Synchronization of A Class of Second-Order Nonlinear Systems

A.P. Mijolaro, L.F.C. Alberto, N.G. Bretas

Abstract — In this paper, the asymptotic behavior of coupled second-order nonlinear dynamical systems is studied. Conditions on the coupling parameters that guarantee synchronization between these systems are provided for every solution starting in a positively invariant region of initial conditions.

1. Introduction

Synchronization is an important concept that encounters applications in several fields of applied sciences such as Electrical and Mechanical Engineering, Biology and Physics. Synchronization ideas have been successfully applied to codification of information on communication systems, see [8]. Experimentally, synchronization has been observed in many different systems, see, for example, [12], for excited coupled chaotic pendulums, [13] for clustering identification applied to scene segmentation and [9] for synchronization between main rhythmic processes in the human cardiovascular system.

There is a great variety of approaches to prove synchronization of nonlinear coupled systems. The authors of [1], for example, prove synchronization for a class of coupled systems, including chaotic systems, with linear coupling. Abstract results, robustness with respect to parameter variation and uniform dissipativeness were obtained in [10] and [1]. For infinite dimensional systems, some results were presented in [10] and [7].

One approach employed to prove synchronization, first proposed in [10], consists of two parts. First, an estimate of the attractor, uniform with respect to the coupling parameter, is obtained. Then, in a second stage, synchronization is studied into this attractor estimate and an estimate of the coupling parameter, which is sufficient to guarantee synchronization, is obtained.

The success of this approach is due to the division of the problem into two stages. In the first stage, uniform dissipativeness is studied while in the second stage the information obtained in the first stage is used to prove synchronization.

In some cases, where uniform dissipativeness does not exist (or cannot be easily proved), this approach cannot be applied. This is especially true when the system is unstable. It will be shown for some class of coupled unstable systems that it is still possible to prove synchronization and even finding estimates on the coupling parameters that guarantee synchronization.

The main concern of this paper is the study of synchronization of a class of nonlinear systems where uniform dissipativeness may not exist. In particular, synchronization between two nonlinear coupled pendulums is studied. These studies were motivated by the analysis of coherency on electrical power systems, see [11], [2] and [5]. The knowledge of coherency or synchronization provides insight into the dynamical behavior of power systems; in particular, it provides important information about instability modes.

Under very mild conditions on the vector field, synchronization is mathematically proven and conditions on the coupling parameters that guarantee synchronization are given. Since synchronization may not be global, the approach used in this paper does not look for ultimate bounds but proves synchronization for a set of initial conditions contained in the synchronization region.

This paper is organized in the following way: Section II presents the class of coupled systems that is studied. Definitions of synchronization are given and the main result of synchronization is presented. An example is presented in Section III. Finally, some conclusions are given in Section IV.

2. Synchronization

This paper is concerned with synchronization of two coupled systems with the following general form:

\[
\begin{align*}
\dot{x}_1 &= f(t, x_1, x_2, \chi_1, \xi_c) \\
\dot{x}_2 &= f(t, x_1, x_2, \chi_2, \xi_c)
\end{align*}
\]

where \(\chi_i\) is a parameter vector of the \(i\)-th system \((i=1,2)\), \(\chi_c\) is a coupling parameter and \((\xi_1, \xi_2) \in \mathbb{R}^2 \times \mathbb{R}^2\).

The solution of this system starting in \((\xi_{10}, \xi_{20})\) at \(t_0\) will be denoted by \((\xi_1(t, \xi_{10}, \xi_{20}, \chi_1, \chi_c), \xi_2(t, \xi_{10}, \xi_{20}, \chi_2, \chi_c))\).

Next definitions state the meaning of synchronization that will be used in this paper. Definition 1 is concerned with synchronization of solutions starting at a particular initial condition while Definition 2 concerns synchronization of solutions for a set of initial conditions.

