Gaussian multipartite bound information

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We show the existence of Gaussian multipartite bound information which is a classical analog of Gaussian multipartite bound entanglement. We construct a tripartite Gaussian distribution from which no secret key can be distilled between any two parties yet it cannot be created by local operations and public communication. This demonstrates a close link between entanglement and secrecy in the domain of random Gaussian continuous variables and underlines the ability of quantum mechanics to develop concepts of classical information theory.

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Quantum entanglement is a fundamental concept of quantum information theory. There are two types of entanglement: free entanglement and bound entanglement [1]. Free entanglement can be distilled [2] by local operations and classical communication into maximal entanglement, and so is a perfect resource for quantum communication. Bound entanglement, on the other hand, is nondistillable and hence useless for standard quantum communication. Nevertheless, it can be activated [3] and renders a secret key [4].

Remarkably, quantum entanglement allows for the solution of classical problems which stems from its close relationship with secret classical correlations [5]. These correlations are related to the scenario in which two honest parties, Alice and Bob, and an adversary Eve, share independent realizations of three random variables $A,B$ and $E$, characterized by the probability distribution $P(A,B,E)$. The distribution contains secret correlations if it cannot be distributed by local operations and public communication (LOPC). As with entanglement, Alice and Bob can sometimes distill secret correlations by LOPC to a secret key, i.e., a common string of random bits about which Eve has practically no information. The distillable secret correlations resemble free entanglement. It is thus natural to ask whether there are also nondistillable secret correlations, referred to as bound information, that act as a classical analogue to bound entanglement [3]. So far, bipartite distributions have been found containing bound information only in the asymptotic limit [7], but surprisingly, an example of tripartite bound information was derived in [8], demonstrating the ability of quantum mechanics to solve an open classical problem.

The example of a distribution carrying the tripartite bound information of Ref. [8] was constructed in the context of discrete random variables. It was obtained as a distribution of outcomes of a measurement performed on a purification of a specific bound entangled state of three two-level systems (qubits). The key step, a proof of the presence of bound information, crucially relied on the utilization of the concept of intrinsic information [9].

In this paper we analyze the concept of bound information within the framework of random Gaussian continuous variables. Specifically, we construct an example of tripartite Gaussian bound information, that is, a Gaussian distribution $P(A,B,C,E)$ shared by three honest parties Alice, Bob and Clare, and an adversary Eve, from which any pair of honest parties cannot distill a secret key, even if they can collaborate with the third party, and yet the distribution cannot be distributed by LOPC [10]. By using powerful Gaussian separability criteria [11, 12] we explicitly decompose the distribution into LOPC with respect to relevant bipartitions which enables us to prove the presence of Gaussian bound information without using the concept of intrinsic information. This also allows to get a deeper insight into the structure of the correlations underlying the phenomenon of Gaussian multipartite bound information.

We consider Gaussian states of quantum systems with infinite-dimensional Hilbert state spaces such as modes of an electromagnetic field. $N$ modes are described by $N$ pairs of canonically conjugate position and momentum quadrature operators, labeled by $\hat{x}_j$ and $\hat{p}_j$, $j = 1,2,\ldots,N$, respectively, satisfying the canonical commutation rules $[\hat{x}_j,\hat{p}_k] = i\delta_{jk}$. $N$-mode quantum states can be described by the Wigner function of $2N$ real variables and Gaussian states are states with a Gaussian Wigner function. Any $N$-mode Gaussian state $\hat{\rho}$ is fully characterized by a real $2N$-dimensional vector $\xi = \text{Tr}(\hat{\rho}\xi)$ of phase-space displacements and by the $2N \times 2N$ real symmetric covariance matrix (CM) $\gamma$ with elements $\gamma_{jk} = 2\text{Re}\text{Tr}[\hat{\rho}(\hat{\xi}_j - \bar{\xi}_j)(\hat{\xi}_k - \bar{\xi}_k)]$, $j,k = 1,\ldots,2N$, where $\hat{\xi} = (\hat{x}_1,\ldots,\hat{x}_N,\hat{p}_1,\ldots,\hat{p}_N)^T$. A Gaussian state can be mapped onto a Gaussian distribution by a Gaussian measurement. The distribution can contain distillable secret correlations depending on the performed measurements and on the entanglement properties of the underlying state. Gaussian entanglement can be verified, e.g., by separability criterion [11]. A Gaussian distribution $P(A,B,E)$ can be distilled to a secret key using the method [13] if it holds that [14]

$$\max(\Delta I_{DR},\Delta I_{RR}) > 0,$$

where $\Delta I_{DR} = I_{AB} - I_{AE}$ and $\Delta I_{RR} = I_{AB} - I_{BE}$ are
differences of mutual information between Alice and Bob and $I_{AE}$ ($I_{BE}$) between Alice (Bob) and Eve.

