \lfloor m/2 \rfloor$ (where $\lfloor \_ \rfloor$ denotes the greatest integer function) and let $S_0, S_1, \ldots, S_{\mu}$ be subsets of $\mathbb{Z}_n$ satisfying the following conditions: (1) $0 \notin S_0 = -S_0$; (2) $\alpha^m S_r = S_r$ for $0 \leq r \leq \mu$; (3) if $m$ is even, then $\alpha^m S_\mu = -S_\mu$. Then we define the $(m,n)$-metacirculant graph $G = MC(m, n, \alpha, S_0, S_1, \ldots, S_{\mu})$ to be the graph with vertex-set $V(G) = \{v_i^j : i \in \mathbb{Z}_m; j \in \mathbb{Z}_n\}$ and edge-set $E(G) = \{v_i^jv_i^{j+r} : 0 \leq r \leq \mu; i \in \mathbb{Z}_m; h, j \in \mathbb{Z}_n; (h-j) \in \alpha^m S_r\}$, where superscripts and subscripts are always reduced modulo $m$ and modulo $n$, respectively.

The concept of $(m,n)$-metacirculant graphs was generally defined in 1982 by Alspach and Parsons [2]. All $(m,n)$-metacirculant graphs are vertex-transitive. Their automorphism groups contain a transitive (on the vertex-set of the graphs) subgroup that is the semi-direct product of two cyclic ones. Thus, this subgroup has a rather simple structure. Even the converse assertion has been proved: if the automorphism group of some graph has the property mentioned above, then this graph is an $(m,n)$-metacirculant [2]. For more information about $(m,n)$-metacirculant graphs see [3, 1, 4, 6].
Let $\Phi(m, n)$ be the family of all cubic $(m, n)$-metacirculant graphs with the condition $S_0 \neq \emptyset$ and let

$$\Phi = \bigcup_{m \geq 1, n \geq 1} \Phi(m, n).$$

Recently, the author [6] has investigated this family $\Phi$. In particular, the components and the automorphism groups of graphs in $\Phi$ are determined. It has also been proved that except the Petersen graph, all connected graphs of the family are hamiltonian.

In [2] Alspach and Parsons have raised the following isomorphism problem: When are two metacirculant graphs $MC(m, n, \alpha, S_0, S_1, \ldots, S_\mu)$ and $MC(m', n', \alpha', S'_0, S'_1, \ldots, S'\mu)$ isomorphic? In this paper, from an algorithmic point of view, we shall consider this isomorphism problem for the family $\Phi$. We shall give here an isomorphism testing algorithm for graphs in $\Phi$ and evaluate the time complexity of this algorithm.

2. Preliminaries

In [6] we established a criterion in terms of the parameters of $(m, n)$-metacirculant graphs that determines whether or not a given $(m, n)$-metacirculant graph belongs to $\Phi$.

**Proposition 1** (Tan [6]). An $(m, n)$-metacirculant graph $G = MC(m, n, \alpha, S_0, S_1, \ldots, S_\mu)$ has $S_0 \neq \emptyset$ and is a cubic graph if and only if one of the following conditions is satisfied:

1. $|S_0| = 3$, $n$ is even and $S_i = \emptyset$ for all $i \in \{1, 2, \ldots, \mu\}$;
2. $|S_0| = 2$, $m$ is even, $|S_\mu| = 1$ and $S_i = \emptyset$ for all $i \in \{1, 2, \ldots, \mu - 1\}$;
3. $|S_0| = 1$, $m$ is odd, $n$ is even, $|S_i| = 1$ for some $i \in \{1, 2, \ldots, \mu\}$ and $S_j = \emptyset$ for all $i \neq j \neq 0$;
4. $|S_0| = 1$, $m$ is even, $n$ is even, $|S_i| = 1$ for some $i \in \{1, 2, \ldots, \mu - 1\}$ and $S_j = S_\mu = \emptyset$ for all $i \neq j \in \{1, 2, \ldots, \mu - 1\}$;
5. $|S_0| = 1$, $m$ is even, $n$ is even, $|S_\mu| = 2$ and $S_i = \emptyset$ for all $i \in \{1, 2, \ldots, \mu - 1\}$.

Let $n$ be a positive integer and $S \subseteq \mathbb{Z}_n \setminus \{0\}$ with the property that $i \in S$ implies $-i \in S$. We define the circulant graph $G = C(n, S)$ to be the graph with vertex-set $V(G) = \{v_i : i \in \mathbb{Z}_n\}$ and edge-set $E(G) = \{v_i v_j : i, j \in \mathbb{Z}_n; (j - i) \in S\}$, where subscripts are always reduced modulo $n$.

For integers $n$ and $k$ with $n \geq 2$ and $1 \leq k \leq n - 1$ we define the generalized Petersen graph $G = GP(n, k)$ to be the graph with vertex-set $V(G) = \{u_i, v_i : i \in \mathbb{Z}_n\}$ and edge-set $E(G) = \{u_i u_{i+1}, u_i v_i, v_i v_{i+k} : i \in \mathbb{Z}_n\}$, where subscripts are always reduced modulo $n$.

Thus, $GP(n, k)$ is a graph of order $2n$ and $GP(5, 2)$ is the well-known Petersen graph.

The following result is proved in [6].
Proposition 2 (Tan [6]). Let $G$ be a cubic $(m, n)$-metacirculant graph with the condition $S_0 \neq \emptyset$. Then its components are isomorphic to each other and to some of the following graphs:

1. a circulant graph $C(2l, S)$ on $2l$ vertices with $S = \{1, -1, l\}$, where $l$ is an integer satisfying $l > 1$;
2. a generalized Petersen graph $GP(d, k)$, where $d$ and $k$ are positive integers satisfying $d > 2$, $k \in \mathbb{Z}^*$ and $k^2 \equiv \pm 1 \pmod{d}$.

In order to describe later clearly our isomorphism testing algorithm for graphs in $\Phi$, we present here an algorithm for determining components of a cubic $(m, n)$-metacirculant graph with the condition $S_0 \neq \emptyset$. This algorithm follows from the proof of Theorem 1 in [6].

Algorithm for determining components of a cubic $(m, n)$-metacirculant graph with the condition $S_0 \neq \emptyset$

Let $G = MC(m, n, x, S_0, S_1, \ldots, S_p)$ be a cubic $(m, n)$-metacirculant graph with the condition $S_0 \neq \emptyset$. By Proposition 1, one of Conditions 1–5 in this proposition is satisfied.

