

ON NEAR-RING IDEALS WITH (σ, τ) -DERIVATION

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ABSTRACT. Let N be a 3-prime left near-ring with multiplicative center Z , a (σ, τ) -derivation D on N is defined to be an additive endomorphism satisfying the product rule $D(xy) = \tau(x)D(y) + D(x)\sigma(y)$ for all $x, y \in N$, where σ and τ are automorphisms of N . A nonempty subset U of N will be called a semigroup right ideal (resp. semigroup left ideal) if $UN \subset U$ (resp. $NU \subset U$) and if U is both a semigroup right ideal and a semigroup left ideal, it be called a semigroup ideal. We prove the following results: Let D be a (σ, τ) -derivation on N such that $\sigma D = D\sigma, \tau D = D\tau$. (i) If U is semigroup right ideal of N and $D(U) \subset Z$ then N is commutative ring. (ii) If U is a semigroup ideal of N and $D^2(U) = 0$ then $D = 0$. (iii) If $a \in N$ and $[D(U), a]_{\sigma, \tau} = 0$ then $D(a) = 0$ or $a \in Z$.

1. INTRODUCTION

H. E. Bell and G. Mason have shown several commutativity theorems for near-rings with derivation in [1]. Bell has proved in [2], that if N be a 3-prime zero symmetric left near-ring and D be a nonzero derivation on N, U is nonzero subset of N such that $UN \subset U$ or $NU \subset U$ and $D(U) \subset Z$ then N is commutative ring. The major purpose of this paper to generalize this result replacing the derivation D by (σ, τ) -derivation.

Throughout this paper, N will denote a zero-symmetric left near-ring and usually will be 3-prime, that is, if $aNb = 0$ then $a = 0$ or $b = 0$, with multiplicative center Z . A nonempty subset U of N will be called a semigroup right ideal (resp. semigroup left ideal) if $UN \subset U$ (resp. $NU \subset U$) and if U is both a semigroup right ideal and a semigroup left ideal, it be called a semigroup ideal. For subsets $X, Y \subset N$ the symbol $[X, Y]$ will denote the set $\{xy - yx \mid x \in X, y \in Y\}$. Let σ, τ be two near-ring automorphisms of N . An additive endomorphism of N with the property that $D(xy) = \tau(x)D(y) + D(x)\sigma(y)$ for all $x, y \in N$ is called (σ, τ) -derivation of N . Given $x, y \in N$, we write $[x, y]_{\sigma, \tau} = x\sigma(y) - \tau(y)x$; in particular $[x, y]_{1,1} = [x, y]$, in the usual sense. As for terminologies used here without mention, we refer to G. Pilz [4].

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2. RESULTS

Lemma 1 ([3, Lemma 2]). *Let D be a (σ, τ) -derivation on the near-ring N . Then*

$$(\tau(x)D(y) + D(x)\sigma(y))\sigma(a) = \tau(x)D(y)\sigma(a) + D(x)\sigma(y)\sigma(a)$$

for all $x, y, a \in N$.

Lemma 2 ([2, Lemma 1.2]). *Let N be a 3-prime near-ring.*

- i) *If $z \in Z \setminus \{0\}$ then z is not a zero divisor.*
- ii) *If $Z \setminus \{0\}$ contains an element z for which $z + z \in Z$, then $(N, +)$ is abelian.*
- iii) *If $z \in Z \setminus \{0\}$ and x is an element of N such that $xz \in Z$ or $zx \in Z$, then $x \in Z$.*

Lemma 3 ([3, Lemma 3]). *Let D be a nonzero (σ, τ) -derivation on prime near-ring N and $a \in N$.*

- i) *If $D(N)\sigma(a) = 0$ then $a = 0$.*
- ii) *If $aD(N) = 0$ then $a = 0$.*

Lemma 4. *Let N be a prime near-ring, D a (σ, τ) -derivation of N and U be nonzero semigroup right ideal (resp. semigroup left ideal). If $D(U) = 0$ then $D = 0$.*

Proof. U be a nonzero semigroup right ideal of N and $D(U) = 0$. For any $x \in N$, $u \in U$, we get

$$0 = D(ux) = \tau(u)D(x) + D(u)\sigma(x)$$

and so,

$$\tau(u)D(x) = 0, \quad \text{for all } u \in U, x \in N.$$

By Lemma 3(ii), we have $D = 0$. If U is a nonzero semigroup left ideal of N , then the proof is similar. \square

