Positive Solutions of Fully Fuzzy Linear Systems

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Abstract

System of linear equations is a problem that may be solved for solving many problems in various areas of applied sciences. Fuzzy methods constitute an important mathematical and computational tool for modeling real-world systems with uncertainties of parameters. The paper discusses fully fuzzy linear systems (shown as FFLS) using embedding approach to find its positive fuzzy number solutions. We investigate an \( n \times n \) (FFLS) and replace the original \( n \times n \) (FFLS) by a \( 2n \times 2n \) parametric linear system and finally, numerical examples are used to illustrate this approach.


1. Introduction

Systems of linear equations are used to solve many problems in various areas such as structural mechanics applications in civil and mechanical structures, heat transport, fluid flow, electromagnetism,... In many applications, at least one of the system's parameters and measurements are vague or imprecise and we can present them with fuzzy numbers rather than crisp numbers. Hence, it is important to develop mathematical models and numerical procedure that would appropriately treat general fuzzy systems and solve them.

The system of linear equations \( AX = b \) where the elements, \( a_{ij} \), of the matrix \( A \) are crisp numbers and the elements, \( \tilde{b}_i \), of \( b \) are fuzzy numbers, is called fuzzy system of linear equation (FSLE). The \( n \times n \) (FSLE) has been studied by many authors [3, 4, 5, 6, 7, 8, 16, 17, 19]. Friedman et al. [16] proposed a general model for solving such fuzzy linear systems by using the embedding approach. Following [16], Allahviranloo et al. [3, 4, 5, 6, 7, 8] and other authors have designed some numerical methods for calculating the solutions of FSLE.

The system of linear equations \( AX = \tilde{b} \) where the elements, \( \tilde{a}_{ij} \), of the matrix \( A \)
and the elements, $\tilde{b}_i$, of the vector $\mathbf{b}$ are fuzzy numbers, is called Fully Fuzzy Linear System $\text{FFLS}$.

The $n \times n$ $\text{FFLS}$ has been studied by some authors [1, 2, 9, 10, 11, 12, 13, 14, 20, 21, 23, 24, 25, 26, 27]. F. Abramovich et al. [1] used approximate formulae of Dubois and Prade [15, 28] to solve a fuzzy linear system of equations. Buckley and Qu in their continuous works [9, 10, 11] suggested different solutions for $\text{FFLS}$. Also, they found relation between these solutions. Based on their works, Muzzioli and Reynaerts in [20, 21] studied $\text{FFLS}$ of the form $A_1X + b_1 = A_2X + b_2$. They clarified the link between interval linear systems and fuzzy linear systems. M. Dehghan et al. [12, 13, 14] studied finite methods for approximately solving $\text{FFLS}$. They represented fuzzy numbers in L.R. form which are defined and used by Dubois and Prade [15, 28] and they have applied approximately operators between fuzzy numbers in this form and found approximated triangular positive fuzzy number solutions of $\text{FFLS}$. Hence procedures for calculating the solutions of $\text{FFLS}$ transformed to calculating the solutions of three crisp systems.

Vroman et al. in their continuous works [25, 26, 27] suggested two methods to solve $\text{FFLS}$. In [25] they propose a method to solve approximately $\text{FFLS}$ and then they prove that their solution is better than Buckley and Qu’s approximated solution vector $X_0$. Furthermore, in [26, 27] they proposed an algorithm and improved it to solve $\text{FFLS}$ by parametric functions.

Allahviranloo et al. [2] suggests a method for solving $\text{FFLS}$ when coefficient matrix is positive and find its non zero fuzzy number solutions.

In this paper, we are going to find positive solutions of $\text{FFLS}$ by using an analytical method. We investigate an $n \times n$ $\text{FFLS}$ and replace the original $n \times n$ $\text{FFLS}$ by a $2n \times 2n$ parametric linear system.

The structure of this paper is organized as follows:

In section 2, some basic definitions and results on interval arithmetic and fuzzy numbers and $\text{FFLS}$ are discussed. In section 3, our procedure for finding positive solutions of $\text{FFLS}$ is introduced and the proposed algorithm is illustrated by solving some numerical examples. Conclusions are drawn in section 4.

