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A COLLOCATION METHOD FOR LANE-EMDEN TYPE EQUATIONS IN TERMS OF GENERALIZED BERNSTEIN POLYNOMIALS

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Abstract

In this study, a collocation method based on Bernstein polynomials defined on the interval $[a, b]$ is developed for approximate solution of the nonlinear differential equations of Lane-Emden type that have an important place in astrophysics and mathematical physics. The proposed method reduces the solution of nonlinear problem to the solution of a system of linear algebraic equations iteratively by using quasilinearization

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technique and collocation points. Some numerical examples are given to illustrate the efficiency, validity and applicability of the method.

1. Introduction

Lane-Emden equations are the most well-known classes of nonlinear second-order ordinary differential equations which model many phenomena in mathematical physics, thermodynamics, fluid mechanics and astrophysics such as the theory of stellar structure, the thermal behavior of a spherical cloud of gas, clusters of galaxies, self-gravitating sphere of polytrophic isothermal gas and theory of thermionic currents. Since the Lane-Emden equations have singularity at $x = 0$, these equations are categorized as singular initial or boundary value problems. Recently, a number of methods have been used to solve the nonlinear Lane-Emden type equations under the initial, boundary or mixed conditions. Some approximate solutions for solving the nonlinear Lane-Emden type equations under the initial conditions have been proposed by using δ-perturbation expansion method [6], Adomian method [32, 33], linearization methods [25], Legendre wavelets [5]; Homotopy perturbation method [9, 29], multistep methods [13], weight fuzzy marginal linearization method [31], Hermite function collocation method [23], radial basis functions [12], Tau method [27], Chebyshev collocation method [28], shifted Jacobi-Gauss collocation method [7], Lie group method [21], homotopy analysis method [15, 30], variational approach [11], series approach [26], Bernstein operational matrix of differentiation [22] and Bernstein polynomial operational matrix of integration [19]. Besides, shifted Jacobi-Gauss collocation method [7], Lie group method [8], modified decomposition method [17] and piecewise quasilinearization methods [24] have been used for approximate solutions of the boundary value problems in Lane-Emden type equations. Moreover, Chebyshev collocation method [3] has been presented for solving Lane-Emden type equations, and Bernstein collocation method [14] has been used only linear type of Lane-Emden equations under the mixed conditions.

The purpose of present paper is to develop a collocation method depending on the Bernstein polynomials for the nonlinear Lane-Emden type equations with initial or boundary conditions. These conditions are general unlike the above referred studies. The proposed method has been produced in the light of the Bernstein collocation method that was introduced for linear differential equations by Akyüz-Daşcıoğlu and Isler [4]. Besides, the nonlinearity of the problem has been removed by means of the quasilinearization method. This technique is based on the linearization of the governing singular ordinary differential equation and require the solution of a linear ordinary differential equation at each iteration. The convergence rate of this method depends on the initial guess of the solution. The quasilinearization technique has been used with great success to study nonlinear ordinary differential equations. Agarwal and Chow [1] have developed the quasilinearization method for solution of the nonlinear ordinary differential equation under the boundary conditions. Mandelzweig and Tabakin [20] have determined general conditions for the quadratic, monotonic and uniform convergence in the quasilinearization method for solving both initial and boundary-value problems in nonlinear ordinary differential equations. In particular, they have applied the quasilinearization method to the singular initial-value problems governed by the Lane-Emden equation.

2. Bernstein Polynomials and Their Basis Form

A Bernstein polynomial defined on the interval $[0, 1]$, originally introduced by Sergei Natanovich Bernstein (1912), is a polynomial that is a linear combination of Bernstein basis polynomials.

The definitions and fundamental properties of the Bernstein polynomials and their basis form that can be easily generalized on the interval $[a, b]$ are given as follows:

Definition 2.1. Generalized Bernstein basis polynomials can be defined on the interval $[a, b]$; by

$$p_{i,n}(x) = \frac{1}{(b-a)^n}\binom{n}{i}(x-a)^i(b-x)^{n-i}; \quad i = 0, 1, \ldots, n.$$

For convenience, we set $p_{i,n}(x) = 0$, if $i < 0$ or $i > n$.

