Radiation effect on boundary layer flow of an Eyring–Powell fluid over an exponentially shrinking sheet

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Abstract The aim of this paper was to examine the steady boundary layer flow of an Eyring–Powell model fluid due to an exponentially shrinking sheet. In addition, the heat transfer process in the presence of thermal radiation is considered. Using usual similarity transformations the governing equations have been transformed into non-linear ordinary differential equations. Homotopy analysis method (HAM) is employed for the series solutions. The convergence of the obtained series solutions is carefully analyzed. Numerical values of the temperature gradient are presented and discussed. It is observed that velocity increases with an increase in mass suction S. In addition, for the temperature profiles opposite behavior is observed for increment in suction. Moreover, the thermal boundary layer thickness decreases due to increase in Prandtl number Pr and thermal radiation R.

1. Introduction

Viscous boundary layer flow due to a stretching/shrinking sheet is of significant importance due to its vast applications. Aerodynamic extrusion of plastic sheets, glass fiber production, paper production, heat treated materials traveling between a feed roll and a wind-up roll, cooling of an infinite metallic plate in a cooling bath and manufacturing of polymeric sheets are some examples for practical applications of non-Newtonian fluid flow over a stretching/shrinking surface. The quality of the final product depends on the rate of heat transfer at the stretching surface. This stretching/shrinking may not necessarily be linear. It may be quadratic, power-law, exponential and so on.

Over the last few decades, in nearly all investigations on the flow past a stretching/shrinking sheet, the flow occurs due to linear stretching/shrinking velocity of the flat sheet. However, boundary layer flow induced by an exponentially stretching/shrinking sheet is not studied much. Crane [1] firstly investigated the steady boundary layer flow of the incompressible flow. Gupta and Gupta [2] and Chen and Char [3] extended the work of Crane under various physical conditions. Hayat et al. [4] examined the unsteady three dimensional flow of couple stress fluid over a stretching surface with chemical reaction.

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Hamad [5] found the analytical solution of natural convection flow of a nanofluid over a linearly stretching sheet in the presence of magnetic field. Bachok et al. [6] described stagnation-point flow over a stretching/shrinking sheet in a nanofluid. Bhattacharyya [7] considered heat transfer analysis in unsteady boundary layer stagnation-point flow toward a shrinking/stretching sheet. Numerical study of MHD boundary layer flow of a Maxwell fluid past a stretching sheet in the presence of nanoparticles has been carried out by Nadeem et al. [8]. Mehmood et al. [9] examined the non-orthogonal flow of a second grade micropolar fluid toward a stretching sheet, they also have taken heat transfer analysis into account. Moreover, Nadeem et al. [10] have recently analyzed the non-orthogonal stagnation point flow of a third order fluid toward a stretching surface in the presence of heat transfer. Magyari et al. [11] obtained the boundary layer flow due to exponentially stretching sheet. Elbashbeshy [12] numerically explained the flow and heat transfer over an exponentially stretching surface considering wall mass suction. The MHD boundary layer flow of a viscous fluid over an exponentially stretching sheet with effects of radiation was studied by Ishaq [13]. Al-Odet et al. [14] examined the effect of magnetic field on thermal boundary layer flow on an exponentially stretching continuous surface with an exponentially temperature distribution. Sajid et al. [15] found the influence of thermal radiation on the boundary layer flow past an exponentially stretching sheet and they reported series solutions for the velocity and temperature by employing HAM. Bhattacharyya [16] discussed the boundary layer and heat transfer over an exponentially shrinking sheet.

Up to date not much study has been carried out for the two dimensional flow of the Eyring–Powell fluid. Although this fluid model has many advantages over the non-Newtonian fluids models. Firstly, it is extracted from the kinetic theory of liquids rather than the empirical relation. Secondy, for low and high shear rates it correctly reduces to Newtonian behavior. Eyring–Powell fluid model [17] a complete mathematical model proposed by Powell and Eyring in 1944. Hayat et al. [18] analyzed steady flow of an Eyring–Powell fluid over a moving surface with convective boundary conditions. Malik et al. [19] presented boundary layer flow of an Eyring–Powell model fluid due to a stretching cylinder with variable viscosity. Nabil et al. [20] studied numerical study of viscous dissipation effect on free convection heat and mass transfer of MHD Eyring–Powell fluid flow through a porous medium. Javed et al. [21] investigated flow of an Eyring–Powell non-Newtonian fluid over a stretching sheet. Characteristics of heating scheme and mass transfer on the peristaltic flow an Eyring–Powell fluid in an endoscope discussed by Nadeem et al. [22]. However, to the best of our knowledge no attempt has been made to study Eyring–Powell fluid over an exponentially shrinking sheet.

Thus current work presents a theoretical study Eyring Powell flow of over an exponentially shrinking sheet. A mathematical model has been prepared in the presence of radiation effects. We developed series solutions for the resulting problems by using the homotopy analysis method [23–27]. Results for the velocity and temperature are constructed. Convergence criteria for the derived series solutions are established. The velocity and temperature are analyzed to gain thorough insight toward the physics of the problem for various parameters of interest. Numerical values of the temperature gradient are presented and discussed.