Definition 1: The solution of (1) given by \((\xi_1(t, t_0, \xi_{10}, \xi_{20}, \chi_1, \chi_c), \xi_2(t, t_0, \xi_{10}, \xi_{20}, \chi_2, \chi_c))\) synchronizes for a certain coupling parameter \(\chi_c\) if

[Further content continues with mathematical details and additional proofs and examples related to synchronization theory.]
lim sup \( t \to \infty \) \( \|x(t, t_0, \xi_0, \eta_0, x_1, \eta_1, x_2, \eta_2) - x(t, t_0, \xi_0, \eta_0, x_1, \eta_1, x_2, \eta_2) \| \leq O(\|x_1 - x_2\|) \)

**Definition 2:** System (1) synchronizes with respect to a nonempty subset \( \Lambda \) of \( \mathbb{R}^2 \times \mathbb{R}^2 \), for a certain parameter \( \Lambda \) if

\[
lim sup \|x(t, t_0, \xi_0, \eta_0, x_1, \eta_1, x_2, \eta_2) - x(t, t_0, \xi_0, \eta_0, x_1, \eta_1, x_2, \eta_2) \| \leq O(\|x_1 - x_2\|) \]

for every initial condition \( (\xi_0, \eta_0) \in \Lambda \), for any \( t_0 \in \mathbb{R} \).

In other words, system (1) synchronizes if the difference between subsystems 1 and 2 tends to become small as time tends to infinite. Moreover, the error has the order of the difference of the parameters, that is, the difference is as small as the difference between subsystems 1 and 2. If subsystems 1 and 2 are equal (have the same parameters), then the difference tends to zero as time goes to infinite.

If \( \Lambda = \mathbb{R}^2 \times \mathbb{R}^2 \), we say that system (1) globally synchronizes. The largest set \( \Lambda \) that satisfies Definition 2 is called the synchronization region of system (1).

In this paper, we will study the following class of second-order nonlinear coupled systems:

\[
\begin{align*}
\dot{x}_i &= y_i, \\
\dot{y}_i &= -h(t, x_i, \eta_i) - D_i y_i - K_1 g(x_i, x_{i+1}) - K_2 (y_i - y_{i+1}) \\
\end{align*}
\]

(2)

where \( \eta_i \) and \( D_i > 0 \) are parameters of system \( i = 1, 2 \) and \( K_1 \) and \( K_2 \) are coupling parameters and \( g \) is a nonlinear coupling function that depends on the positions \( x_i \) and \( x_{i+1} \).

Next theorem proves, under mild conditions on the functions \( h \) and \( g \), that subsystems 1 and 2 of (2) synchronize, if their parameters are sufficiently close and the coupling parameters \( K_1 \) and \( K_2 \) are sufficiently large.

**Theorem 1:** Consider system (2), and suppose the following assumptions are satisfied:

(i) There exists a continuous function \( H \) such that

\[
h(t, x, \eta) - h(t, x, \eta) = -H(t, x_1, x_2, \eta) \cdot (x_1 - x_2),
\]

(ii) There exists a continuous function \( G : \mathbb{R}^2 \to \mathbb{R}^2 \) such that

\[
g(x_1, x_2) = G(x_1, x_2) (x_1 - x_2),
\]

(iii) \( h(t, x_1, \eta_1) - h(t, x_2, \eta_2) \leq O(\|\eta_1 - \eta_2\|) \) \( \forall x \in \mathbb{R}^2 \);

(iv) \( |y_i(t)| < \infty, \forall t \geq 0 \);

(v) There exist an open subset \( \Lambda \subset \mathbb{R}^2 \times \mathbb{R}^2 \) with nonempty intersection with the diagonal set \( D := \{(x_1, y_1, x_2, y_2) \in \mathbb{R}^2 \times \mathbb{R}^2 : x_1 = x_2\} \).

(vi) There exist positive numbers: \( \mu, \mu, M \) and \( s \) such that

\[
0 < \mu < m < -\alpha(t, x_1, x_2, \eta) < M \quad \text{for any} \quad K_1 > s \quad \text{and any} \quad (x_1, y_1, x_2, y_2) \in \Lambda, \quad \text{where} \quad -\alpha(t, x_1, x_2, \eta) := H(t, x_1, x_2, \eta) / |K_1| + 2G(x_1, x_2);
\]

Then, every solution lying on \( \Lambda \) for \( t \geq 0 \) synchronizes, in the sense of Definition 1, if \( K_1 > s \) and \( K_2 > -D_i / 2 + 1 / 2 \left[ \max(4K_1, 2K_2(M + s)) \right] \).