Definition 2.2. Let $y : [a, b] \to \mathbb{R}$ be continuous function on the interval $[a, b]$. Generalized Bernstein polynomials of nth-degree are defined by

$$B_n(y; x) = \sum_{i=0}^{n} y\left(a + \frac{(b-a)i}{n}\right)p_{i,n}(x).$$

Theorem 2.1. *If* $y \in C^k[a, b]$, *for some integer* $m \geq 0$, *then*

$$\lim_{n\to\infty} B_n^{(k)}(y;\, x) = y^{(k)}(x); \quad k = 0, 1, \ldots, m$$

converges uniformly.

Bernstein polynomials and their basis forms also have many useful properties such as the positivity, continuity, recursion's relation, symmetry, unity partition of the basis set over the interval $[a, b]$, and these polynomials are differentiable and integrable. For more information about Bernstein polynomials, see [10, 16].

Theorem 2.2 [2]**.** *There is a relation between generalized Bernstein basis polynomials matrix and their derivatives in the form*

$$\mathbf{P}^{(k)}(x) = \mathbf{P}(x)\mathbf{N}^k; \quad k = 1, \ldots, m$$

such that

$$\mathbf{P}(x) = [p_{0,n}(x) \quad p_{1,n}(x) \ldots p_{n,n}(x)].$$

Here the elements of $(n+1)\times(n+1)$ *matrix* $\mathbf{N} = (d_{ij}),\ \ i, j = 0, 1, \ldots, n$ *are defined by*

$$d_{ij} = \frac{1}{b-a}\begin{cases} n-i; & \text{if } j = i+1 \\ 2i-n; & \text{if } j = i \\ -i; & \text{if } j = i-1 \\ 0; & \text{otherwise.} \end{cases}$$

3. Method of Solution

In this section, let us consider the nonlinear Lane-Emden type equation of the form

$$y''(x) + \frac{\alpha}{x}y'(x) + f(x, y) = 0; \quad a \leq x \leq b, \quad a, \alpha > 0, \tag{1}$$

under the initial conditions

$$y(a) = \lambda, \quad y'(a) = \mu \tag{2}$$

or boundary conditions

$$A_1 y(a) + A_2 y'(a) = \beta, \quad B_1 y(b) + B_2 y'(b) = \gamma, \tag{3}$$

where α is a constant, $f(x, y)$ is a nonlinear function of x and y and $y(x)$ is unknown function. Then, a collocation method is derived iteratively to get the approximate solution of this equation by means of quasilinearization technique and generalized Bernstein polynomials:

$$y(x) \cong B_n(y; x) = \sum_{i=0}^{n} y\left(a + \frac{(b-a)i}{n}\right) p_{i,n}(x).$$

Theorem 3.1. *Let* $x_s \neq 0 \in [a, b]$ *be collocation points. The nonlinear Lane-Emden type equation* (1) *has the following matrix form*:

$$[\mathbf{PN}^2 + \alpha\mathbf{QPN} + \mathbf{F}_r\mathbf{P}]\mathbf{Y}_{r+1} = \mathbf{H}_r; \quad r = 0, 1, ... \tag{4}$$

Here

$$\mathbf{Q} = diag\left[\frac{1}{x_s}\right], \quad \mathbf{F}_r = diag[f_{y_r}(x_s, y_r)], \quad \mathbf{P} = [p_{i,n}(x_s)]$$

are $(n+1)\times(n+1)$ *matrices, and*

$$\mathbf{Y}_{r+1} = \left[y_{r+1}\left(a + \frac{(b-a)i}{n}\right)\right], \quad \mathbf{H}_r = [y_r(x_s) f_{y_r}(x_s, y_r) - f(x_s, y_r)]$$

are $(n+1)\times 1$ *matrices for* $i, s = 0, 1, ..., n.$

Proof. First, by applying the quasilinearization method to the nonlinear Lane-Emden type equation (1), we obtain a sequence of linear differential equations:

$$y''_{r+1}(x) + \frac{\alpha}{x} y'_{r+1}(x) + [f(x, y_r) + f_{y_r}(x, y_r)(y_{r+1}(x) - y_r(x))] = 0, \tag{5}$$

where the expression f_{y_r} is a partial differentiation of the function f with respect to y_r. Here $y_0(x)$ is a reasonable initial approximation of the function $y(x)$ and $y_r(x)$ is always considered known and $y_{r+1}(x)$ is obtained from previous iteration.