Case 1: Condition 1 is satisfied, $S_0 = \{s, -s, n/2\}$ with $0 \neq s \neq n/2$. Let $d$ be the order of the subgroup $\langle s \rangle$ of the additive group of $\mathbb{Z}_n$. If $d = 2l$ with $l > 1$, then the components of $G$ are isomorphic to $C(d, S)$ with $S = \{1, -1, l\}$, otherwise, they are isomorphic to $GP(d, 1)$.

Case 2: Condition 2 is satisfied, $S_0 = \{s, -s\}$ with $s \neq -s \neq 0$. Let $k = x^{m/2}$ and $d$ be the order of the subgroup $\langle s \rangle$ of the additive group of $\mathbb{Z}_n$. Then the components of $G$ are isomorphic to $GP(d, k)$.

Case 3: Condition 3 or Condition 4 is satisfied, $S_i = \{s\}$ with $i \in \{1, 2, \ldots, p\}$ if Condition 3 holds and $i \in \{1, 2, \ldots, p - 1\}$ if Condition 4 holds. Let $c$ be the order of the subgroup $\langle i \rangle$ of the additive group of $\mathbb{Z}_m$ and $d$ be the order of the subgroup of the additive group of $\mathbb{Z}_n$ generated by $(x^{c-1} + x^{c-2} + \ldots + x + 1)s \pmod{n}$. If $d = 0$, then the components of $G$ are isomorphic to $GP(c, 1)$. If $d > 0$ and $d = 2l$ with $l > 1$, then the components of $G$ are isomorphic to $C(cd, S)$ with $S = \{1, -1, cl\}$; otherwise, they are isomorphic to $GP(cd, 1)$.

Case 4: Condition 5 is satisfied, $S_\mu = \{s, r\}$ with $s \neq r \pmod{n}$. Let $d$ be the order of the subgroup $\langle s - r \rangle$ of the additive group of $\mathbb{Z}_n$. If $d$ is even, then the components of $G$ are isomorphic to $C(2d, S)$ with $S = \{1, -1, d\}$; otherwise, they are isomorphic to $GP(2d, 1)$.

We consider now the time complexity of this algorithm. It is not difficult to see that the order $d$ of the subgroup $\langle s \rangle$ of the additive group of $\mathbb{Z}_n$ is equal to $n/\gcd(n, s)$. We can compute $\gcd(n, s)$ using Euclid’s algorithm. Therefore, if we denote by $\lceil x \rceil$ the least integer $z \geq x$, then $d$ can be computed in at most $2\lceil \log_2 n \rceil + 1$ divisions. The exponent $x$ can be computed with the algorithm that uses repeated squaring method. So to compute $x^i$ we need at most $2\lceil \log_2 i \rceil$ multiplications.
Thus, if case 1 or case 4 holds, then we need at most $2^\lceil \log_2 n \rceil + 2$ arithmetic operations to determine the components of $G$. Analogously, in case 2, we need at most $2^\lceil \log_2 (m/2) \rceil + 1 = 2^\lceil \log_2 m \rceil - 1$ operations to compute $k$ and $2^\lceil \log_2 n \rceil + 1$ to compute $d$. In total, at most $2^\lceil \log_2 m \rceil - 1 + 2^\lceil \log_2 n \rceil + 1 \leq 2^\lceil \log_2 (mn) \rceil$ operations are needed. Finally, suppose that case 3 is valid. To compute $d$, we must know the element 

$$(\alpha^{(c-1)i} + \alpha^{(c-2)i} + \ldots + \alpha^i + 1)s \pmod{n}. \quad (1)$$

We have $\alpha^{(c-1)i} + \alpha^{(c-2)i} + \ldots + \alpha^i + 1 = ((\alpha^c - 1)/(\alpha^i - 1))$ with $c \leq m$ and $1 \leq i \leq \lfloor m/2 \rfloor$. Therefore, we can compute the element (1) in at most $2^\lceil \log_2 (m/2) \rceil + 2^\lceil \log_2 m \rceil + 6 = 4^\lceil \log_2 m \rceil + 4$ operations. Consequently, we need at most $4^\lceil \log_2 m \rceil + 4 + 2^\lceil \log_2 n \rceil + 1 = 4^\lceil \log_2 m \rceil + 2^\lceil \log_2 n \rceil + 5$ operations to compute $d$ and $2^\lceil \log_2 m \rceil + 1$ operations to compute $c$. In total, to determine the components of $G$ in this case at most $4^\lceil \log_2 m \rceil + 2^\lceil \log_2 n \rceil + 5 + 2^\lceil \log_2 m \rceil + 1 \leq 8^\lceil \log_2 (m+n) \rceil = 8^\lceil \log_2 (mn) \rceil$ operations are needed.

Thus, in any case we need at most $O(\log_2 (mn))$ arithmetical operations to determine the components of $G$. Using the fact that every arithmetical operation on $k$-bit numbers requires at most $O(k^2)$ bit operations, we conclude that the time complexity of our algorithm is $O(\log_2 (mn))O(\log_2^2 (mn)) = O(\log_2^3 (mn))$.

3. An isomorphism testing algorithm for graphs in $\Phi$

We consider now the isomorphism problem for graphs in $\Phi$. Since $\Phi$ is contained in the family $\Psi$ of graphs, each of which is either the union of a finite number of disjoint copies of $C(2l, S)$ with $l > 1$ and $S = \{1, -1, l\}$ or the union of a finite number of disjoint copies of $GP(d, k)$ with $d \geq 2$, $k \in \mathbb{Z}_*^+$ and $k^2 \equiv \pm 1 \pmod{d}$, we first consider the same problem for graphs in $\Psi$.

**Proposition 3.** Let $G$ be the union of $t$ disjoint copies of $C(2l, S)$ with $l > 1$, $S = \{1, -1, l\}$ and $G'$ be the union of $t'$ disjoint copies of $C(2l', S')$ with $l' > 1$, $S' = \{1, -1, l'\}$. Then $G'$ and $G$ are isomorphic if and only if $t' = t$ and $l' = l$.