2 Preliminaries

2.1 Systems of Linear interval equations

An interval is defined as a subset of the set of real numbers $\mathbb{R}$ such that

$$\hat{x} = [\underline{x}, \overline{x}] = \{x \in \mathbb{R} | \underline{x} \leq x \leq \overline{x}\}$$

where $\underline{x} \leq \overline{x}$ and $\underline{x} = \inf x$, $\overline{x} = \sup x$ are endpoints of the interval $\hat{x}$. If $\underline{x} > 0$ an interval $\hat{x}$ is called positive interval. The set of all (real) intervals is denoted by $\mathbb{IR}$ and called a (real) interval space.

For many operations, including a standard arithmetic operations of addition, subtraction, multiplication and division, the resulting set is also an interval that can be conveniently defined in terms of endpoint of the argument intervals:
\[ \hat{x} + \hat{y} = [x + y, x + y], \]
\[ -\hat{x} = [-x, -x], \]
\[ \hat{x} - \hat{y} = [x - y, x - y], \]
\[ \hat{x}, \hat{y} = [\min\{x, y, \bar{x}, \bar{y}, x, y\}, \max\{x, y, \bar{x}, \bar{y}, x, y\}] \quad (1) \]

For any bounded set of real numbers \( S \) we can define a smallest interval enclosure of the set, called also(interval) hull of the set:

\[ \text{hull } S = [\inf S, \sup S]. \quad (2) \]

Thus, when an application of some operations or functions produces a set which is not an interval, the hull of the set can be taken if there is a need to stay within interval arithmetic all the time (which is usually the case).

Most operations on intervals can be extended to interval matrices, by applying them componentwise to all matrix elements. An interval matrix \( \hat{A} \in \mathbb{IR}^{m \times n} \) can be also considered as a set of real matrices, or as a matrix interval:

\[ \hat{A} = \{ A \in \mathbb{IR}^{m \times n} | A_{ij} \leq \hat{A}_{ij} \leq \bar{A}_{ij} \} = [\hat{A}, \bar{A}], \quad \hat{A}, \bar{A} \in \mathbb{IR}^{m \times n}. \quad (3) \]

Matrix multiplication is defined as for real matrices, with the hull operation used as the final step. The boundary (or the vertex set) of an interval matrix is a set of real matrices consisting of \( 2^t \) elements, where \( t \) is the number of interval coefficients of \( A \), and defined as:

\[ \text{vert } \hat{A} = \{ A \in \hat{A} | a_{ij} \in \text{vert } \hat{A}_{ij} \} = \{ A \in \hat{A}, a_{ij} \in \{ a_{ij}, \bar{a}_{ij} \} \}. \quad (4) \]

Of course, for an interval \( x \), \( \text{vert } x = \{ \bar{x}, \bar{x} \} \).

Consider a linear interval system of equations with an interval coefficient matrix \( \hat{A} \in \mathbb{IR}^{m \times n} \) and an interval right-hand side vector \( \hat{b} \in \mathbb{IR}^n \), \( \hat{A} \hat{X} = \hat{b} \).

There are different concepts of a solution to this system such as: Algebraic Solution (AS), United Solution Set (USS)

\[ \sum (\hat{A}, \hat{b}) = \sum (\hat{A}, \hat{b}) = \{ X \in \mathbb{IR}^n | (\exists A \in \hat{A}, \exists b \in \hat{b}) AX = b \} = \{ X \in \mathbb{IR}^n | AX \bigcap b \neq \emptyset \}, \quad (5) \]

Tolerable Solution Set (TSS)

\[ \sum (\hat{A}, \hat{b}) = \sum (\hat{A}, \hat{b}) = \{ X \in \mathbb{IR}^n | (\forall A \in \hat{A}, \exists b \in \hat{b}) AX = b \} = \{ X \in \mathbb{IR}^n | AX \subseteq b \}, \quad (6) \]

Controllable Solution Set (CSS)

\[ \sum (\hat{A}, \hat{b}) = \sum (\hat{A}, \hat{b}) = \{ X \in \mathbb{IR}^n | (\forall b \in \hat{b}, \exists A \in \hat{A}) AX = b \} = \{ X \in \mathbb{IR}^n | AX \supseteq b \}, \quad (7) \]

and Vertex Solution Set (VSS)
The VSS is a discrete and finite set of points, with number of elements equal to $2^t \leq 2^{n^2+n}$, where $t = t_A + t_B$ is the number of intervals in matrix $\hat{A}$ and $\hat{b}$ of the system.