Denoting

$$h_r(x) = y_r(x) f_{y_r}(x, y_r) - f(x, y_r),$$

then the equation (5) can be reformulated as

$$y''_{r+1}(x) + \frac{\alpha}{x} y'_{r+1}(x) + f_{y_r}(x, y_r) y_{r+1}(x) = h_r(x). \tag{6}$$

Let an unknown function $y_{r+1}(x)$ be the Bernstein polynomial solution. Then the function and its derivatives can be expressed by

$$y_{r+1}(x) \simeq B_n(y_{r+1}; x) = \mathbf{P}(x)\mathbf{Y}_{r+1},$$

$$y'_{r+1}(x) \simeq B'_n(y_{r+1}; x) = \mathbf{P}'(x)\mathbf{Y}_{r+1},$$

$$y''_{r+1}(x) \simeq B''_n(y_{r+1}; x) = \mathbf{P}''(x)\mathbf{Y}_{r+1}.$$

By utilizing Theorem 2.2 and collocation points, the above relations become

$$y_{r+1}(x_s) = \mathbf{P}(x_s)\mathbf{Y}_{r+1},$$

$$y'_{r+1}(x_s) = \mathbf{P}(x_s)\mathbf{N}\mathbf{Y}_{r+1},$$

$$y''_{r+1}(x_s) = \mathbf{P}(x_s)\mathbf{N}^2\mathbf{Y}_{r+1}. \tag{7}$$

Substituting the collocation points and relation (7) into equation (6), we obtain linear algebraic system

$$\mathbf{P}(x_s)\mathbf{N}^2\mathbf{Y}_{r+1} + \frac{\alpha}{x_s}\mathbf{P}(x_s)\mathbf{N}\mathbf{Y}_{r+1} + f_{y_r}(x_s, y_r)\mathbf{Y}_{r+1} = h_r(x_s). \tag{8}$$

Considering the matrices

$$\mathbf{Q} = \begin{bmatrix} 1/x_0 & 0 & \cdots & 0 \\ 0 & 1/x_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/x_n \end{bmatrix},$$

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}(x_0) \\ \mathbf{P}(x_1) \\ \vdots \\ \mathbf{P}(x_n) \end{bmatrix} = \begin{bmatrix} p_{0,n}(x_0) & p_{1,n}(x_0) & \cdots & p_{n,n}(x_0) \\ p_{0,n}(x_1) & p_{1,n}(x_1) & \cdots & p_{n,n}(x_1) \\ \vdots & & & \vdots \\ p_{0,n}(x_n) & p_{1,n}(x_n) & \cdots & p_{n,n}(x_n) \end{bmatrix},$$

$$\mathbf{F}_r = \begin{bmatrix} f_{y_r}(x_0, y_r) & 0 & \cdots & 0 \\ 0 & f_{y_r}(x_1, y_r) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f_{y_r}(x_n, y_r) \end{bmatrix}$$

and

$$\mathbf{H}_r = \begin{bmatrix} y_r(x_0) f_{y_r}(x_0, y_r) - f(x_0, y_r) \\ y_r(x_1) f_{y_r}(x_1, y_r) - f(x_1, y_r) \\ \vdots \\ y_r(x_n) f_{y_r}(x_n, y_r) - f(x_n, y_r) \end{bmatrix},$$

the equation (8) can be written as desired matrix form (4). This is completed the proof.

The equation (4) can be written simply in the form

$$\mathbf{W}_r \mathbf{Y}_{r+1} = \mathbf{H}_r \text{ or } [\mathbf{W}_r; \mathbf{H}_r]$$

so that

$$\mathbf{W}_r = \mathbf{PN}^2 + \alpha \mathbf{QPN} + \mathbf{F}_r \mathbf{P}.$$

Now, we can solve the nonlinear Lane-Emden type equation (1) under the initial (2) or boundary (3) conditions as follows:

Step 1. The augmented matrix $[\mathbf{W}_r;\, \mathbf{H}_r]$ corresponds to a system of linear algebraic equations with n-unknown coefficients $y_{r+1}\left(a + \frac{(b-a)i}{n}\right)$. To determine the $\mathbf{W}_r$ and $\mathbf{H}_r$, the first iteration function $y_0(x)$ should satisfy the given conditions.