**Proof.** It is trivial that $G'$ and $G$ are isomorphic if $t' = t$ and $l' = l$. Conversely, suppose that $G'$ and $G$ are isomorphic. Since $C(2l', S')$ and $C(2l, S)$ are connected graphs, $G'$ consists of $t'$ components each of which is isomorphic to $C(2l', S')$ and $G$ consists of $t$ components each of which is isomorphic to $C(2l, S)$. Consequently, $t'$ has to be equal to $t$ and $C(2l', S')$ has to be isomorphic to $C(2l, S)$. But $C(2l', S')$ is a graph of order $2l'$ and $C(2l, S)$ is a graph of order $2l$. Therefore, $l' = l$ and Proposition 3 is proved. $\Box$

**Proposition 4.** Let $G$ be the union of $t$ disjoint copies of $C(2l, S)$ with $l > 1$, $S = \{1, -1, l\}$ and $G'$ be the union of $t'$ disjoint copies of $GP(d, k)$ with $d \geq 2$, $k \in \mathbb{Z}_*^+$ and $k^2 \equiv \pm 1 \pmod{d}$. Then $G$ is not isomorphic to $G'$.
Proof. It is not difficult to see that $C(2l, S)$ with $l > 1$, $S = \{1, -1, l\}$ and $GP(d, k)$ with $d > 2$, $k \in \mathbb{Z}_d^+$, $k^2 \equiv \pm 1(\text{mod} \ d)$ are connected graphs. Therefore, $G$ consists of $t$ components each of which is isomorphic to $C(2l, S)$ and $G'$ consists of $t'$ components each of which is isomorphic to $GP(d, k)$.

Suppose that $G$ is isomorphic to $G'$. Then $t$ has to be equal to $t'$ and $C(2l, S)$ has to be isomorphic to $GP(d, k)$. Hence, $l = d \geq 2$ because the order of $C(2l, S)$ is equal to $2l$ and the order of $GP(d, k)$ is equal to $2d$.

We have $C(4, S) = K_4$ (the complete graph of order 4) and $GP(2, 1) = C_4$ (the cycle of length 4). So $C(4, S)$ is not isomorphic to $GP(2, 1)$. The graph $C(6, S)$ has girth 4, and $GP(3, 1)$ and $GP(3, 2)$ have girth 3. Hence, $C(6, S)$ is not isomorphic to $GP(3, k)$ ($k = 1, 2$), either. All this implies that $l = d \geq 4$.

The graph $C(2l, S)$, with $l \geq 4$, $S = \{1, -1, l\}$, has the cycles $v_i v_{i+1} v_{i+1} v_i$ ($i = 0, \ldots, 2l - 1$) of length 4 and it does not have any cycles of length 3. This means that the girth of $C(2l, S)$ with $l \geq 4$ is 4.

Consider $GP(d, k)$ with $d \geq 4$, $k \in \mathbb{Z}_d^+$ and $k^2 \equiv \pm 1(\text{mod} \ d)$. If $d = 4$, then only $k = 1$ or $k = 3$ satisfy the conditions $k \in \mathbb{Z}_d^+$ and $k^2 \equiv \pm 1(\text{mod} \ d)$. Suppose that $d \geq 4$. If some cycle of $GP(d, k)$ does not contain edges of the type $u_i v_i$ (see the definition of $GP(n, k)$), then this cycle contains either all the vertices $u_i$ and no vertices $v_i$ or all the vertices $v_i$ and no vertices $u_i$ with $i = 0, \ldots, d - 1$. Hence it has length $d \geq 5$. Therefore, if some cycle $X$ of $GP(d, k)$ has length $l \leq 4$, then it has to contain two edges of the type $u_i v_i$. This implies that $X$ has to be a cycle of the form $v_i u_i u_{i+1} v_{i+1}$. So $v_i$ is adjacent to $v_{i+1}$ and $k \equiv \pm 1(\text{mod} \ d)$. This means that if $d \geq 4$ and $k \not\equiv \pm 1(\text{mod} \ d)$, then the girth of $GP(d, k)$ is at least 5 and $GP(d, k)$ cannot be isomorphic to $C(2l, S)$.

Thus, if $C(2l, S)$, with $l > 1$ and $S = \{1, -1, l\}$, is isomorphic to $GP(d, k)$, with $d > 2$, $k \in \mathbb{Z}_d^+$ and $k^2 \equiv \pm 1(\text{mod} \ d)$, then $l = d \geq 4$ and $k \equiv \pm 1(\text{mod} \ d)$.

We show now that the vertex-independence number $\beta_0(C(2l, S))$ of $C(2l, S)$ is odd. Since the vertex-independence number of a cycle of length $p$ is $\left\lfloor p/2 \right\rfloor$, we have $\beta_0(C(2l, S)) \leq \beta_0(C(2l, \{1, -1\}) = \left\lfloor 2l/2 \right\rfloor = l$. If $l$ is odd, then $\{v_0, v_2, v_4, \ldots, v_{2l-2}\}$ is a set of independent vertices of $C(2l, S)$. Therefore, $\beta_0(C(2l, S)) = l$ is odd. Suppose that $l$ is even. It is clear that every set of independent vertices of $C(2l, S)$ is also a set of independent vertices of $C(2l, \{1, -1\})$ and only the sets $\{v_0, v_2, v_4, \ldots, v_{2l-2}\}$ and $\{v_1, v_3, v_5, \ldots, v_{2l-1}\}$ of independent vertices of $C(2l, \{1, -1\})$ have $\beta_0(C(2l, \{1, -1\}))$ vertices. Therefore, if $\beta_0(C(2l, S)) = l$, then at least one of the sets $\{v_0, v_2, v_4, \ldots, v_{2l-2}\}$ and $\{v_1, v_3, v_5, \ldots, v_{2l-1}\}$ is a set of independent vertices of $C(2l, S)$. This is impossible because $v_i \in \{v_0, v_2, v_4, \ldots, v_{2l-2}\}$, $v_i$ is adjacent to $v_0$, $v_{i+1} \in \{v_1, v_3, \ldots, v_{2l-1}\}$ and $v_{i+1}$ is adjacent to $v_i$. Hence, $\beta_0(C(2l, S)) \leq l - 1$. But the set $\{v_0, v_2, \ldots, v_{2l-2}, v_{i+1}, v_{i+3}, \ldots, v_{2l-3}\}$ is a set of independent vertices of $C(2l, S)$ and it has $(l - 1)$ vertices. So $\beta_0(C(2l, S)) = l - 1$ is odd.

It follows from above that if $C(2l, S)$ is isomorphic to $GP(d, k)$, then $\beta_0(GP(d, k))$ is also odd.

On the other hand, any set $R$ of independent vertices of $GP(d, k)$ with $k \equiv \pm 1(\text{mod} \ d)$ is partitioned into two subsets $R_U$ and $R_V$. The first subset $R_U$ consists of all vertices $u_i \in R$ and the second subset $R_V$ consists of all vertices $v_i \in R$. 


