Closed 2-cell embeddings of graphs with no $V_8$-minors

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Abstract

A closed 2-cell embedding of a graph embedded in some surface is an embedding such that each face is bounded by a cycle in the graph. The strong embedding conjecture says that every 2-connected graph has a closed 2-cell embedding in some surface. In this paper, we prove that any 2-connected graph without $V_8$ (the Möbius 4-ladder) as a minor has a closed 2-cell embedding in some surface. As a corollary, such a graph has a cycle double cover. The proof uses a classification of internally-4-connected graphs with no $V_8$-minor (due to Kelmans and independently Robertson), and the proof depends heavily on such a characterization. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

The closed 2-cell embedding (called strong embedding in [2,4], and circular embedding in [6]) conjecture says that every 2-connected graph $G$ has a closed 2-cell embedding in some surface, that is, an embedding in a compact closed 2-manifold in which each face is simply connected and the boundary of each face is a cycle in the graph (no repeated vertices or edges on the face boundary). The cycle double cover conjecture says that every connected graph without cut edges has a cycle double cover, i.e., a list of cycles in the graph with each edge contained in exactly two of these cycles. Clearly, the existence of a closed 2-cell embedding implies the existence of a cycle double cover for the graph; the face boundaries are the cycles of a double cover.

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It is well known that any spherical embedding of any 2-connected planar graph is a closed 2-cell embedding. Negami [5], and independently Robertson and Vitray [8] showed that every 2-connected projective planar graph has a closed 2-cell embedding either in the sphere or in the projective plane. For cubic graphs, the existence of a cycle double cover is equivalent to the existence of a closed 2-cell embedding. Hence, all the cubic graphs which have cycle double covers also have closed 2-cell embeddings. Alspach and Zhang [1,11] proved that any cubic graph without a Petersen-minor has a cycle double cover, therefore, has a closed 2-cell embedding in some surface. Richter et al. [7] proved that every 3-connected planar graph has a closed 2-cell embedding in some surface other than the sphere; also they characterized those planar graphs which have this property. Zha [9,10] proved that every 2-connected doubly toroidal graph or 5 cross-cap embeddable graph has a closed 2-cell embedding in some surface. Zhang [11] showed that every 2-connected graph without $K_5$-minor has a closed 2-cell embedding in some surface.

The graph $V_8$ is the Möbius ladder on eight vertices, or a graph with a Hamilton cycle of length 8, and each pair of antipodal vertices on this Hamilton cycle is joined by an edge. In this paper, we use a characterization of graphs with no $V_8$-minor (due to Kelmans, and independently Robertson) to prove the following result.

**Theorem 1.1.** Any 2-connected graph containing no $V_8$-minor has a closed 2-cell embedding in some surface.

**Corollary 1.2.** The above graph has a cycle double cover.

Note that a $V_8$-free graph may have $P_{10}$-minor or $K_5$-minor.

2. Some reductions

We make some reductions in this section. Let $G$ be a 2-connected graph (so loops are not allowed). First of all, we may assume $G$ is simple, since, if we have a closed 2-cell embedding of $G$, we can subdivide the faces by the multiple edges and still obtain a closed 2-cell embedding. Next, if $G$ has a nontrivial 2-vertex separation, say $G = G_1 \cup G_2$, $G_1 \cap G_2 = \{x, y\} \subset V(G)$, then we separate $G$ at $x$ and $y$, add a virtual edge $e$ to $G_1$ and $G_2$, respectively, such that $e$ is incident to $x$ and $y$. Suppose $G_1$ and $G_2$ have closed 2-cell embeddings in surfaces $\Sigma_1$ and $\Sigma_2$, respectively. Then for $i = 1, 2$, choose a closed disc $D_i$ in $\Sigma_i$ meeting $G_i$ in $e$ with $x, y$ on the boundary and $e$ in its interior. Cut off $D_i$ from $\Sigma_i$, and then identify two boundaries of $D_1$ and $D_2$ (the connected-sum of $\Sigma_1$ and $\Sigma_2$). The resulting embedding is also a closed 2-cell embedding.