Calculating (and representing) the exact solution set $\sum(\hat{A}, \hat{b})$ may be quite hard and impractical, especially for larger $n$. Therefore, for many practical purposes we are satisfied with various approximations to this set. The natural approximation is the interval enclosure of the set. The smallest enclosure is the hull of the set:

$$hull(\sum(\hat{A}, \hat{b})) = \inf \sum(\hat{A}, \hat{b}), \sup \sum(\hat{A}, \hat{b}) = hull(\sum(\vert \hat{A}, \vert \hat{b}))$$

Hence, when we want to solve interval linear system, we try to find its approximated USS i.e. $hull(\sum(\vert \hat{A}, \vert \hat{b}))$.

### 2.2 Fuzzy Background

The set of all fuzzy numbers is denoted by $\mathcal{E}$ and defined as follows:

**Definition 1** A fuzzy number $\tilde{u}$ is a pair $(u(r), \tilde{u}(r))$ of functions $u(r), \tilde{u}(r); 0 \leq r \leq 1$ which satisfy the following requirements:

- $u(r)$ is a bounded monotonic increasing left continuous function;
- $\tilde{u}(r)$ is a bounded monotonic decreasing left continuous function;
- $u(r) \leq \tilde{u}(r), 0 \leq r \leq 1$.

A crisp number $k$ is simply represented by $\tilde{k}(r) = k(r), 0 \leq r \leq 1$ and called singleton.

A fuzzy number $\tilde{a}$ can be represented by its $\lambda$-cuts $(0 < \lambda \leq 1)$:

$$\tilde{a}^\lambda = \{x \mid x \in \mathbb{R}, \tilde{a}(x) \geq \lambda\}.$$

And

$$supp \tilde{a} = \tilde{a}^0 = Cl(\{x \mid x \in \mathbb{R}, \tilde{a}(x) > 0\}) = [a(0), \bar{a}(0)].$$

Note that the $\lambda$- cuts of a fuzzy number are closed and bounded intervals. The fuzzy arithmetic based on the Zadeh extension principle can also be calculated by interval arithmetic applied to the $\lambda$-cuts.

For fuzzy number $\tilde{u} = (u(r), \tilde{u}(r)), 0 \leq r \leq 1$, we will write (1)$\tilde{u} > 0$ if $u(0) > 0,(2)\tilde{u} \geq 0$ if $u(0) \geq 0,(3)\tilde{u} < 0$ if $\tilde{u}(0) > 0,(4)\tilde{u} \leq 0$ if $\tilde{u}(0) \leq 0$.

For arbitrary $\tilde{u} = (u(r), \tilde{u}(r)), \tilde{v} = (v(r), \tilde{v}(r))$, we define addition $(\tilde{u} + \tilde{v})$, subtraction $(\tilde{u} - \tilde{v})$, multiplication $(\tilde{u} \cdot \tilde{v})$ as follows:

**Addition:**

$$(u + v)(r) = u(r) + v(r), \quad (\tilde{u} + \tilde{v})(r) = \tilde{u}(r) + \tilde{v}(r),$$

(10)
Subtraction:
\[ (u - v)(r) = u(r) - \tilde{v}(r), \quad (\tilde{u} - v)(r) = \tilde{u}(r) - v(r), \] (11)

Multiplication:
\[ (uv)(r) = \min\{u(r)v(r), u(r)v(r), u(r)v(r), u(r)v(r)\}, \quad (\tilde{u}v)(r) = \max\{u(r)v(r), u(r)v(r), u(r)v(r), u(r)v(r)\} \] (12)