Then $R_U$ is a set of independent vertices of the cycle $U = u_0u_1 \ldots u_{d-1}$ and $R_V$ is a set of independent vertices of the cycle $V = v_0v_1 \ldots v_{d-1}$. Hence, $|R| = |R_U| + |R_V| \leq \beta_0(U) + \beta_0(V) = 2\lfloor d/2 \rfloor$. But the set $\{u_0, u_2, \ldots, u_{2\lfloor d/2 \rfloor - 1}, v_0, v_2, \ldots, v_{2\lfloor d/2 \rfloor - 1}\}$ is a set of independent vertices of $GP(d, k)$. Thus, $eta_0(GP(d, k)) = 2\lfloor d/2 \rfloor$ is even.

This contradiction shows that our suggestion about the isomorphism between $G$ and $G'$ is false. Thus, $G$ and $G'$ are not isomorphic and Proposition 4 is proved.

**Proposition 5.** Let $G$ be the union of $t$ disjoint copies of $GP(d, k)$ with $d > 2$, $k \in \mathbb{Z}_+^*$, $k^2 \equiv \pm 1 \pmod{d}$ and $G'$ be the union of $t'$ disjoint copies of $GP(d', k')$ with $d' > 2$, $k' \in \mathbb{Z}_+^*$, $k'^2 \equiv \pm 1 \pmod{d'}$. Then $G'$ and $G$ are isomorphic if and only if $t' = t$, $d' = d$ and $k' \equiv \pm k \pmod{d}$.

This proposition follows from the following Lemmas 1–4.

**Lemma 1.** Let $GP(d, k)$ be one of the graphs $GP(4, 1)$, $GP(4, 3)$, $GP(5, 2)$, $GP(5, 3)$, $GP(8, 3)$, $GP(8, 5)$, $GP(10, 3)$, $GP(10, 7)$, $GP(12, 5)$, $GP(12, 7)$, $GP(24, 5)$ and $GP(24, 19)$. Then a graph $GP(d', k')$, with $d' > 2$, $k' \in \mathbb{Z}_+^*$, and $k'^2 \equiv \pm 1 \pmod{d'}$, is isomorphic to $GP(d, k)$ if and only if $d' = d$ and $k' \equiv \pm k \pmod{d}$.

**Proof.** Suppose that $GP(d', k')$, with $k' > 2$, $k' \in \mathbb{Z}_+^*$, and $k'^2 \equiv \pm 1 \pmod{d'}$, is isomorphic to $GP(d, k)$, where $(d, k)$ is one of the pairs $(4, 1), (4, 3), (5, 2), (5, 3), (8, 3), (8, 5), (10, 3), (10, 7), (12, 5), (12, 7), (24, 5)$ and $(24, 19)$. Then, since the order of $GP(d', k')$ is $2d'$ and the order of $GP(d, k)$ is $2d$, we must have $d' = d$. If $k' \not\equiv \pm k \pmod{d}$, then by the results in [5] the automorphism group $\text{Aut}(GP(d', k'))$ has order $4d'$. But as shown in [5], $|\text{Aut}(GP(d, k))| > 4d$. This means that $GP(d', k')$ and $GP(d, k)$ cannot be isomorphic. Thus, $k' \equiv \pm k \pmod{d}$.

Conversely, if $d' = d$ and $k' \equiv \pm k \pmod{d}$, then it is easy to see that $GP(d, k)$ and $GP(d', k')$ have the same vertex sets and the same edge sets. So they are trivially isomorphic, hence Lemma 1 is proved.

**Lemma 2.** Let $d$ and $k$ be integers, $d > 2$, $1 \leq k \leq d - 1$, $k \in \mathbb{Z}_+^*$, $k^2 \equiv 1 \pmod{d}$ and let $\Gamma$ be the following group:

$$\Gamma = \langle \rho, \delta, \alpha : \rho^d = \delta^2 = \alpha^2 = 1, \delta \alpha = \alpha \delta, \delta \rho \delta = \rho^{-1}, \alpha \rho \alpha = \rho k \rangle.$$  

Then for any normal cyclic subgroup $\langle \sigma \rangle$ of order $d$ of $\Gamma$ and for any element $\gamma \in \Gamma$, one of the following equalities holds:

$$\gamma \sigma^{-1} = \sigma, \quad \gamma \sigma^{-1} = \sigma^{-1}, \quad \gamma \sigma^{-1} = \sigma^k, \quad \gamma \sigma^{-1} = \sigma^{-k}.$$

**Proof.** It follows from the definition of $\Gamma$ that every element of $\Gamma$ can be represented in one of the forms: $\rho^l, \delta \rho^l, \alpha \rho^l, \alpha \delta \rho^l$, where $l = 0, 1, \ldots, d - 1$.  

Let $d = m2^n$ with $m$ odd. There are several cases to consider.

Case 1: $n = 0$. Then $d$ is odd. Since $\Gamma$ is a group of order $4d$, the subgroup $\langle \rho \rangle$ is the unique normal cyclic subgroup of order $d$ of $\Gamma$. Therefore, the assertion of Lemma 2 is trivially true.

Case 2: $n > 2$. Then, since $k \in \mathbb{Z}_4^*$ and $d$ is even, $k$ has to be odd. Therefore, $k = 4t - 1$ or $k = t - 3$, where $t \geq 1$ is a positive integer. For all $l = 0, 1, \ldots, d - 1$, we have $(\delta^l)^{2^m} = (\delta^l)^{2^m} = (\delta^l)^{2^m} = 1$. Therefore, any element of order $d$ of $\Gamma$ not contained in $\langle \rho \rangle$ has to have the form $\alpha \rho^l$ or $\alpha \delta \rho^l$.

(2.1) Suppose that $k = 4t - 3$ with $t \geq 1$. In this subcase, for any $l = 0, 1, \ldots, d - 1$, we have

$$\alpha \rho^l \cdot m2^n - 1 = (\alpha \delta^l)^{m2^n} = (\alpha \delta^l)^{m2^n} = 1.$$

So any element of order $d$ of $\Gamma$ not contained in $\langle \rho \rangle$ has to have the form $\alpha \rho^l$ if $k = 4t - 3$ with $t \geq 1$.