In a 3-connected graph, a 3-vertex separation is trivial if it is formed by three neighbor vertices of a trivalent vertex. A graph $G$ is called internally-4-connected if either $G$
is $K_{3,3}$, or all 3-vertex separations are trivial, and for each 3-vertex separation $A$, $G \setminus A$ consists of only two components. Now, we look at all nontrivial 3-vertex separations. Suppose $G$ has a 3-vertex separation, say $G = G_1 \cup G_2, G_1 \cap G_2 = \{x, y, z\} \subset V(G)$, and $|V(G_1)| \geq 4, |V(G_2)| \geq 4$. We adjoin a triad $T$ to $G_1$ ($G_2$, respectively) by adding a new vertex $u$ joined to $x, y, z$ by three new edges $ux, uy, uz$. If there exist some pendant edges after the separation, we shall treat each of the pendant edges and the adjacent new edge as a subdivided edge (topologically equivalent to a single edge, for embeddings). If we have closed 2-cell embeddings for both $G_1 \cup T$ and $G_2 \cup T$ in surfaces $\Sigma_1$ and $\Sigma_2$, respectively, we can obtain a closed 2-cell embedding of $G$ by removing the interior of a circuit $C_i$ in $\Sigma_i$ meeting $G_i \cup T$ in $x, y, z$ and bounding a closed disc containing only $T$ and taking the connected sum of $\Sigma_1$ and $\Sigma_2$ with $x, y, z$ identified. When forming the connected sum, we must observe whether the respective orders of $x, y$ and $z$ on the boundaries of disks in $\Sigma_1$ and $\Sigma_2$ match each other or not. If they do not match, we can rectify this discrepancy by taking the mirror embedding in $\Sigma_2$. Therefore, we may always assume that $G$ is (topologically) internally-4-connected.

We now consider graphs without $V_8$ as a minor. In order to find closed 2-cell embeddings of such graphs, we need to apply the following structure theorem by Kelmans [3] and Robertson [7].

**Theorem 2.1.** Let $G$ be an internally-4-connected graph with no $V_8$-minor. Then one of the following holds:

1. $G$ is a planar graph;
2. $G$ has two adjacent vertices $x$ and $y$ such that the subgraph induced by $V(G) \setminus \{x, y\}$ is a cycle;
3. $G$ has a set $B$ of four vertices such that the subgraph induced by $V(G) \setminus B$ is edgeless;
4. $G = L(K_{3,3})$, the line graph of $K_{3,3}$;
5. $|V(G)| \leq 7$.

Since the graphs in (1) are planar, the graphs in (2) are projective planar and the graphs in (4) and (5) are toroidal, it is known they all have closed 2-cell embeddings [5,8,9]. The only remaining case is the graphs in (3). Thus, we can assume $G$ is an internally 4-connected graph with a set $B$ of four vertices so that every edge is incident with a vertex in $B$. Call these four vertices base vertices.

Let $G$ be such a graph with set $B$ of base vertices, where $B = \{x_1, x_2, x_3, x_4\}$. For $i = 1, 2, 3, 4$, let $V(i) = \{x \in V(G) \setminus B | x$ is not adjacent to $x_i\}$. Since $G$ is internally-4-connected, $|V(i)| \leq 1$ for $i = 1, 2, 3, 4$, i.e., for any three base vertices, there is at most one non-base vertex that is only adjacent to these three base vertices. Since there may be some edges connecting these base vertices, we may finally assume that $G = K_{4,n} \cup E \setminus M_j$, where $K_{4,n}$ is a complete bipartite graph, $E$ consists of some of edges connecting base vertices, and $M_j$ is a $j$-matching from the four base vertices to the other $n$ non-base vertices, for $j = 1, 2, 3, 4$. 
3. The proof of Theorem 1.1

In this section we construct closed 2-cell embeddings for \( K_{4,n} \cup E \setminus M_j, n \geq 3, j = 1, 2, 3, 4 \), where \( E \) and \( M_j \) are as in the previous section.