**Definition 2** The \( n \times n \) linear system of equations
\[
\begin{align*}
\tilde{a}_{11}\tilde{x}_1 + \tilde{a}_{12}\tilde{x}_2 + \cdots + \tilde{a}_{1n}\tilde{x}_n &= \tilde{b}_1, \\
\tilde{a}_{21}\tilde{x}_1 + \tilde{a}_{22}\tilde{x}_2 + \cdots + \tilde{a}_{2n}\tilde{x}_n &= \tilde{b}_2, \\
& \vdots \\
\tilde{a}_{n1}\tilde{x}_1 + \tilde{a}_{n2}\tilde{x}_2 + \cdots + \tilde{a}_{nn}\tilde{x}_n &= \tilde{b}_n,
\end{align*}
\] (13)

where the elements \( \tilde{a}_{ij} \), of the coefficient matrix \( A \), \( 1 \leq i, j \leq n \) and the elements \( \tilde{b}_i \), of the vector \( \tilde{b} \) are fuzzy numbers is called a fully fuzzy linear system of equations FFLS.

**Definition 3** For any FFLS \( AX = \tilde{b} \) and for all \( \lambda \in [0,1] \), the \( n \times n \) interval linear system of equations
\[
\begin{align*}
\tilde{a}_{11}\tilde{x}_1^\lambda + \tilde{a}_{12}\tilde{x}_2^\lambda + \cdots + \tilde{a}_{1n}\tilde{x}_n^\lambda &= \tilde{b}_1^\lambda, \\
\tilde{a}_{21}\tilde{x}_1^\lambda + \tilde{a}_{22}\tilde{x}_2^\lambda + \cdots + \tilde{a}_{2n}\tilde{x}_n^\lambda &= \tilde{b}_2^\lambda, \\
& \vdots \\
\tilde{a}_{n1}\tilde{x}_1^\lambda + \tilde{a}_{n2}\tilde{x}_2^\lambda + \cdots + \tilde{a}_{nn}\tilde{x}_n^\lambda &= \tilde{b}_n^\lambda,
\end{align*}
\] (14)

where \( \tilde{a}_{ij}^\lambda, \tilde{x}_j^\lambda, \tilde{b}_i^\lambda \), \( 1 \leq i, j \leq n, \tilde{a}_{ij}^\lambda, \tilde{x}_j^\lambda, \tilde{b}_i^\lambda \) are \( \lambda \)-cut sets of fuzzy numbers \( \tilde{a}_{ij}, \tilde{x}_j, \tilde{b}_i \) is called \( \lambda \)-cut system of linear system and represented by \( A^\lambda X^\lambda = \tilde{b}^\lambda, 0 \leq \lambda \leq 1 \).

Any \( \lambda \)-cut system is interval linear system.

**Definition 4** A fuzzy number vector \( \tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)' \) given by
\[ \tilde{x}_i = (x_i(r), \bar{x}_i(r)) \quad 1 \leq i \leq n, \quad 0 \leq r \leq 1 \] is called a solution of FFLS, if
\[
\begin{align*}
\sum_{j=1}^{n} a_{ij} x_j(r) &= b_i(r), \quad \sum_{j=1}^{n} a_{ij} \bar{x}_j(r) = \bar{b}_i(r), \quad i = 1, \ldots, n.
\end{align*}
\] (15)
Now, we define positive fuzzy number solution of FFLS as follows:

**Definition 5** A fuzzy number solution vector \((\bar{x}_1, \bar{x}_2, ..., \bar{x}_n)^T\) of FFLS is called positive fuzzy number solution if for all \(i, (i = 1, 2, ..., n) \bar{x}_i > 0\).

Necessary and sufficient condition for the existence of a positive fuzzy number solution of FFLS is:

**Theorem:** If FFLS \(AX = b\) has a fuzzy number solution, then FFLS \(AX = b\) has positive fuzzy number solution if and only if \(0 - \text{cut} \) system of linear system represented by \(A^0X^0 = b^0\) has positive solution.

**Proof:** Let \((\bar{x}_1, \bar{x}_2, ..., \bar{x}_n)^T\) is fuzzy number solution of \(AX = b\). \(AX = b\) has positive solution, if and only if, \(\bar{x}_i > 0, i = 1, 2, ..., n\), if and only if \(\text{supp } \bar{x}_i = \{x | x \in \mathbb{R} \text{ and } \bar{x}_i(x) > 0\} = [\bar{x}_i(0), \bar{x}_i(0)]\) are positive intervals if and only if \(A^0X^0 = b^0\) has positive interval solution.