Step 2. From expression (7), the matrix forms of initial (2) and boundary (3) conditions can be written respectively

$$\mathbf{P}(a)\mathbf{Y}_{r+1} = \lambda, \quad \mathbf{P}(a)\mathbf{N}\mathbf{Y}_{r+1} = \mu, \tag{9}$$

$$A_1\mathbf{P}(a)\mathbf{Y}_{r+1} + A_2\mathbf{P}(a)\mathbf{N}\mathbf{Y}_{r+1} = \beta, \quad B_1\mathbf{P}(b)\mathbf{Y}_{r+1} + B_2\mathbf{P}(b)\mathbf{N}\mathbf{Y}_{r+1} = \gamma. \tag{10}$$

Besides, equations (9) and (10) can be denoted by the augmented matrices

$$[\mathbf{P}(a);\, \lambda], \quad [\mathbf{P}(a)\mathbf{N};\, \mu] \tag{11}$$

and

$$[A_1\mathbf{P}(a) + A_2\mathbf{P}(a)\mathbf{N};\, \beta], \quad [B_1\mathbf{P}(b) + B_2\mathbf{P}(b)\mathbf{N};\, \gamma]. \tag{12}$$

Step 3. To obtain the solution of nonlinear Lane-Emden type equation (1) under the initial (2) or boundary (3) conditions, we add the elements of the row matrices (11) or (12) to the end of the augmented matrix $[\mathbf{W}_r;\, \mathbf{H}_r]$. In this way, we have the new augmented matrix $[\tilde{\mathbf{W}}_r;\, \tilde{\mathbf{H}}_r]$ that is $(n+3)\times(n+1)$ rectangular matrix. Alternatively, two rows of the augmented matrix $[\mathbf{W}_r;\, \mathbf{H}_r]$ can be replaced with the rows of the augmented matrix (11) or (12). In this case, we have an $(n+1)\times(n+1)$ square matrix and we denote the new augmented matrix by $[\mathbf{W}_r^*;\, \mathbf{H}_r^*]$.

Step 4. If $rank(\tilde{\mathbf{W}}_r) = rank[\tilde{\mathbf{W}}_r;\, \tilde{\mathbf{H}}_r] = n+1$, then unknown coefficients y_{r+1} are uniquely determined for each iteration r. This kind of systems can be solved by the Gauss Elimination, Generalized Inverse and QR factorization methods.

4. Numerical Results

In this section, nonlinear Lane-Emden type equations with initial or boundary value problems are considered for testing applicability of the presented method. On the collocation points $x_s \neq 0 \in [a, b]$, absolute error

$$| e(x_s) | = | y(x_s) - B_n(y_r;\, x_s) |,$$

maximum error

$$E = \max_{x_s \in [a,\, b]} | e(x_s) |$$

and absolute error of the *r*th iteration function

$$| e_r(x_s) | = | B_n(y_{r+1};\, x_s) - B_n(y_r;\, x_s) |,$$

maximum error of the *r*th iteration function

$$E_r = \max_{x_s \in [a,\, b]} | e_r(x_s) |$$

are given for the nonlinear Lane-Emden type problems. Here $B_n(y_r;\, x)$ is a Bernstein approximation of the *r*th iteration function and $y(x)$ is an exact solution. The numerical results, computed in MATLAB 7.1 and $tol = 10^{-10}$ by applying the one of the techniques referred to Step 3, are compared with the other methods to illustrate the high accuracy and efficiency of the proposed method.

Example 4.1. Consider the nonlinear Lane-Emden equation

$$y''(x) + \frac{2}{x} y'(x) + y^5(x) = 0; \quad 0 < x \le 1. \tag{13}$$

The equation has exact solution $y(x) = \left(1 + \frac{x^2}{3}\right)^{-1/2}$ under the initial conditions

$$y(0) = 1, \quad y'(0) = 0.$$

Let $y_0 = 1$ be the first iteration function.