The bound entangled state of three qubits used in to construct bound information is separable with respect to two of three possible bipartite splittings of the systems into two groups which is also a necessary prerequisite for entanglement distribution by separable states. Here we report on the construction of Gaussian bound information using the continuous-variable (CV) analog of the latter phenomenon and the mapping of it onto a classical protocol being a CV analog of the discrete protocol for secrecy distribution by non-secret correlations. We show that a Gaussian distribution in step 2 of the CV classical protocol being a CV analog of the discrete protocol involves two of Eve’s modes $E_1, E_2$ encoding information on variables $v$ and $u$. It reads as

$$|\psi\rangle = \int \sqrt{P(u,v)} \frac{\exp[-(u^2 + v^2)/(4x)]}{(4\pi x)} \, dudv,$$

where $|\alpha \pm r\rangle = \exp(\alpha d^\dagger - \alpha^* d_0) |\pm r\rangle$ is the displaced squeezed state and $|\psi^{(x)}\rangle$ and $|\psi^{(y)}\rangle$ are the eigenvectors of position and momentum quadratures, respectively. The state (7) is a proper quantum state normalized to unity that satisfies $\rho_1 = \text{Tr}_{E_1}(|\psi\rangle\langle\psi|)$. Note also, that the state is designed such that it does not contain correlations between position and momentum quadratures.

The purification (7) is fully characterized by its CM, that we get in the form $X_1 = X_1 \oplus (X_1)^{-1}$ where

$$X_1 = \begin{pmatrix}
  e^{2x} & 0 & 0 & 0 \\
  0 & 1 + 4x & 2\sqrt{2}x & 4x & 1 - 2x \\
  0 & 2\sqrt{2}x & e^{-2x} + 2x & 2\sqrt{2}x & -\sqrt{2}x \\
  0 & 4x & 2\sqrt{2}x & 4x & 2x \\
  y & -1 & -2x & -2x & y
\end{pmatrix},$$

with $y = x + x + \frac{1}{4x} + \frac{2x^2}{x^2}$ and $\det X_1 = 1$.

Now we want to associate with the state (7) a Gaussian distribution $\Pi_1$ that contains no secret correlations across any bipartition of honest parties. The distribution can be obtained by measuring position quadratures on all five modes of the purification (7). This gives

$$\Pi_1(\eta) = (\pi - \frac{x^2}{2} e^{-\eta^T(X_1)^{-1}}\eta,$$

where $\eta = (x_A, x_B, x_C, x_{E_1}, x_{E_2})^T$ is the vector of measurement outcomes and the classical covariance matrix (CCM) $X_1$ is given in Eq. (8).

The distribution (9) does not contain secret correlations across any of the splittings $A-\{BC\}$, $B-\{AC\}$ and $C-\{AB\}$. To show this, consider the reduced distribution $\pi_1(\tau) = \exp[-\tau^T(C_1)^{-1}\tau] / (\pi^T\sqrt{\det C_1})$ obtained from distribution (9) by throwing away Eve’s variable $x_{E_2}$. Here $\tau = (x_A, x_B, x_C, x_{E_1})^T$ and CCM $C_1$ is obtained from CCM (8) by dropping the last row and column.
The reduced distribution can be prepared by LOPC as follows. First, say, Alice privately draws two uncorrelated random variables $x_A$ and $x_{E_1}$ from Gaussian distributions with variances $2(x_A^2) = e^{2r}$ and $2(x_{E_1}^2) = 4x$ and sends the variable $x_{E_1}$ through a public channel to Bob and Clare. They then privately draw random variables $x_B$ and $x_C$ from Gaussian distributions with variances $2(x_B^2) = 1$ and $2(x_C^2) = e^{-2r}$ and all the participants perform displacements like in Eq. (9), where the quadrature operators are replaced by the corresponding classical variables and the displacement $v$ is replaced by the variable $x_{E_1}$. Calculating the CCM with elements $\{\{\tau'_i, \tau'_j\}\}$, where $\tau' = (x_A, x'_B, x'_C, x_{E_1})^T$ by expressing the primed variables using right-hand sides (RHS) of Eq. (8) and averaging over the distribution with CCM $e^{2r} I \otimes e^{-2r} \otimes 4x$ we get CCM $C_1$. That is, the distribution $\pi_1$ was created by LOPC. Consequently, since there are no secret correlations across any bipartition, even if Eve throws away a part of her information, the original distribution (9) also cannot possess them, which is what we set out to prove.