### 3. Positive Solutions of FFLS

In this section, we are going to find positive fuzzy number solutions of FFLS. We suppose that \(\tilde{a}_{ij}\) the elements of \(A\) are in the three forms: 1- \(\tilde{a}_{ij} \geq 0\) 2- \(\tilde{a}_{ij} \leq 0\) 3- \(\tilde{a}_{ij}(r) \leq 0 \text{ and } \tilde{a}_{ij}(r) \geq 0\).

Let \(AX = b\) be FFLS. Consider \(i\)–th equation of this system:

\[
\sum_{j=1}^{n} \tilde{a}_{ij} \bar{x}_j = b_i, \quad i = 1, ..., n. \tag{16}
\]

Since we suppose that elements of coefficient matrix are in three forms, we may define three \(n \times n\) matrix of fuzzy numbers \(A_1, A_2, A_3\) as follows:

\[
(A_1)_{ij} = \begin{cases} \tilde{a}_{ij} & \text{if } \tilde{a}_{ij} \geq 0 \\ 0 & \text{if } O.w \end{cases} \quad (A_2)_{ij} = \begin{cases} \tilde{a}_{ij} & \text{if } \tilde{a}_{ij} \leq 0 \\ 0 & \text{if } O.w \end{cases} \quad (A_3)_{ij} = \begin{cases} \tilde{a}_{ij} & \text{if } \tilde{a}_{ij}(r) \leq 0 \text{ and } \tilde{a}_{ij}(r) \geq 0 \\ 0 & \text{if } O.w \end{cases} \tag{17}
\]

It is clear that \(A = A_1 + A_2 + A_3\),

\[
AX = A_1X + A_2X + A_3X. \tag{18}
\]
Hence the i-th equation of system $AX = b$ is transformed to this equation:

$$
\tilde{b}_i = \sum_{j=1}^{n} \tilde{a}_{ij} \tilde{x}_j = \sum_{j=1}^{n} \tilde{a}_{ij} \tilde{x}_j + \sum_{j=1}^{n} \tilde{a}_{2ij} \tilde{x}_j + \sum_{j=1}^{n} \tilde{a}_{3ij} \tilde{x}_j, \quad i = 1, \ldots, n,
$$

(20)

If we use definition 4, and if $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n)'$ is fuzzy number solution of $AX = b$, following equations must be true:

$$
\tilde{b}_j(r) = \sum_{j=1}^{n} \tilde{a}_{ij} x_j(r) + \sum_{j=1}^{n} \tilde{a}_{2ij} x_j(r) + \sum_{j=1}^{n} \tilde{a}_{3ij} x_j(r), \quad i = 1, \ldots, n,
$$

(21)

and

$$
\tilde{b}_j(r) = \sum_{j=1}^{n} \tilde{a}_{ij} x_j(r) + \sum_{j=1}^{n} \tilde{a}_{2ij} x_j(r) + \sum_{j=1}^{n} \tilde{a}_{3ij} x_j(r), \quad i = 1, \ldots, n.
$$

(22)

Now, let $AX = b$ has a positive fuzzy number solution, since $\tilde{z}_j \geq 0, \quad (1 \leq j \leq n)$ and by applying $(1 - 4)$ we can write:

$$
a_{ij} x_j(r) = a_{ij}(r) x_j(r) \quad \tilde{a}_{ij} x_j(r) = \tilde{a}_{ij}(r) x_j(r)
$$

$$
a_{2ij} x_j(r) = a_{2ij}(r) x_j(r) \quad \tilde{a}_{2ij} x_j(r) = \tilde{a}_{2ij}(r) x_j(r)
$$

$$
a_{3ij} x_j(r) = a_{3ij}(r) x_j(r) \quad \tilde{a}_{3ij} x_j(r) = \tilde{a}_{3ij}(r) x_j(r)
$$

(23)