If $\gcd(m, 2t - 1) = m' > 1$, then for any $l = 0, 1, \ldots, d - 1$,

$$\alpha \rho^l \cdot m2^n = ((\alpha \rho^l \cdot m2^n)^{m/m'}) = (\rho^{i(k + l)})^{m2^n - 1} = (\rho^{i(k + l)})^{m2^n - 1} = 1.$$

Hence, $\langle \rho \rangle$ is the unique normal cyclic subgroup of order $d$ of $\Gamma$ and Lemma 2 is again true in this situation.

Suppose now that $\gcd(m, 2t - 1) = 1$. If $l$ is even, then

$$\alpha \rho^l \cdot m2^n = ((\alpha \rho^l \cdot m2^n)^{m2^n - 1} = (\rho^{i(k + l)})^{m2^n - 1} = 1.$$

If $\gcd(m, l) = l' > 1$, then again

$$\alpha \rho^l \cdot m2^n = ((\alpha \rho^l \cdot m2^n)^{m2^n - 1} = (\rho^{i(k + l)})^{m2^n - 1} = 1.$$

Hence, if $\alpha \rho^l$ has order $d$, then $l$ has to be odd and $\gcd(m, l) = 1$.

Conversely, if $n \geq 2$, $k = 4t - 3$, $\gcd(m, 2t - 1) = 1$, $\gcd(m, l) = 1$ and $l$ is odd, then it is not difficult to verify that $\alpha \rho^l$ has order $d$ and $\langle \alpha \rho^l \rangle = \{\alpha \rho^l, \alpha \rho^3, \ldots, \alpha \rho^{2t-1}, \ldots, 1, \rho^2, \rho^4, \ldots, \rho^{2t-2}, \ldots : i \geq 1\}$. This means that in this situation $\Gamma$ has just two distinct normal cyclic subgroups of order $d$: the groups $\langle \rho \rangle$ and $\langle \alpha \rho \rangle$. Moreover, we have $k^2 = (4t - 3)(4t - 3) = 16t^2 - 24t + 9 = 1 + m2^n$ for some integer $s$. This is equivalent to $8(t - 1)(2t - 1) = m2^n$. Therefore, $m2^n - 3$ divides $(t - 1)(2t - 1)$ if $n \geq 3$ and $m$ divides $(t - 1)(2t - 1)$ if $n = 2$. But $\gcd(m, 2t - 1) = 1$ and $\gcd(2^n - 3, 2t - 1) = 1$. Hence, $m2^n - 3$ divides $(t - 1)$ if $n \geq 3$ and $m$ divides $(t - 1)$ if $n = 2$. Therefore, $t = m2^n - 3 + 1$, $k = 4t - 3 = 2n - 1 + 1$ if $n \geq 3$ and $t = mr + 1$,
If \( k = 4t - 3 = 4mr + 1 \) if \( n = 2 \). But since \( 1 \leq k \leq m2^n - 1 \), we have \( k = 1 \) or \( k = m2^n - 1 + 1 \) if \( n \geq 3 \) and \( k = 1 \) if \( n = 2 \).

If \( k = 1 \), then \( \rho(\alpha \rho)^{-1} = \alpha \rho \), \( \delta(\alpha \rho) \delta = \alpha \rho^{-1} = (\alpha \rho)^{-1} \) and \( \alpha(\alpha \rho) \alpha = \alpha \rho \). If \( k = m2^n - 1 + 1 \), then

\[
\rho(\alpha \rho)^{-1} = \rho \alpha = \alpha \rho^{m2^n - 1 + 1} = (\alpha \rho)^{m2^n - 1 + 1} = (\alpha \rho)^k,
\]
\[
\delta(\alpha \rho) \delta = \alpha \rho^{-1} = (\alpha \rho)^{-m2^n - 1 + 1} = (\alpha \rho)^{-k},
\]
\[
\alpha(\alpha \rho) \alpha = \rho \alpha = \alpha \rho^{m2^n - 1 + 1} = (\alpha \rho)^{m2^n - 1 + 1} = (\alpha \rho)^k.
\]

Therefore, for any element \( \gamma \in \Gamma \), one of the following equalities holds: either \( \psi \sigma^{-1} = \sigma, \psi \sigma^{-1} = \sigma^{-1}, \psi \sigma^{-1} = \sigma^k \), or \( \psi \sigma^{-1} = \sigma^{-k} \), where \( \sigma \) is any generator of \( \langle \rho \rangle \) or \( \langle \alpha \rho \rangle \), \( k = 1 \) or \( m2^n - 1 + 1 \). Thus, Lemma 2 is proved in the case \( k = 4t - 3 \).

(2.2) Suppose that \( k = 4t - 1 \) with \( t \geq 1 \). In this subcase, for any \( l = 0, \ldots, d - 1 \), we have

\[
(\alpha \rho^l)^{m2^n - 1} = ((\alpha \rho^l)(\alpha \rho^l))^{m2^n - 1} = (\rho^{l(k + 1)})^{m2^n - 2} = (\rho^{4lt})^{m2^n - 2} = (\rho^{m2^n})^{lt} = 1.
\]

Therefore, if \( k = 4t - 1 \), then any element of order \( d \) of \( \Gamma \) not contained in \( \langle \rho \rangle \) has to have the form \( \alpha \delta \rho^l \).

If \( \gcd(m, 2t - 1) > 1 \) or \( l \) is even or \( \gcd(m, l) > 1 \), then \( \alpha \delta \rho^l \) has order less than \( d \). This can be proved as in the case \( k = 4t - 3 \). So if \( n \geq 2 \) and \( k = 4t - 1 \), then an element \( \alpha \delta \rho^l \) has order \( d \) only if \( \gcd(m, 2t - 1) = 1 \), \( \gcd(m, l) = 1 \) and \( l \) is odd. Conversely, if \( n \geq 2, k = 4t - 1, \gcd(m, 2t - 1) = 1, \gcd(m, l) = 1 \) and \( l \) is odd, then it is not difficult to verify that \( \alpha \delta \rho^l \) has order \( d \) and \( \langle \alpha \delta \rho^l \rangle = \{ \alpha \delta \rho, \alpha \delta \rho^3, \ldots, \alpha \delta \rho^{2i-1}, \ldots, 1, \rho^2, \rho^4, \ldots, \rho^{2i-2}, \ldots : i \geq 1 \} \). This means that in this situation, \( \Gamma \) has just two distinct normal cyclic subgroups of order \( d \): the groups \( \langle \rho \rangle \) and \( \langle \alpha \delta \rho \rangle \). Moreover, we have \( k^2 = (4t - 1)(4t - 1) = 16t^2 - 8t + 1 = 1 + m2^n \) for some integer \( s \). Hence, \( m2^n \) divides \( 8t(2t - 1) \). Therefore, \( m2^{n-3} \) divides \( t(2t - 1) \) if \( n \geq 3 \) and \( m \) divides \( t(2t - 1) \) if \( n = 2 \). Since \( \gcd(m, 2t - 1) = 1 \) and \( \gcd(2^n - 3, 2t - 1) = 1 \), this implies that \( m2^{n-3} \) divides \( t \) if \( n \geq 3 \) and \( m \) divides \( t \) if \( n = 2 \). Hence, \( t = m2^{n-3}r \) if \( n \geq 3 \) and \( t = mr \) if \( n = 2 \) for some integer \( r \). Therefore, \( k = m2^{n-1}r - 1 \) if \( n \geq 3 \) and \( k = 4mr - 1 \) if \( n = 2 \). But \( 1 \leq k \leq m2^n - 1 \). So \( k = m2^{n-1} - 1 \) or \( k = m2^n - 1 \) if \( n \geq 3 \) and \( k = 4m - 1 \) if \( n = 2 \).