Lemma 5. *Let N be a 3-prime near-ring, D a nonzero (σ, τ) -derivation of N and U be nonzero semigroup ideal.*

- i) *If $x \in N$ and $D(U)\sigma(x) = 0$ then $x = 0$.*
- ii) *If $x \in N$ and $xD(U) = 0$ then $x = 0$.*

Proof. i) Suppose U a nonzero semigroup ideal of N and $D(U)\sigma(x) = 0$. By Lemma 1 for all $u, v \in U$, we get

$$0 = D(uv)\sigma(x) = \tau(u)D(v)\sigma(x) + D(u)\sigma(v)\sigma(x).$$

Using the hypothesis and σ is an automorphism of N , we have

$$\sigma^{-1}(D(u))Ux = 0, \quad \text{for all } u \in U.$$

Since D is a nonzero (σ, τ) -derivation of N , this relation gives us $x = 0$ by [2, Lemma 1.4 (i)].

- ii) A similar argument works if $xD(U) = 0$. \square

Theorem 1. *Let N be a 3-prime near-ring, D be a nonzero (σ, τ) -derivation of N and U a nonzero semigroup right ideal. If $D(U) \subset Z$, then N is commutative ring.*

Proof. For all $u, v \in U$, we get

$$D(uv) = \tau(u)D(v) + D(u)\sigma(v) \in Z$$

and commuting this element with $\sigma(v)$ gives us

$$(\tau(u)D(v) + D(u)\sigma(v))\sigma(v) = \sigma(v)(\tau(u)D(v) + D(u)\sigma(v)).$$

Using Lemma 1 and $D(u) \in Z$, we have

$$\tau(u)D(v)\sigma(v) + D(u)\sigma(v)\sigma(v) = \sigma(v)\tau(u)D(v) + D(u)\sigma(v)\sigma(v)$$

and so,

$$D(v)[\tau(u), \sigma(v)] = 0, \quad \text{for all } u, v \in U.$$

Since $D(v) \in Z$, we obtain

$$D(v) = 0 \quad \text{or} \quad [\tau(u), \sigma(v)] = 0, \quad \text{for all } u, v \in U.$$

Suppose that $D(v) = 0$, then $D(uv) = \tau(u)D(v) + D(u)\sigma(v) = D(u)\sigma(v) \in Z$. This gives us $v \in Z$ from Lemma 2(iii). For any cases we conclude that

$$(2.1) \quad [\tau(u), \sigma(v)] = 0, \quad \text{for all } u, v \in U.$$

Since τ is an automorphism of N , the relation (2.1) yields

$$[U, \tau^{-1}(\sigma(v))] = 0, \quad \text{for all } v \in U$$

and so, $U \subset Z$ from [2, Lemma 1.3 (iii)]. Thus, we obtain that N is commutative ring by [2, Lemma 1.5]. \square

Theorem 2. *Let N be a 3-prime near-ring, D a nonzero (σ, τ) -derivation of N and U be nonzero semigroup left ideal. If $D(U) \subset Z$, then N is commutative ring.*

Proof. If we use the same argument in the proof of Theorem 1, we conclude that

$$(2.2) \quad D(v) = 0 \quad \text{or} \quad [\tau(u), \sigma(v)] = 0, \quad \text{for all } u, v \in U.$$

Suppose that $D(v) = 0$, then $D(uv) = \tau(u)D(v) + D(u)\sigma(v) = D(u)\sigma(v) \in Z$, for all $u \in U$, so that

$$D(u)\sigma(v)x = xD(u)\sigma(v) = D(u)x\sigma(v), \quad \text{for all } u \in U, x \in N.$$

Thus, $D(u)[\sigma(v), x] = 0$, for all $u \in U, x \in N$. By Lemma 2 (i) and Lemma 4, we have show that

$$(2.3) \quad \text{If } v \in U \quad \text{and} \quad D(v) = 0, \quad \text{then} \quad v \in Z.$$