Let try to find the numerical solution of the above problem for $n = 2$, $r = 3$ and collocation points $x_0 = \frac{1}{3}$, $x_1 = \frac{2}{3}$, $x_2 = 1$. In this case, the main matrix equation of equation (13) is obtained as

$$\mathbf{W}_3\mathbf{Y}_4 = \mathbf{H}_3$$

such that $\mathbf{W}_3 = \mathbf{PN}^2 + 2\mathbf{QPN} + \mathbf{F}_3\mathbf{P}$. Here the matrices are as follows:

$$\mathbf{H}_3 = \begin{bmatrix} 4y_3^5(1/3) \\ 4y_3^5(2/3) \\ 4y_3^5(1) \end{bmatrix} = \begin{bmatrix} 3.671 \\ 2.813 \\ 1.744 \end{bmatrix}, \quad \mathbf{P} = \begin{bmatrix} 4/9 & 4/9 & 1/9 \\ 1/9 & 4/9 & 4/9 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{N} = \begin{bmatrix} -2 & 2 & 0 \\ -1 & 0 & 1 \\ 0 & -2 & 2 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{F}_3 = \begin{bmatrix} 5y_3^4(1/3) & 0 & 0 \\ 0 & 5y_3^4(2/3) & 0 \\ 0 & 0 & 5y_3^4(1) \end{bmatrix} = \begin{bmatrix} 4.669 & 0 & 0 \\ 0 & 3.773 & 0 \\ 0 & 0 & 2.574 \end{bmatrix}.$$

Here y_3 is known function from the former iteration. The main matrix equation can be written in the augmented matrix form $[\mathbf{W}_3;\ \mathbf{H}_3]$. Then, deleting last two rows of the augmented matrix $[\mathbf{W}_3;\ \mathbf{H}_3]$ and replacing with the rows of augmented matrix $[1 \quad 0 \quad 0 \quad ; \quad 1]$ and $[-2 \quad 2 \quad 0 \quad ; \quad 0]$ obtained by initial conditions, new augmented matrix is obtained as $[\mathbf{W}_3^*;\ \mathbf{H}_3^*]$. If this system is solved by using the algorithm in Matlab 7.1, unknown coefficient is uniquely determined as

$$y_4 = (x-1)^2 - 2x(x-1) + (7629317x^2)/9007199.$$

Table 1. Comparison of the absolute errors for Example 4.1

	Presented method		Chebyshev series method [3]	
x_s	$n = 2$	$n = 15$	$n = 2$	$n = 15$
0.001	$1.4e-008$	$9.9e-017$	$7.0e-008$	$5.0e-017$
0.01	$1.4e-006$	$6.6e-015$	$7.0e-006$	$4.0e-016$
0.1	$1.4e-004$	$6.6e-009$	$7.0e-004$	$2.0e-015$

The comparison of absolute error difference between obtained solution and exact solution with the Chebyshev series method [3] is given for the first iteration, in Table 1. As seen from Table 1, the numerical results of the proposed method is better than the other method for $n = 2.$ The numerical results of the proposed method are calculated with deleting last two rows of the main augmented matrix and replacing with the rows of augmented matrix obtained by initial conditions on the collocation points $x_s = a + \frac{(b-a)s}{n+1};\ s = 1, \ldots, n+1.$ Moreover, the absolute errors and the maximum error of third iteration are obtained respectively as $|e_3(1/3)| = 3.8e-010,$ $|e_3(2/3)| = 1.5e-009,$ $|e_3(1)| = 3.4e-009$ and $E_3 = 3.7e-009.$ Besides, Pandey and Kumar [22] presented the graph of absolute errors by using Bernstein operational matrix of differentiation for $n = 5$ and 6 nearly 10^{-6}. Therefore, we can say that the proposed method can be applied simply to the nonlinear Lane-Emden type problem and the numerical results have higher accuracy for smaller values n and increasing iteration points r.