If \( k = m2^n - 1 \), then

\[
\rho(\alpha \delta \rho)^{-1} = \alpha \delta \rho, \quad \delta(\alpha \delta \rho) \delta = \alpha \delta \rho^{-1} = (\alpha \delta \rho)^{-1}
\]
and

\[
\alpha(\alpha \delta \rho) \alpha = \alpha \delta \rho^{-1} = (\alpha \delta \rho)^{-1}.
\]

If \( k = m2^n - 1 - 1 \), then

\[
\rho(\alpha \delta \rho)^{-1} = \alpha \delta \rho^{m2^n - 1} = (\alpha \delta \rho)^{m2^n - 1 - 1} = (\alpha \delta \rho)^k,
\]
\[
\delta(\alpha \delta \rho) \delta = \alpha \delta \rho^{-1} = (\alpha \delta \rho)^{1 - m2^n - 1} = (\alpha \delta \rho)^{-k},
\]
\[
\alpha(\alpha \delta \rho) \alpha = \alpha \delta \rho^{m2^n - 1 - 1} = (\alpha \delta \rho)^{-1}.
\]
Therefore, for any element \( \gamma \in \Gamma \) one of the following equalities holds: either 
\[ \gamma \sigma^{-1} = \sigma, \quad \gamma \sigma^{-1} = \sigma^{-1}, \quad \gamma \sigma^{-1} = \sigma^k, \] or 
\[ \gamma \sigma^{-1} = \sigma^{-k}, \] where \( \sigma \) is any generator of \( \langle \rho \rangle \) or \( \langle \alpha \rho \rangle \) and \( k = m2^n - 1 \) or \( k = m2^n - 1 - 1 \). Thus, Lemma 2 is also proved in the case \( k = 4t - 1 \).

Case 3: \( n = 1 \). If \( m = 1 \), then \( d = 2 \) and the lemma is trivially true. So we can suppose from now on that \( m > 1 \), i.e., \( d = 2m > 2 \). For any \( l = 0, 1, \ldots, d - 1 \), we have 
\[ (\delta \rho)^2 = (\delta \rho \delta) \rho^l = \rho^{-1} \rho^l = 1. \] Therefore, any element of order \( d \) of \( \Gamma \) not contained in \( \langle \rho \rangle \) has to have either the form \( \alpha \rho^l \) or the form \( \alpha \delta \rho^l \).

Let \( \sigma \) be either \( \alpha \rho^l \) or \( \alpha \delta \rho^l \) such that \( \langle \sigma \rangle \) is a normal cyclic subgroup of order \( 2m \) of \( \Gamma \). If \( \langle \sigma \rangle \langle \rho \rangle = \Gamma \), then \( \langle \sigma \rangle \langle \rho \rangle / \langle \rho \rangle = \Gamma / \langle \rho \rangle \cong \langle \alpha, \delta \rangle \), which is an elementary abelian group of order 4. On the other hand, \( \langle \sigma \rangle \langle \rho \rangle / \langle \rho \rangle \cong \langle \sigma \rangle / (\langle \sigma \rangle \cap \langle \rho \rangle) \), which is a cyclic subgroup. This contradiction shows that \( \langle \sigma \rangle \langle \rho \rangle \) is a proper subgroup of \( \Gamma \). The group \( \langle \sigma \rangle \langle \rho \rangle \neq \langle \rho \rangle \), since otherwise, \( \alpha \) or \( \alpha \delta \) are in \( \langle \rho \rangle \), which is impossible. Therefore, \( |\langle \sigma \rangle \langle \rho \rangle / \langle \rho \rangle| = |\langle \sigma \rangle / (\langle \sigma \rangle \cap \langle \rho \rangle)| = 2 \). Thus, \( \langle \sigma \rangle \) contains \( \langle \rho^2 \rangle \). But \( \langle \sigma \rangle \) is cyclic. So, if \( \sigma = \alpha \rho^l \), then \( \alpha \delta \rho^{l+2} = (\alpha \rho^l) \rho^2 = \rho^2 (\alpha \rho^l) = (\rho^2 \alpha) \rho = \alpha \rho^{2k+1} \). This implies \( \rho^{2k+1} = \rho^{l+2} \). Since \( \rho \) is of order \( 2m \), \( 2(k - 1) = 2m \) for some integer \( s \). Hence, either \( k - 1 = 0 \) or \( k - 1 = m \). On the other hand, \( \alpha \delta \rho \) has to be odd (because \( k \in \mathbb{Z}_2^+ \)). Therefore, \( k - 1 = m \) does not hold and so \( k = 1 \). Hence, our lemma is trivially true in this case.