Thus, the relation (2.2) yields

$$[\tau(u), \sigma(v)] = 0, \quad \text{for all } u, v \in U.$$

By the hypothesis, we get $D(wu) \in Z$, for all $u, w \in U$. That is,

$$D(wu) = \tau(w)D(u) + D(w)\sigma(u) \in Z.$$

Commuting this element with $\sigma(v)$, one can obtain

$$(\tau(w)D(u) + D(w)\sigma(u))\sigma(v) = \sigma(v)(\tau(w)D(u) + D(w)\sigma(u)).$$

Applying Lemma 1 and using $\sigma(v)\tau(w) = \tau(w)\sigma(v)$, $D(u) \in Z$, we have

$$D(u)\tau(w)\sigma(v) + D(w)\sigma(u)\sigma(v) = D(u)\tau(w)\sigma(v) + D(w)\sigma(v)\sigma(u)$$

and so,

$$D(w)\sigma([u, v]) = 0, \quad \text{for all } u, v, w \in U.$$

Since $D(w) \in Z$, we have

$$D(w) = 0 \quad \text{or} \quad [u, v] = 0, \quad \text{for all } u, v, w \in U.$$

In the first case, we find that $D = 0$, a contradiction. So, we must have U is commutative.

Now, we assume that $U \cap Z \neq \{0\}$. Then U contains a nonzero central element w , we have

$$(wx)u = (xw)u = u(xw) = u(wx) = (uw)x = (wu)x$$

that is,

$$w[x, u] = 0, \quad \text{for all } u \in U.$$

Since w is nonzero central element, we have $U \subset Z$. Thus N is commutative ring by [2, Lemma 1.5].

If $U \cap Z = \{0\}$, in which case (2.3) shows that $D(u) \neq 0$ for all $u \in U \setminus \{0\}$. For each such u , $D(u^2) = \tau(u)D(u) + D(u)\sigma(u) = D(u)(\tau(u) + \sigma(u)) \in Z$. Since $D(u)$ is noncentral element of N , we get

$$\tau(u) + \sigma(u) \in Z, \quad \text{for all } u \in U.$$

Suppose that $\tau(u) + \sigma(u) = 0$, for all $u \in U \setminus \{0\}$. By the hypothesis, $D(u^3) \in Z$, we get

$$\begin{aligned} D(u^3) &= \tau(u)D(u^2) + D(u)\sigma(u^2) \\ &= \tau(u)\tau(u)D(u) + \tau(u)D(u)\sigma(u) + D(u)\sigma(u^2) \in Z. \end{aligned}$$

Using $\tau(u) = -\sigma(u)$ and $D(u) \in Z$, we get

$$\tau(u)\tau(u)D(u) - \tau(u)\tau(u)D(u) + D(u)\sigma(u^2) \in Z$$

which implies $u^2 \in Z$. Hence we get $u^2 \in U \cap Z = \{0\}$, and so $u^2 = 0$.

Now, for any $x \in N$, $D(xu) = \tau(x)D(u) + D(x)\sigma(u) \in Z$. Hence,

$$\sigma(u)(\tau(x)D(u) + D(x)\sigma(u)) = (\tau(x)D(u) + D(x)\sigma(u))\sigma(u).$$

Using Lemma 1 and $u^2 = 0$, we have

$$\sigma(u)(\tau(x)D(u) + D(x)\sigma(u)) = \tau(x)D(u)\sigma(u).$$

Left-multiplying this relation by $\sigma(u)$ and using $u^2 = 0$, we obtain

$$\sigma(u)\tau(x)D(u)\sigma(u) = 0, \quad \text{for all } u \in U, x \in N.$$

The primeness of the near-ring N , we obtain that $D(u)\sigma(u) = 0$. Since $0 \neq D(u) \in Z$, we have $\sigma(u) = 0$, a contradiction. Thus, we must have $z = \sigma(u_0) + \tau(u_0) \in Z \setminus \{0\}$ for an $u_0 \in U$. Since $D(zu) = \tau(z)D(u) + D(z)\sigma(u) \in Z$, for any $u \in U$, we have

$$(\tau(z)D(u) + D(z)\sigma(u))\sigma(y) = \sigma(y)(\tau(z)D(u) + D(z)\sigma(u)), \quad \text{for all } y \in N$$

and so,

$$\tau(z)D(u)\sigma(y) + D(z)\sigma(u)\sigma(y) = \sigma(y)\tau(z)D(u) + \sigma(y)D(z)\sigma(u).$$