Let \( \Psi \) be a closed 2-cell embedding of a 2-connected graph \( G \). Suppose \( u \in V(G) \), \( \deg(u) = 4 \), and \( u x_i, i = 1, 2, 3, 4 \) are four edges incident with \( u \). We say \( u \) has a pseudo 4-wheel neighborhood with four base vertices \( \{x_1, x_2, x_3, x_4\} \) and two multiple vertices \( v, w \), if the local embedding of \( u \) and four incident faces is homeomorphic to the embedding shown in Fig. 1(a), namely, the four faces incident with \( u \) form almost a 4-wheel except two repeated vertices \( v \) and \( w \) on the rim. Denote this pseudo-4-wheel as \( W_u(B; v, w) \), where \( B = \{x_1, x_2, x_3, x_4\} \) is the set of four base vertices.

Suppose \( \Psi \) is a closed 2-cell embedding, \( u \) is a vertex in the graph, and \( u \) has a pseudo-4-wheel \( W_u(B; v_0, v_k) \), for some \( k > 0 \). We define the following operation \( \Psi_u(v_0, v_k+1) \): Remove the vertex \( u \) and four edges \( u x_i, i = 1, 2, 3, 4 \); cut a small disk inside this pseudo-4-wheel centered at \( u \), and replace it by a cross-cap (identifying the antipodal points on the boundary of the removed disk); redraw the vertex \( u \) and a new vertex \( v_{k+1} \) near the cross-cap and draw eight new edges \( u x_i, v_{k+1} x_i, i = 1, 2, 3, 4 \), to obtain a local embedding of a new graph, as shown in Fig. 1(b). The embedding \( \Psi_u(v_0, v_k+1) \) is a closed 2-cell embedding such that \( u \) and \( v_{k+1} \) have pseudo-4-wheel neighborhoods \( W_u(B; v_0, v_{k+1}) \), and \( W_{u (v_{k+1})} (B; u, v_k) \), respectively. If the clockwise vertex-rotation around \( u \) in \( \Psi \) is \( x_1, x_2, x_3, x_4 \), then the clockwise vertex-rotations around \( u \) and \( v_{k+1} \) in \( \Psi_u(v_0, v_{k+1}) \) are \( x_1, x_2, x_3, x_4 \), and \( x_1, x_4, x_2, x_3 \), respectively. If the clockwise vertex-rotation around \( u \) in \( \Psi \) is \( x_1, x_2, x_3, x_4 \), then the clockwise vertex-rotations around \( u \) and \( v_{k+1} \) in \( \Psi_u(v_0, v_{k+1}) \) are \( x_1, x_4, x_2, x_3 \), and \( x_1, x_2, x_3, x_4 \), respectively.

Let \( \Phi_1 \) be the embedding of \( K_{4,3} \), as shown in Fig. 2. Define \( \Phi_2 = \Phi_1 \oplus \Psi_u(v_0, v_2) \) by performing the operation \( \Psi_u(v_0, v_2) \) on \( \Phi_1 \) at the vertex \( u \). Inductively, let \( \Phi_{i+1} = \Phi_i \oplus \Psi_u(v_0, v_{i+1}) \), by performing the operation \( \Psi_u(v_0, v_{i+1}) \) on \( \Phi_i \) at the vertex \( u \), for
Fig. 2.

$t \geq 2$. Then $\Phi_t$ is a quadrilateral embedding, and therefore a closed 2-cell embedding of $K_{4,t+2}$, $t \geq 1$. The clockwise vertex-rotation around $u$ in $\Phi_1$ is $x_1,x_4,x_2,x_3$. Therefore, by observation in the previous paragraph, the clockwise vertex-rotation around $u$ in $\Phi_t$ is $x_1,x_4,x_2,x_3$ if $t$ is odd, and is $x_1,x_2,x_4,x_3$ if $t$ is even. The clockwise vertex-rotation around $v_i$ in $\Phi_t$ is $x_1,x_2,x_3,x_4$ if $t$ is odd, and is $x_1,x_4,x_3,x_2$ if $t$ is even.

Claim 3.1. $\Phi_t \cup E$ has a closed 2-cell embedding, for $t \geq 1$, where $E$ consists of some of edges connecting base vertices $x_i, i = 1, \ldots, 4$.