If \( \sigma = \alpha \delta \rho^l \), then \( \alpha \delta \rho^{l+2} = (\alpha \delta \rho^l) \rho^2 = \rho^2 (\alpha \delta \rho^l) = \alpha \delta \rho^{2k-1} \). This implies \( \rho^{2k-1} = \rho^{l+2} \). It follows that \( k + 1 = m \) or \( k + 1 = 2m \) (because \( 1 \leq k \leq d - 1 \)). But \( k \) is odd. So \( k + 1 = m \) is impossible. Thus, \( k = 2m - 1 \). It is easy to verify that the lemma is again true in this case. Lemma 2 is proved. \( \square \)

Lemma 3. Let \( d \) and \( k \) be integers, \( d > 2 \), \( 1 \leq k \leq d - 1 \), \( k \in \mathbb{Z}_2^* \), \( k^2 \equiv -1 \mod d \) and let \( \Gamma \) be the group \( \langle \rho, \alpha : \rho^d = 1, \alpha^4 = 1, \alpha \rho^{-1} = \rho^k \rangle \). Then for any normal cyclic subgroup \( \langle \sigma \rangle \) of order \( d \) of \( \Gamma \) and for any element \( \gamma \in \Gamma \), one of the following equalities holds:

\[ \gamma \sigma^{-1} = \sigma, \quad \gamma \sigma^{-1} = \sigma^{-1}, \quad \gamma \sigma^{-1} = \sigma^k, \quad \gamma \sigma^{-1} = \sigma^{-k}. \]

Proof. It follows from the definition of \( \Gamma \) that every element of \( \Gamma \) can be represented in one of the forms: \( \rho^l, \alpha \rho^l, \alpha^2 \rho^l, \alpha^3 \rho^l \), where \( l = 0, 1, \ldots, d - 1 \).

Let \( d = m2^n \) with an odd integer \( m \). There are several cases to consider.

Case 1: \( n = 0 \). Then \( d \) is odd and since \( |\Gamma| = 4d \), the group \( \langle \rho \rangle \) is the unique normal cyclic subgroup of order \( d \) of \( \Gamma \). Therefore, the lemma is trivially true.

Case 2: \( n \neq 0 \). Then \( d \) is even and since \( k \in \mathbb{Z}_2^* \), \( k \) is odd. Let \( k = 2t - 1 \) with \( t \geq 1 \). We have \( k^2 = (2t - 1)(2t - 1) = 4t^2 - 4t + 1 = -1 + sm2^n \) for some integer \( s \). Therefore, \( 4t^2 - 4t + 2 = sm2^n \iff 2(2t^2 - 2t + 1) = sm2^n \iff 2t^2 - 2t + 1 = sm2^n - 1 \). Since \( 2t^2 - 2t + 1 \) is odd, \( n - 1 \) has to be 0 and so \( n = 1 \).
If \( m = 1 \), then \( d = 2 \). Thus, \( \Gamma \) is abelian and the lemma is true. If \( m \neq 1 \), then \( d \geq 6 \).
In this subcase, for any \( l = 0, 1, \ldots, d - 1 \), we have
\[
(ax^l)^2 = x^2(x^{l+2})x^l = x^{l+2} = 1,
\]
\[
(ax^l)^4 = ((ax^l)(ax^l))^2 = (x^2x^{-l})^2 = 1,
\]
\[
(ax^l)^8 = (ax^l)(ax^l)^2 = (x^2x^{2l})x^l = x^{2l+1} = 1.
\]
Therefore, \( \langle \rho \rangle \) is the unique normal cyclic subgroup of order \( d \) of \( \Gamma \). Hence, the lemma is also true. Lemma 3 is completely proved. 

**Lemma 4.** Let \((d, k)\), with \( d > 2 \), \( k \in \mathbb{Z}_d^* \), \( k^2 \equiv \pm 1 \pmod{d} \), and \((d', k')\), with \( d' > 2 \), \( k' \in \mathbb{Z}_{d'}^* \), \( k'^2 \equiv \pm 1 \pmod{d'} \), be ordered pairs not belonging to \((4, 1), (4, 3), (5, 2), (5, 3), (8, 3), (8, 5), (10, 10), (12, 5), (12, 7), (24, 5)\) and \((24, 19)\). Then \( \text{GP}(d, k) \) and \( \text{GP}(d', k') \) are isomorphic if and only if \( d' = d \) and \( k' \equiv \pm k \pmod{d} \).

**Proof.** If \( d' = d \) and \( k' \equiv \pm k \pmod{d} \), then it is easy to see that \( \text{GP}(d, k) \) and \( \text{GP}(d', k') \) are isomorphic.

Conversely, suppose that \( \text{GP}(d, k) \) and \( \text{GP}(d', k') \) are isomorphic. It is trivial that the equality \( d' = d \) holds because the order of \( \text{GP}(d, k) \) is \( 2d \) and the order of \( \text{GP}(d', k') \) is \( 2d' \). Let \( \tau \) be an isomorphism from \( \text{GP}(d, k) \) to \( \text{GP}(d', k') \). Then \( \tau \) is a permutation on \( V(\text{GP}(d, k)) = V(\text{GP}(d', k')) \) such that
\[
\tau \text{Aut}(\text{GP}(d, k))\tau^{-1} = \text{Aut}(\text{GP}(d', k')).
\]

**Case 1:** \( k^2 \equiv k'^2 \pmod{d} \). In this case, either \( k^2 \equiv 1 \pmod{d} \) and \( k'^2 \equiv -1 \pmod{d} \), or \( k^2 \equiv -1 \pmod{d} \) and \( k'^2 \equiv 1 \pmod{d} \). Without loss of generality, we may assume that \( k^2 \equiv 1 \pmod{d} \) and \( k'^2 \equiv -1 \pmod{d} \). Since \((d, k)\) and \((d', k')\) are different from \((4, 1), (4, 3), (5, 2), (5, 3), (8, 3), (8, 5), (10, 3), (10, 7), (12, 5), (12, 7), (24, 5)\) and \((24, 19)\), we have \( \text{Aut}(\text{GP}(d, k)) = \langle \rho, \delta, x : \rho^4 = \delta^2 = x^2 = 1, \delta \rho = \delta x, \delta \rho \delta = \rho^{-1}, \delta x \rho = \rho^k \rangle \) and \( \text{Aut}(\text{GP}(d', k')) = \langle \rho, \delta, x' : \rho^4 = x'^4 = 1, \delta x' \rho x'^{-1} = \rho^k \rangle \) (see [5]). It is not difficult to show that \( \text{Aut}(\text{GP}(d, k)) \) is not isomorphic to \( \text{Aut}(\text{GP}(d', k')) \). This contradicts (2). Thus, case 1 cannot happen.