An interesting question arises as to whether Eve’s variable $x_{E_1}$ can also be obtained as a result of LOPC. Remarkably, our analysis indicates that this is not the case. Indeed, the only way we found for the honest parties to prepare it is to compose it as
\[
\langle \hat{a} \rangle \langle \hat{b} \rangle = \langle \hat{a} \rangle \langle \hat{b} \rangle = \sum_{i,j} \frac{\pi_i}{\sqrt{\pi_i}} \sum_{i,j} \frac{\pi_j}{\sqrt{\pi_j}} \sqrt{\pi_i \pi_j} \pi_{ij}
\]
where $\pi_1 = \exp[-\varepsilon^2(C_2)^{-1} \varepsilon / (\pi^2 \sqrt{\det(C_2)})]$ obtained from distribution (10) by discarding Eve’s variable $x_{E_1}$. Here $\varepsilon = (x_A, x_B, x_C, x_{E_1})^T$ and CCM $C_2$ is obtained from the CCM (11) by dropping the fourth row and column. Such a distribution can be prepared by LOPC with respect to $C - (AB)$ splitting. To prove this, consider the reduced distribution $\pi_2(\varepsilon) = \sum_{i,j} \frac{\pi_i}{\sqrt{\pi_i}} \sum_{i,j} \frac{\pi_j}{\sqrt{\pi_j}} \sqrt{\pi_i \pi_j} \pi_{ij}$ splitting follows. First, say, Alice privately draws two uncorrelated variables $x_A$ and $x_B$ from a Gaussian distribution with CCM (12). They also draw a third variable $x_{E_1}$ from a Gaussian distribution with variance $2(x_{E_1}^2) = y$ and send it to Clare through a public channel. She privately generates a random variable $x_{E_1}$ obeying a Gaussian distribution with variance $2(x_{E_1}^2) = 1$ and all participants perform displacements (13), where the quadrature operators are replaced by the corresponding classical variables and the displacement $v$ is replaced by $x_{E_1}$. If we now calculate the CCM with elements $\{\{\tau'_i, \tau'_j\}\}$, $i, j = 1, 2, \ldots, 4$, where $\epsilon' = (x_A, x'_B, x'_C, x_{E_1})^T$ by expressing the primed variables using the RHS of Eq. (13) and averaging over

\[
\Pi_2(\eta) = (\pi)^{-\frac{1}{2}} e^{-\eta^T \epsilon \eta}, (10)
\]
the Gaussian distribution with CCM $X_{AB} \oplus 1 \oplus \bar{y}$ we get the CCM $C_2$ of the distribution $\pi_2$. The distribution was created by LOPC and therefore has no secret correlations across $C - (AB)$ splitting. Consequently, as Clare does not share secret correlations with Alice and Bob, even if Eve discards the part of her information contained in the variable $x_{E_1}$, there could not be secret correlations across the $C - (AB)$ splitting even in the original distribution $\Pi_1$. Similar to the case of distribution $\Pi_0$, the remaining variable $x_{E_1}$ cannot be created by LOPC with respect to $C - (AB)$ splitting. Namely, it can be composed as

$$x_{E_1} = (e^{2r} - 1) \left[ \chi_{E_1} - \frac{x_C}{e^{2r}} - \frac{x_{E_2}}{y} \right] + \frac{x_A}{2y} + \frac{e^{2r} x_B}{y},$$

where the random variable $\chi_{E_1}$ is uncorrelated with the remaining variables and obeys a Gaussian distribution with variance $2(e^{2r} - 1)$. As with the variable $x_{E_1}$ in the first step of our protocol, the variable $x_{E_1}$ contains variables belonging to both parts of the $C - (AB)$ splitting and thus private communication across the splitting is needed to reveal the variable to Eve.