Example 4.2. Nonlinear Lane-Emden equation is given as

$$y''(x) + \frac{2}{x}y'(x) + 4(2e^{y} + e^{y/2}) = 0; \quad 0 < x \leq 1$$

under the initial conditions

$$y(0) = y'(0) = 0,$$

where the exact solution is $y(x) = -2\ln(1+x^2)$. Let first iteration function be $y_0(x) = 0.$

Table 2. Absolute errors of Example 4.2

x_s	$n = 5$	$n = 10$	$n = 15$	$n = 30$
0.1	$2.0e-006$	$1.5e-008$	$5.4e-012$	$2.8e-015$
0.2	$1.6e-005$	$2.0e-008$	$5.0e-012$	$2.6e-015$
0.25	$3.2e-005$	$1.1e-008$	$9.8e-012$	$2.6e-015$
0.3	$4.2e-005$	$2.1e-008$	$3.4e-013$	$2.9e-015$

0.4	$4.3e-006$	$1.0e-008$	$1.9e-012$	$2.3e-015$
0.5	$1.1e-004$	$4.5e-008$	$3.9e-012$	$1.8e-015$
0.6	$1.6e-004$	$5.0e-008$	$5.0e-012$	$2.9e-015$
0.7	$1.7e-005$	$9.9e-009$	$1.1e-011$	$8.4e-014$
0.75	$1.8e-004$	$7.8e-008$	$1.9e-011$	$4.5e-013$
0.8	$3.1e-004$	$3.6e-008$	$2.2e-011$	$2.1e-012$
0.9	$1.5e-004$	$4.2e-008$	$5.4e-011$	$3.3e-011$
1.0	$3.6e-003$	$2.1e-006$	$5.2e-010$	$3.5e-010$

Table 3. Comparisons of the absolute errors with other methods

x_s	Presented method			Legendre Wavelets method [5]	HFC method [23]	SJC method [7]
	$n = 5$	$n = 15$	$n = 30$	$n = 5$	$n = 30$	$n = 30$
0.1	$2.0e-006$	$5.4e-012$	$2.8e-015$	$1.6e-004$	$2.9e-006$	$2.1e-007$
0.5	$1.1e-004$	$3.9e-012$	$1.8e-015$	$2.0e-004$	$3.0e-006$	$4.3e-007$
1.0	$3.6e-003$	$5.2e-010$	$3.5e-010$	$2.7e-002$	$9.3e-007$	$5.6e-008$

In Table 2, the absolute errors are listed by using proposed method on the collocation points $x_s = \left(1 - \cos\left(\frac{\pi s}{n+1}\right)\right)/2;\ \ s = 1, \ldots, n+1$ for different values n and iteration point $r = 3.$ The numerical results are computed with deleting last two rows of the main augmented matrix and replacing with the rows of augmented matrix obtained by initial conditions. Table 2 shows that the approximate solution of the proposed method converges to the exact solution consistently for increasing values n. The comparisons of the absolute errors obtained by proposed method with the Legendre wavelets approximations [5], Hermite functions collocation method [23] and shifted Jacobi-Gauss collocation spectral method [7] are given in Table 3. As seen from Table 3, the numerical results of the presented method are better and more efficiency than the other methods for smaller values n.

Example 4.3. Consider the isothermal gas spheres equation

$$y''(x) + \frac{1}{x} y'(x) + e^{y} = 0; \quad 0 < x \leq 1$$

under the following boundary conditions

$$y'(0) = 0, \quad y(1) = 0.$$

This problem has the exact solution $y(x) = 2\ln\left(\frac{c+1}{cx^2+1}\right)$, where $c = 3 - 2\sqrt{2}$. Let the first iteration function be $y_0(x) = 0$.