Now, if we replace above expressions in (21) and (22), they can be rewritten as:

$$
\sum_{j=1}^{n} a_{ij} x_j(r) = \sum_{j=1}^{n} a_{ij}(r) x_j(r) + \sum_{j=1}^{n} (a_{2ij}(r) + a_{3ij}(r)) x_j(r) = \tilde{b}_j(r), \quad i = 1, \ldots, n,
$$

(24)

and

$$
\sum_{j=1}^{n} a_{ij} x_j(r) = \sum_{j=1}^{n} a_{ij}(r) x_j(r) + \sum_{j=1}^{n} (a_{2ij}(r) + a_{3ij}(r)) x_j(r) = \tilde{b}_j(r), \quad i = 1, \ldots, n.
$$

(25)

Hence i-th equation of $AX = b$ (16) is transformed to two parametric equations (24) and (25). Now we illustrate these equations in matrix forms.

If $C_l \quad l = 1,2,3,4$ are parametric $n \times n \times n$ matrices by elements

$$(C_1)_{ij} = a_{ij}(r), \quad (C_2)_{ij} = a_{2ij}(r) + a_{3ij}(r),
$$

$$(C_3)_{ij} = a_{ij}(r) + a_{3ij}(r), \quad (C_4)_{ij} = a_{2ij}(r).
$$

(26)

and if $X_1, X_2, B_1$ and $B_2$ are parametric $n$-vectors by elements.
The FFLS $AX = b$ can be represented in matrix form as:

$$\begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

(28)

where this matrix of coefficients is represented in $2n \times 2n$. If we solve this system, its solution is positive solution of FFLS.

Note that before solving $FFLS AX = b$ we must have information about its solution such as: 1- Does this system has a positive solution? and 2- Is $\bar{x}_j$ positive fuzzy number? Hence, we must solve 0-cut system of $AX = b$ to find $FFLS AX = b$ solution supports. After solving this interval system we can find our questions'answers. If that interval system has positive interval solution, we can transform $FFLS AX = b$ to $2n \times 2n$ parametric linear system as form (28).

The algorithm for solving FFLS and to find its positive solution is illustrated as follows:

Positive Solution of FFLS 's Algorithm

Suppose $AX = b$ is a FFLS.

• 1. Solve $A^0 X^0 = b^0$ system and find its $\text{hull} \sum (\text{vert} A, \text{vert} b)$. If this system has positive solution then go to 2 else go to 5.

• 2. Transform $AX = b$ to (28) system using $A^0 X^0 = b^0$ solutions.

• 3. Solve $2n \times 2n$ parametric system (28).

• 4. If the solutions of (28) can define fuzzy numbers, this solution is a positive fuzzy number solution of $FFLS AX = b$ go to 6.

• 5. This system has a non positive solution or this system does not have any solution and this algorithm can not solve it.

• 6. End.

**Example 1** [12] Consider the fully fuzzy linear system $AX = b$ where

$$A = \begin{pmatrix} (4 + r, 6 - r) & (5 + r, 8 - 2r) \\ (6 + r, 7) & (4, 5 - r) \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} (40 + 10r, 67 - 17r) \\ (43 + 5r, 55 - 7r) \end{pmatrix}$$

Dehghan et al. in [12] have solved this system approximately and their approximated solutions are $x_1 = \left(\frac{43}{11} + \frac{1}{11} r, 4\right)$ and $x_2 = \left(\frac{54}{11} + \frac{1}{11} r, \frac{21}{4} - \frac{1}{4} r\right)$ that are triangular fuzzy numbers.

We solve this system by our algorithm as follow's:

Its 0-cut system is
\[
\begin{bmatrix}
[4,6] & [5,8] \\
[6,7] & [4,5]
\end{bmatrix}
\begin{bmatrix}
\overline{x}(0), \underline{x}(0) \\
\overline{x}(0), \underline{x}(0)
\end{bmatrix}
= 
\begin{bmatrix}
[40,67] \\
[43,55]
\end{bmatrix}
\]

Its vertex hull solution is:

\[x^0_1 = \left[ \frac{55}{14}, \frac{105}{26} \right], \quad x^0_2 = \left[ \frac{34}{7}, \frac{139}{26} \right]\]

This interval system has positive solution.