We have shown that distribution $\Pi_0$ does not contain secret correlations across the $B - (AC)$ and $C - (AB)$ splitting. This prevents any two parties from establishing a secret key even in the case when Alice is allowed to perform joint operations with either Bob or Clare. The distribution, however, cannot be created by LOPC as it contains secret correlations across the $A - (BC)$ splitting. The correlations are not detected by the criterion (1) because its left-hand side is negative (see Fig. 1) but can be activated by allowing Bob and Clare to perform the following joint operation: $x_{B,C} \rightarrow (x_B \pm x_C)/\sqrt{2}$. Thus the CCM (11) transforms to CCM $X_2$ and the distribution $\Pi_1$ attains the form $\Pi_2(\eta) = (\pi)^{-1/2} e^{-\eta^2(X_2)^2}$. The presence of secret correlations can be seen from a part of the CCM $X_2$ corresponding to Alice’s, Bob’s and Eve’s variables that reads

$$\hat{X} = \begin{pmatrix}
\hat{a} & \frac{2\hat{x} + 1}{\sqrt{2}} & 2\hat{x} & \hat{x} \\
\frac{2\hat{x} + 1}{\sqrt{2}} & \hat{x} & \sqrt{2}\hat{x} & \hat{x} \\
2\hat{x} & \sqrt{2}\hat{x} & \frac{2r}{2\hat{x}} & \hat{x} \\
\hat{x} & \hat{x} & \frac{2r}{2\hat{x}} & \frac{2r + 1}{\sqrt{2}} \\
\end{pmatrix}.$$

(14)

From the CCM we can calculate the information differences $\Delta I_{DR}$ and $\Delta I_{RR}$ arising in condition (11). They can be expressed as [13]

$$\Delta I_{DR} = \frac{1}{2} \log_2 \left( \frac{\det \hat{X}_{AE} \det \hat{X}_{AB}}{\det \hat{X}_{AE} \det \hat{X}_{AB}} \right),$$

(15)

$$\Delta I_{RR} = \frac{1}{2} \log_2 \left( \frac{\det \hat{X}_{RE} \det \hat{X}_{AB}}{\det \hat{X}_{RE} \det \hat{X}_{AB}} \right),$$

(16)

where $\det \hat{X}_{AE} \det \hat{X}_{AB} = (\hat{X}_{13} \hat{X}_{44} - \hat{X}_{34}^2)(\hat{X}_{11} \hat{X}_{22} - \hat{X}_{12}^2)$, and the CCMs $\hat{X}_{AE}$ ($\hat{X}_{RE}$) are obtained from the CCM (11) by dropping its second (first) row and column. The information differences are depicted in Fig. 1. The figure reveals, that $\Delta I_{RR} > 0$ for sufficiently large squeezing $r$ ($r > r_{\Delta I_{RR}} = 0.1656$ as can be calculated numerically). Therefore Alice and Bob can distill a secret key using the reverse reconciliation protocol [22]. This would be impossible without the presence of secret correlations across the $A - (BC)$ splitting in distribution $\Pi_0$ which accomplishes the construction of Gaussian bound information.

In conclusion, we derived a tripartite Gaussian distribution having secret correlations across only one bipartition which is an example of Gaussian bound information. Its secrecy properties were proved using the tools of Gaussian entanglement theory capable of uncovering the structure of secret correlations which confirms Gaussian CVs to be a relevant platform for their investigation. The present result also demonstrates an intimate link between entanglement and secrecy for Gaussian CVs and gives further evidence of the ability of quantum mechanics to solve classical problems.

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[10] A distribution $P(A, B, C, E)$ cannot be distributed by LOPC if many realizations of the random variables $A, B$ and $C$ obeying marginal probability distribution $P(A, B, C)$ cannot be distributed among Alice, Bob and Clare if their public communication is constrained to contain at most information of the variable $E$ [5].