Since $\Phi_t$ is a closed 2-cell embedding, if we can use edges in $E$ to subdivide faces of $\Phi_t$, then the resulting embedding is a closed 2-cell embedding of $\Phi_t \cup E$. The edge set $E$ consists of at most six edges. Therefore, we may assume $E$ consists of all possible six edges on base vertices $x_1,x_2,x_3,x_4$ (if $E$ consists of less edges, then just use these edges to subdivide part of those six faces). If $t$ is odd, then place edges $x_1x_3,x_1x_4,x_2x_3,x_2x_4$ in faces incident with $u$, and place edges $x_1x_2,x_3x_4$ in faces incident with $v_i$. If $t$ is even, then place edges $x_1x_2,x_1x_3,x_2x_4,x_3x_4$ in faces incident with $u$, and place edges $x_1x_4,x_2x_3$ in faces incident with $v_i$. The resulting embedding is a closed 2-cell embedding. Thus Claim 3.1 is true.

Claim 3.2. $\Phi_t \backslash M_j$, $j = 1,2,3,4$ has a closed 2-cell embedding, for $t \geq 3$, where $M_j$ is a matching of size $j$ from the four base vertices $x_1,x_2,x_3,x_4$ to those non-base vertices.

Without loss of generality, we may assume $j = 4$. For $t \geq 3$, let $M_4 = \{x_1v_1,x_2v_1,x_3v_2,x_4u\}$ for every even $t$, and $M_4 = \{x_1v_1,x_2v_2,x_3v_3,x_4u\}$ for every odd $t$. First, we assume $t$ is even and $t \geq 4$. After deletion of two edges $x_2v_i$ and $x_4u$, the two new faces in $\Phi_t \backslash \{x_2v_i,x_4u\}$ are bounded by two cycles $x_2v_i,x_3v_1u$ and $x_3v_4v_3v_2u$, respectively. Therefore, $\Phi_t \backslash \{x_2v_i,x_4u\}$ is a closed 2-cell embedding. Notice that, by the construction of the operation $\Phi(\psi_4(t_0,t_1))$, the boundary of the pseudo-4-wheel $W_u(B; v_0,v_1)$ in the previous embedding becomes the boundary of the union of two new pseudo-4-wheels $W_u(B; v_0,v_i)$ and $W_u(B; v_0,v_1)$. Therefore, these two pseudo-4-wheels $W_u(B; v_0,v_i)$
and $W_t(B; u, v_{t-1})$ are edge disjoint from the pseudo-4-wheels $W_{e_1}(B; v_0, v_2)$ and $W_{e_2}(B; v_1, v_3)$ (see Fig. 3). Hence, deleting edges $x_1v_1$ and $x_3v_2$ does not create non-closed 2-cell faces. Thus, we obtain a closed 2-cell embedding of $\Phi_t \setminus M_4$ for every even $t$, $t \geq 4$.

Now, we assume $t$ is odd, and $t \geq 3$. After deletion of two edges $x_3v_t$ and $x_4u$, the two new faces in $\Psi_t \setminus \{x_3v_t, x_4u\}$ are bounded by two cycles $ux_3v_{t-1}x_4v_tx_2u$ and $ux_1v_3v_0x_2u$, respectively. Therefore, $\Psi_t \setminus \{x_3v_t, x_4u\}$ is a closed 2-cell embedding. Again since the two pseudo-4-wheels $W_u(B; v_0, v_t)$ and $W_t(B; u, v_{t-1})$ are edge disjoint from the pseudo-4-wheels $W_{e_1}(B; v_0, v_2)$ and $W_{e_2}(B; v_1, v_3)$, we may delete edges $x_1v_1$ and $x_2v_3$, and the resulting embedding is still a closed 2-cell embedding. Thus Claim 3.2 is true.

By Claims 3.1 and 3.2, we obtain a closed 2-cell embedding for $K_{4,n} \cup E \setminus M_j$, for $n \geq 5$ and $j = 1, 2, 3, 4$ (the orders of first adding edges in $E$ to $\Phi_t$ and then deleting edges in $M_j$ from $\Phi_t \cup E$ or vice versa does not affect the resulting embedding being a closed 2-cell embedding). Notice that $K_{4,n} \cup E \setminus M_j, n = 3, 4$ has at most eight vertices, and therefore is a doubly toroidal graph (the orientable genus of the complete graph on eight vertices is 2). Hence $K_{4,n} \cup E \setminus M_j, n = 3, 4, j = 1, 2, 3, 4$, has a closed 2-cell embedding in some surface [9]. This completes the proof of Theorem 1.1. $\Box$

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