The prove and estimates are based on well-known results in nonlinear literature as the Variation of Constants Formula and the Gronwall Inequality [4].

In spite of the apparent complexity of Theorem 1, synchronization is proved using very mild conditions that can be easily checked for many physical systems. If the coupling function \( G(x, x) > 0 \) for any \( x \), then the coupling function \( g \) acts as a synchronizing force in the neighborhood of the diagonal set \( D \cap \Lambda \). Thus, for sufficiently large \( K_1 > 0 \), the synchronizing force of coupling function \( g \) wins the possible asynchronous force due to function \( h \). This observation shows that assumptions (v) and (vi) of Theorem 1 will be satisfied for a large number of practical systems. The third condition requires similarity between subsystems 1 and 2 and the fourth assumes the velocities are bounded. Conditions (i) and (ii) are also satisfied for a large number of systems.

**Remark:** There is a tradeoff between the choice of \( \Lambda \) and the required coupling \( K_1 \) and \( K_2 \) that guarantee synchronization. Larger sets \( \Lambda \) will require larger \( K_1 \) and \( K_2 \).

**Numerical estimates of these parameters will be given in the example.**

**3. Example**

Consider the following coupled nonlinear pendulums which arise from dynamical studies of electrical power systems [5]

\[
\begin{align*}
\dot{x}_1 &= y_1 \\
\dot{y}_1 &= P_1 - C_1 \sin x_1 - K_1 \sin(x_1 - x_2) - D_1 y_1 - K_2 (y_1 - y_2) \\
\dot{x}_2 &= y_2 \\
\dot{y}_2 &= P_2 - C_2 \sin x_2 - K_1 \sin(x_2 - x_1) - D_2 y_2 - K_2 (y_2 - y_1)
\end{align*}
\]

(3)

(4)

Physically, \( x_i(t) \) and \( y_i(t) \) represent angles of pendulums 1 and 2, respectively, and \( y_i(t) \) and \( y_2(t) \) are their angular speeds. Now, we will show that the coupled systems (3) and (4) satisfy the assumptions of Theorem 1.

First of all, let us prove that assumption (vi) of Theorem 1 is satisfied, that is \( |y_i(t)| < \infty, \forall t \geq 0 \). For this purpose, \( \dot{y}_1 \) and \( \dot{y}_2 \) are rewritten as

\[
\begin{pmatrix}
\dot{y}_1 \\
\dot{y}_2
\end{pmatrix} = \begin{pmatrix}
-D_1 y_1 - K_2 (y_1 - y_2) \\
-D_2 y_2 - K_2 (y_2 - y_1)
\end{pmatrix} + \begin{pmatrix}
u_1(x_1, x_2) \\

\v_2(x_1, x_2)
\end{pmatrix}
\]

(5)

where \( \nu_1, \nu_2 \) are clearly bounded functions given by

\[
\begin{align*}
\nu_1(x_1, x_2) &= P_1 - C_1 \sin x_1 - K_1 \sin(x_1 - x_2) \\
\nu_2(x_1, x_2) &= P_2 - C_2 \sin x_2 - K_1 \sin(x_2 - x_1)
\end{align*}
\]

The eigenvalues of the matrix of the linear part of equation (5) have negative real part. Thus, applying the Variation of Constants Formula, one proves that \( y_1(t), y_2(t) \) are bounded for \( t > 0 \). Hence, assumption (vi) of Theorem 1 is satisfied.
Identifying \( h(t, x_1, \eta_1) := -P_1 - C_1 \sin x_1 \) and 
\( h(t, x_2, \eta_2) := -P_2 - C_2 \sin x_2 \), one obtains 
\[ H(x_1, x_2, \eta_1, \eta_2) = -2C_1 \sin \frac{\pi x_1}{2} \, x. \]

Then, assumption (i) of Theorem 1 is satisfied. 
Assumption (ii) of theorem 1 is satisfied choosing 
\[ (x_1, x_2) = (x_1, x_1) / (x_1 - x_2) = \sin x / x. \]

Since 
\[ h(t, x_1, \eta_1) - h(t, x_2, \eta_2) = \sin x \big[-(P_1 - P_2) - (C_1 - C_2)\big] \leq \|P_1 - P_2\| - (C_1 - C_2) \leq \|O\| \|x_1 - x_2\|, \]
then assumption (iii) also holds.