And \(C_i\), \(i = 1, \ldots, 4\), are defined as:

\[C_1 = \begin{pmatrix}
4 + r & 5 + r \\
6 + r & 4
\end{pmatrix}, \quad C_3 = \begin{pmatrix}
6 - r & 8 - 2r \\
7 & 5 - r
\end{pmatrix}, \quad C_2 = C_4 = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}\]

And matrix of coefficients is

\[
\begin{pmatrix}
C_1 & C_2 \\
C_4 & C_3
\end{pmatrix} = 
\begin{pmatrix}
4 + r & 5 + r & 0 & 0 \\
6 + r & 4 & 0 & 0 \\
0 & 0 & 6 - r & 8 - 2r \\
0 & 0 & 7 & 5 - r
\end{pmatrix}
\]

Where it is \(4 \times 4\) parametric linear system and its solutions are:

\[z_1(r) = \overline{x}(r) = \frac{5r^2 + 28r + 55}{r^2 + 7r + 14}, \quad z_1(r) = \underline{x}(r) = \frac{3r^2 + 14r - 105}{r^2 + 3r - 26}, \]
\[z_2(r) = \overline{x}(r) = \frac{5r^2 + 37r + 68}{r^2 + 7r + 14}, \quad z_2(r) = \underline{x}(r) = \frac{7r^2 + 22r - 139}{r^2 + 3r - 26}. \]

Since \(\overline{x}(r)\) and \(\overline{x}(r)\) are bounded monotonic increasing left continuous functions. \(\underline{x}(r)\) and \(\underline{x}(r)\) are bounded monotonic decreasing left continuous functions. \(\overline{x}_1(r) \leq \underline{x}_1(r), (0 \leq r \leq 1)\) and \(\overline{x}_2(r) \leq \underline{x}_2(r), (0 \leq r \leq 1)\)

Parametric system solutions define fuzzy numbers and positive fuzzy number solutions of this FFLS are:

\[x_1 = \left( \frac{5r^2 + 28r + 55}{r^2 + 7r + 14}, \frac{3r^2 + 14r - 105}{r^2 + 3r - 26} \right) \text{ and } x_2 = \left( \frac{5r^2 + 37r + 68}{r^2 + 7r + 14}, \frac{7r^2 + 22r - 139}{r^2 + 3r - 26} \right).\]

**Example 2** Consider the fully fuzzy linear system \(AX = b\) where

\[
A = \begin{pmatrix}
-5 + r & -1 - r \\
3 + r^2 & -2r^2 \\
2r & 5 - 3r \\
1 + 3r^2 & 6 - 2r
\end{pmatrix}
\]
We solve this system by our algorithm as follows:

Its 0-cut system is

\[
\begin{bmatrix}
[-5, -1] & [-3, 0] \\
[0, 5] & [1, 6]
\end{bmatrix}
\begin{bmatrix}
[x_1(0), \bar{x}_1(0)] \\
[x_2(0), \bar{x}_2(0)]
\end{bmatrix}
=
\begin{bmatrix}
[-15, 20] \\
[-19, 30]
\end{bmatrix}
\]

And its vertex hull solution is:

\[x_1^0 = [-4, 0], \quad x_2^0 = [1, 5]\]

Since \(x_1^0\) is non positive interval, this system does not have positive solution our algorithm does not have solution and can not solve it.

4. Conclusion
In this paper, we found positive solution of fully fuzzy linear system of equations, analytically. For this means an algorithm is introduced. In this Algorithm, \(n \times n\) system \(AX = b\) is transformed to \(2n \times 2n\) parametric system. For this aim, We first solve 0-cut system of fully fuzzy linear system of equations and find its solutions. If its solutions are positive, we solve \(2n \times 2n\) parametric form linear system and finally, if its solutions can define fuzzy numbers; we can say that our algorithm has find positive fuzzy numbers solutions. Note that may be \(A^0X^0 = b^0\) has non positive solution, this system does not have positive solution our algorithm does not have solution and can not solve it and hence, we can not solve fully fuzzy linear system of equations. Our algorithm does not have any restricted in parametric form of fuzzy numbers.

References