**Case 2:** \( k^2 \equiv k'^2 \equiv 1 \pmod{d} \). Then \( \text{Aut}(\text{GP}(d, k)) = \langle \rho, \delta, x : \rho^4 = \delta^2 = x^2 = 1, \delta \rho = \delta x, \delta \rho \delta = \rho^{-1}, \delta x \rho = \rho^k \rangle \) and \( \text{Aut}(\text{GP}(d', k')) = \langle \rho, \delta, x' : \rho^4 = x'^4 = 1, \delta x' \rho x'^{-1} = \rho^k \rangle \) (see [5]). Hence, for any \( \gamma \in \text{Aut}(\text{GP}(d, k)) \), by Lemma 2 one of the following equalities holds: either \( \gamma \rho \gamma^{-1} = \rho \), \( \gamma \rho \gamma^{-1} = \rho^{-1} \), \( \gamma \rho \gamma^{-1} = \rho^k \), or \( \gamma \rho \gamma^{-1} = \rho^{-k} \). Therefore, for any \( \gamma \in \text{Aut}(\text{GP}(d, k)) \), one of the following equalities holds:
\[
(\tau \gamma \tau^{-1})(\tau \rho \tau^{-1})(\tau \gamma \tau^{-1})^{-1} = \tau \rho \tau^{-1},
\]
\[
(\tau \gamma \tau^{-1})(\tau \rho \tau^{-1})(\tau \gamma \tau^{-1})^{-1} = \tau \rho^{-1} \tau^{-1} = (\tau \rho \tau^{-1})^{-1},
\]
\[
(\tau \gamma \tau^{-1})(\tau \rho \tau^{-1})(\tau \gamma \tau^{-1})^{-1} = \tau \rho^k \tau^{-1} = (\tau \rho \tau^{-1})^k,
\]
\[
(\tau \gamma \tau^{-1})(\tau \rho \tau^{-1})(\tau \gamma \tau^{-1})^{-1} = \tau \rho^{-k} \tau^{-1} = (\tau \rho \tau^{-1})^{-k}.
\]
Because of equality (2), for any $\gamma' \in \text{Aut}(GP(d, k'))$, one of the following equalities holds:

$$
\gamma'(\tau \rho^{-1})\gamma'^{-1} = \tau \rho^{-1}, \quad \gamma'(\tau \rho^{-1})\gamma'^{-1} = (\tau \rho^{-1})^{-1}, \\
\gamma'(\tau \rho^{-1})\gamma'^{-1} = (\tau \rho^{-1})^k, \quad \gamma'(\tau \rho^{-1})\gamma'^{-1} = (\tau \rho^{-1})^{-k}.
$$

On the other hand, since $\langle \rho \rangle$ is a normal cyclic subgroup of order $d$ of $\text{Aut}(GP(d, k))$, $\tau \langle \rho \rangle \tau^{-1} = \langle \tau \rho^{-1} \rangle$ is a normal cyclic subgroup of order $d$ of $\text{Aut}(GP(d, k'))$. Therefore, again by Lemma 2, for any $\gamma' \in \text{Aut}(GP(d, k'))$, one of the following equalities holds:

$$
\gamma'(\tau \rho^{-1})\gamma'^{-1} = \tau \rho^{-1}, \quad \gamma'(\tau \rho^{-1})\gamma'^{-1} = (\tau \rho^{-1})^{-1}, \\
\gamma'(\tau \rho^{-1})\gamma'^{-1} = (\tau \rho^{-1})^k, \quad \gamma'(\tau \rho^{-1})\gamma'^{-1} = (\tau \rho^{-1})^{-k}.
$$

Hence, $k' \equiv k (\text{mod } d)$ or $k' \equiv -k (\text{mod } d)$.

**Case 3:** $k \equiv k' (\text{mod } d)$. Again by [5], we have

$$
\text{Aut}(GP(d, k)) = \langle \rho, \alpha: \rho^d = \alpha^4 = 1, \alpha \rho \alpha^{-1} = \rho^k \rangle, \\
\text{Aut}(GP(d, k')) = \langle \rho, \alpha': \rho^d = \alpha'^4 = 1, \alpha' \rho \alpha'^{-1} = \rho^{k'} \rangle.
$$

By using Lemma 3 instead of Lemma 2, the proof of case 3 is now exactly the same as the proof of case 2. Lemma 4 is proved. □

From Lemmas 1 and 4, Proposition 5 follows immediately, and from Propositions 2–5 above we have the following theorem.

**Theorem 1.** Let $G$ and $G'$ be graphs in the family $\mathcal{G}$. Then $G$ and $G'$ are isomorphic if and only if either they are unions of equal numbers of disjoint copies of the same $C(2l, S)$, where $l$ is an integer satisfying $l > 1$ and $S = \{1, -1, l\}$ or they are unions of equal numbers of disjoint copies of $GP(d, k)$ or $GP(d, d - k)$, where $d$ is an integer satisfying $d > 2$, $k \in \mathbb{Z}^*_d$ and $k^2 \equiv -1 (\text{mod } d)$.

Suppose now that graphs in $\Phi$ are given by their parameters. Then, for the given cubic metacirculant graphs $G = MC(m, n, \alpha, S_0, S_1, \ldots, S_\mu)$ with $S_0 \neq \emptyset$ and $G' = MC(m', n', \alpha', S'_0, S'_1, \ldots, S'_\mu')$ with $S'_0 \neq \emptyset$, we can test whether or not they are isomorphic by using the algorithm for determining components of cubic $(m, n)$-metacirculant graphs with the condition $S_0 \neq \emptyset$ (see Section 2) and Theorem 1 above.

**Isomorphism testing algorithm for graphs in $\Phi$**

1. Check $mn = m'n'$. If the equality does not hold, then these graphs are not isomorphic. Otherwise, continue.

2. Find components of $MC(m, n, \alpha, S_0, S_1, \ldots, S_\mu)$ by the algorithm for determining components of a cubic $(m, n)$-metacirculant graph with the condition $S_0 \neq \emptyset$ (see Section 2).
3. Find components of $MC(m', n', \alpha', S_0', S_1', \ldots, S_{\alpha'}')$ by the same algorithm.

4. Use Theorem 1 above and the results of Steps 2 and 3 to conclude whether or not the given graphs are isomorphic.

We already know (Section 2) that the time complexity of the algorithm for determining components of a cubic $(m, n)$-metacirculant graphs with the condition $S_0 \neq \emptyset$ is $O(\log^3(mn))$. It is easy to see that the time complexity of our isomorphism testing algorithm for graphs in $\Phi$ is the same, i.e., is $O(\log^3(mn))$.

Acknowledgements

I would like to thank Drs. Mari-Jo P. Ruiz, Evelyn L. Tan and other organizers of 'Graph Theory and Combinatorics, Manila 1991' for their hospitality and financial support, which enabled me to present this work at the Conference.

I would also like to thank the referee for many improvements in the text.

References