Consider the set 
\[ \Lambda := \{(x_1, y_1, x_2, y_2) \in \mathbb{R}^2 \times \mathbb{R}^2 : |x_1 - x_2| < a\} \text{ with } a > 0. \]

Since the diagonal set \( D \) is contained in \( \Lambda \), assumption (v) of Theorem 1 is satisfied.

Choosing \( a = 2\pi / 3 \) and \( \mu = 1 \), all solutions of (3)-(4) is unbounded.

In order to find set A, the results of uniform stability presented for non-autonomous systems in [3] were used.

Following that paper, we consider the next energy function candidate:
\[ V(x, y) = \frac{y^2}{2} - 2K_1 \cos x - 4C \cos \frac{\pi}{2} - \tau (-2C \sin \frac{\pi}{2} - 2K_1 \sin x) y. \]

It is easy to check that
\[ -\dot{V} = \begin{bmatrix} y & P \end{bmatrix} \begin{bmatrix} K - \tau \left[ C \cos \frac{x - \pi}{2} + 2K_1 \cos x \right] - \frac{K}{\tau} \end{bmatrix} \begin{bmatrix} y \\ P \end{bmatrix} + 2CP_1 \tau \sin \frac{x}{2} \left(1 - \cos \frac{\pi}{2} - 2C \sin \frac{x}{2} \right) \geq c(x, y) \]
where \( P_1 := -2C \sin \frac{x}{2} - 2K_1 \sin x \).

As a consequence of Sylvester’s criterion, the quadratic term 
\[ \begin{bmatrix} y & P \end{bmatrix} \left[ K - \tau \left[ C \cos \frac{x - \pi}{2} + 2K_1 \cos x \right] - \frac{K}{\tau} \right] \begin{bmatrix} y \\ P \end{bmatrix} \]
is positively defined if \( \tau \leq K \left(1 + 2K_1 + \frac{K}{\tau} \right) \).

The level curves of function \( V \) are depicted in Figure 2. The positively invariant set \( S(L) \) is shown in a different color. This set is contained in set 
\[ \Lambda := \{(x_1, y_1, x_2, y_2) \in \mathbb{R}^2 \times \mathbb{R}^2 : |x_1 - x_2| < a\}. \]

Therefore, system (3)-(4) synchronizes with respect to \( A = S(L) \). At the same figure, we also project two trajectories whose initial conditions are:

T1: \( (x_1, x_2, y_1, y_2) = (6.0, 5.5, 3.5, -2.5) \) and T2: \( (x_1, x_2, y_1, y_2) = (-5.5, -3.5, -3.5, -2.5) \). The first one starts out of the positively invariant set \( S(L) \), but after entering into this set, it does not leave this set anymore. Trajectory 2 begins inside the positively invariant set and so it does not leave.
the set over positive time. Both trajectories approach the point \((x,y)=(0,0)\) indicating synchronization.

Fig. 3 shows the trajectories that were projected in Fig. 2. Note that solutions synchronize when \(t \to \infty\).

Fig. 2 – Level curves of function \(V\) with \(C=2; P=3, D=1, K_1=2.1, K_2=2.55\) and \(\tau=0.06\). Two projected trajectories of system (23)-(24) are shown. They synchronize when \(t \to \infty\).

Fig. 3. The two solutions of system (23)-(24) corresponding to the projected trajectories that were depicted in Fig.2. In both cases, synchronization is observed.

4. Conclusions
Synchronization of a class of coupled second-order systems was studied in this paper. Using very mild conditions on the vector field, conditions on the coupling parameters that guarantee synchronization are given.

The proposed approach is new and can be used to study even systems that do not globally synchronize. In these cases, an estimate of the synchronization region is given.

An example was presented to illustrate the application of our result. In that, synchronization between two pendulums with a nonlinear coupling is studied. Estimates on the minimum coupling parameter values that guarantee synchronization are given for situations where solutions are unbounded.

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References