Using $z, D(u) \in Z$, we get

$$D(u)\tau(z)\sigma(y) + D(z)\sigma(u)\sigma(y) = D(u)\tau(z)\sigma(y) + \sigma(y)\sigma(u)D(z).$$

That is,

$$D(z)\sigma([u, y]) = 0, \quad \text{for all } u \in U, y \in N.$$

Since $D(z)$ is a nonzero central element of N , we have $U \subset Z$. Thus, we conclude that N is commutative ring from [2, Lemma 1.5]. \square

Theorem 3. *Let N be a 3-prime near-ring, D be a (σ, τ) -derivation of N such that $\sigma D = D\sigma, \tau D = D\tau$ and U a nonzero semigroup ideal of N . If $D^2(U) = 0$ then $D = 0$.*

Proof. For arbitrary $u, v \in U$, we have $0 = D^2(uv) = D(D(uv)) = D(\tau(u)D(v) + D(u)\sigma(v)) = \tau^2(u)D^2(v) + D(\tau(u))\sigma(D(v)) + \tau(D(u))D(\sigma(v)) + D^2(u)\sigma^2(v)$. Using the hypothesis and $\sigma D = D\sigma, \tau D = D\tau$, we get

$$2D(\tau(u))D(\sigma(v)) = 0, \quad \text{for all } u, v \in U$$

and so,

$$2\sigma^{-1}(\tau(D(u)))D(U) = 0, \quad \text{for all } u \in U.$$

From Lemma 5(ii), we obtain that

$$2D(u) = 0, \quad \text{for all } u \in U.$$

Now for any $y \in N$ and $u \in U$, $0 = D^2(yu) = D(\tau(y)D(u) + D(y)\sigma(u)) = \tau^2(y)D^2(v) + 2\tau(D(y))\sigma(D(u)) + D^2(y)\sigma^2(u)$. Hence,

$$D^2(y)\sigma^2(u) = 0, \quad \text{for all } y \in N, u \in U.$$

Using σ is an automorphism on N and [2, Lemma 1.4 (i)], we can take $D^2(N) = 0$. That is $D = 0$ by [3, Lemma 4]. \square

Theorem 4. *Let N be a 3-prime near-ring, D be a (σ, τ) -derivation of N such that $\sigma D = D\sigma, \tau D = D\tau$ and U a nonzero semigroup ideal of N . If $a \in N$ and $[D(U), a]_{\sigma, \tau} = 0$ then $D(a) = 0$ or $a \in Z$.*

Proof. For $u \in U$, we get $D(au)\sigma(a) = \tau(a)D(au)\tau(a)$. Expanding this equation, one can obtain,

$$\tau(a)D(u)\sigma(a) + D(a)\sigma(u)\sigma(a) = \tau(a)\tau(a)D(u) + \tau(a)D(a)\sigma(u).$$

By the hypothesis, we get $D(u)\sigma(a) = \tau(a)D(u)$. Hence,

$$\tau(a)\tau(a)D(u) + D(a)\sigma(u)\sigma(a) = \tau(a)\tau(a)D(u) + \tau(a)D(a)\sigma(u).$$

That is,

$$(2.4) \quad D(a)\sigma(u)\sigma(a) = \tau(a)D(a)\sigma(u), \quad \text{for all } u \in U.$$

Replacing $ux, x \in N$ by u in (2.4) and using (2.4), we have

$$D(a)\sigma(u)\sigma(x)\sigma(a) = \tau(a)D(a)\sigma(u)\sigma(x) = D(a)\sigma(u)\sigma(a)\sigma(x)$$

and so,

$$D(a)\sigma(u)\sigma([x, a]) = 0, \quad \text{for all } x \in N, u \in U.$$

Since σ is an automorphism of N , this relation can be written,

$$\sigma^{-1}(D(a)U[x, a]) = 0, \quad \text{for all } x \in N.$$

Thus we conclude that $a \in Z$ or $D(a) = 0$ by [2, Lemma 1.4 (i)]. \square

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