Table 4. Maximum errors of Example 4.3

n	$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 5$	$r = 6$
6	1.6*e* – 002	1.4*e* – 005	2.7*e* – 006	2.7*e* – 006	2.7*e* – 006	2.7*e* – 006
9	2.0*e* – 002	2.1*e* – 005	1.4*e* – 009	1.4*e* – 009	1.4*e* – 009	1.4*e* – 009
12	2.3*e* – 002	2.4*e* – 005	2.6*e* – 011	1.7*e* – 011	1.7*e* – 011	1.1*e* – 011
15	2.6*e* – 002	2.7*e* – 005	2.7*e* – 011	5.0*e* – 015	4.3*e* – 015	4.4*e* – 015
17	2.8*e* – 002	2.8*e* – 005	2.9*e* – 011	2.0*e* – 015	2.2*e* – 015	2.4*e* – 015
20	3.0*e* – 002	3.1*e* – 005	3.2*e* – 011	1.2*e* – 015	1.2*e* – 015	1.1*e* – 015
24	3.3*e* – 002	3.4*e* – 005	3.5*e* – 011	1.3*e* – 015	1.1*e* – 015	1.0*e* – 015
30	3.7*e* – 002	3.8*e* – 005	3.9*e* – 011	1.7*e* – 015	2.2*e* – 015	2.3*e* – 015
33	3.9*e* – 002	3.9*e* – 005	4.1*e* – 011	2.3*e* – 015	2.0*e* – 015	2.8*e* – 015
65	5.4*e* – 002	5.5*e* – 005	5.7*e* – 011	1.4*e* – 014	1.2*e* – 014	1.2*e* – 014

Table 5. Comparisons of the maximum errors

n	Presented method	C^1-linearization method [24]	Finite-difference method [18]
9	2.7*e* – 006	3.1*e* – 003	2.0*e* – 004
17	2.4*e* – 015	5.3*e* – 004	5.0*e* – 005
33	2.8*e* – 015	1.5*e* – 004	1.2*e* – 005
65	1.2*e* – 014	6.8*e* – 005	3.1*e* – 006

In Table 4, by considering collocation points $x_s = \left(1 - \cos\left(\frac{\pi s}{n+1}\right)\right)/2;$ $s = 1, \ldots, n+1,$ maximum errors of the proposed method is presented for different values n and iteration points r. The numerical results are computed with adding two rows of augmented matrix obtained by boundary conditions to the main augmented matrix. Table 4 shows that the better numerical results of the proposed method are obtained for sufficiently large values n, and increasing iteration points r. The comparisons of the maximum errors obtained by proposed method for iteration $r = 6$ with both the C^1-linearization method [24] and finite-difference method [18] are given in Table 5. As seen from Table 5, the numerical results of the proposed method provide high accuracy for increasing values n and iteration points r than the numerical results of other methods. The absolute errors of the proposed method are also obtained as $|e(0.4)| = 3.5e - 016,$ $|e(0.8)| = 1.1e - 016,$ for $n = 16$ and $r = 5,$ whereas the absolute errors of the shifted Jacobi-Gauss collocation spectral method [14] and the modified decomposition method in combination with the cubic B-spline collocation technique [32] are respectively $|e(0.4)| = 3.9e - 016,$ $|e(0.8)| = 2.2e - 016$ and $|e(0.4)| = 1.8e - 006,$ $|e(0.8)| = 9.2e - 007$ for $n = 20.$ We can say that the numerical results obtained by proposed method are more efficiency, higher accuracy, and have faster convergence than the other methods for smaller values n and chosen iteration point r.

5. Conclusions

The main goal of this paper, to develop an efficient and accurate numerical shame based on a collocation method by using generalized Bernstein polynomials and quasilinearization technique for approximate solution of nonlinear Lane-Emden type equations as singular initial and boundary value problems. Some numerical examples have been given to demonstrate the validity and applicability of the method. One of the advantage for solving the nonlinear Lane-Emden type equations as initial value problems is to delete last two rows of augmented matrix and adding the rows of augmented matrix obtained by initial conditions. Because the numerical results of the proposed method are obtained effectively for smaller values n. The second advantage of the Lane-Emden type equations is to write the augmented matrix as a rectangular matrix with adding technique, since it provides high accuracy for increasing values n and iteration points r. Moreover, comparisons of the numerical results obtained by proposed method with the other methods have been made. They have shown that the present work provides acceptable approach for nonlinear Lane-Emden type equations. Important concerns of the proposed method are easy to compute, rapid convergence and arbitrarily high accuracy by taking the truncation n to be sufficiently large. In view of all this, the presented method could be utilized to solve nonlinear singular initial or boundary value problems in practical application.

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