2. Mathematical formulation

Consider the steady two dimensional boundary layer flow of an Eyring–Powell fluid with heat transfer in the presence of thermal radiation over an exponentially shrinking sheet (see Fig. 1). The stress tensor of an Eyring–Powell model [28] is expressed as

$$A = -p d + \tau$$  \hspace{1cm} (1)

where extra stress tensor $\tau_{ij}$ is given by

$$\tau_{ij} = \mu \frac{\partial u_i}{\partial x_j} + \frac{1}{\beta} \sinh^{-1} \left( \frac{1}{C_0} \frac{\partial u_i}{\partial x_j} \right).$$  \hspace{1cm} (2)

In accordance with the boundary layer approximations, the governing equations for the flow are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$  \hspace{1cm} (3)

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \left( \gamma + 1 \right) \frac{\partial^2 u}{\partial y^2} - \frac{1}{2\rho d^2} \left( \frac{\partial u}{\partial y} \right)^2$$  \hspace{1cm} (4)

$$\frac{\partial v}{\partial y} = 0$$  \hspace{1cm} (5)

Eqs. (4) and (5) show that the pressure is independent of $y$. Since the lateral velocity is zero far away from the sheet and the pressure is uniform, then Eq. (4) takes the form

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \left( \gamma + 1 \right) \frac{\partial^2 u}{\partial y^2} - \frac{1}{2\rho d^2} \left( \frac{\partial u}{\partial y} \right)^2.$$  \hspace{1cm} (6)

The equations representing temperature with heat radiation may be written in usual notation as below:

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{\kappa}{\rho C_p} \frac{\partial^2 T}{\partial y^2} - \frac{1}{\rho C_p} \frac{\partial q}{\partial y}$$  \hspace{1cm} (7)

where $u$ and $v$ are the velocity components, $\gamma$ is the kinematic viscosity, $\rho$ is the fluid density, $\beta$ and $d$ are the fluid parameters of Eyring–Powell model, $d$ has the dimension of $(time)^{-1}$. $\kappa$ is the fluid thermal conductivity and $C_p$ is the specific heat constant. The radiative heat flux, $q$, is given by $q_r = -\frac{d q}{d x}$, where $\sigma$ is Stefan–Boltzmann constant and $K$ is Rosseland mean absorption coefficient.

The boundary conditions are given by

$$u = U_0(x), \quad v = v_0, \quad at \quad y = 0; \quad u \rightarrow 0 \quad as \quad y \rightarrow \infty$$  \hspace{1cm} (8)

Fig. 1 Physical model of the fluid.
where \( c > 0 \) is shrinking constant. Here \( v_w \) is the constant with \( v_w < 0 \) for mass suction and \( v_w > 0 \) for mass injection. Now, the stream function \( \psi(x, y) \) has been introduced as
\[
u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}
\]
The continuity Eq. (1) is identically satisfied by Eq. (11), the momentum and energy equations have been reduced to the forms
\[
\frac{\partial \psi}{\partial y} \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x} \frac{\partial^2 \psi}{\partial x \partial y} = \left( \nu + \frac{1}{\rho \beta d} \right) \frac{\partial^2 \psi}{\partial y^2} - \frac{1}{\nu^2 \rho \beta d^2} \left( \frac{\partial^2 \psi}{\partial y^2} \right)^2
\]
\[
\frac{\partial T}{\partial y} \frac{\partial^2 T}{\partial x \partial y} - \frac{\partial T}{\partial x} \frac{\partial^2 T}{\partial x \partial y} = \frac{\kappa c_p}{\rho c_p} \left( \frac{\partial^2 T}{\partial y^2} \right) - 1 \frac{\partial T}{\partial y}
\]
The boundary conditions in Eq. (8) for the velocity components take the form
\[
\frac{\partial \psi}{\partial y} = U_w(x), \quad \frac{\partial \psi}{\partial x} = -v_w \quad \text{at} \quad y = 0; \quad \frac{\partial \psi}{\partial x} \rightarrow 0 \quad \text{as} \quad y \rightarrow \infty.
\]
The non-linear coupled differential Eqs. (16) and (17) along with the boundary conditions Eqs. (18) and (19) have been solved by employing HAM.

3. Series solutions

The auxiliary linear operators and the initial approximations for the homotopy analysis solutions are chosen as solutions of Eqs. (16) and (17) with the boundary conditions Eqs. (18) and (19).
\[
f_0 = -1 + e^{-\eta} + S, \theta_0 = e^{-\eta}
\]
\[
L_1(f) = f' - f
\]
\[
L_2(\theta) = \theta' - \theta
\]
We note that the auxiliary linear operators in the above equations satisfy the following properties
\[
L_1[c_1 + c_2 e^\eta + c_3 e^{-\eta}] = 0, \quad L_2[c_4 e^\eta + c_5 e^{-\eta}] = 0
\]
The associated zeroth-order deformation problems are
\[
(1 - \beta) L_1[\tilde{f}(\eta, p) - f_0(\eta)] = ph N_1[\tilde{f}(\eta, p)]
\]
\[
(1 - \beta) L_2[\tilde{\theta}(\eta, p) - \theta_0(\eta)] = ph N_2[\tilde{\theta}(\eta, p)]
\]
\[
\tilde{f}(0, p) = S, \quad \tilde{\theta}(0, p) = 1, \tilde{f}(\infty, p) = 0,
\]
\[
\tilde{\theta}(0, p) = 0, \quad \tilde{\theta}(\infty, p) = 0
\]
The boundary conditions for this deformation are based in Eqs. (16) and (17), the non-linear operators \( N_1 \) and \( N_2 \) are introduced as
\[
N_1(\tilde{f}(\eta, p)) = (1 + N) \frac{\partial^2 \tilde{f}(\eta, p)}{\partial \eta^2} - \tilde{f}(\eta, p) \frac{\partial^2 \tilde{f}(\eta, p)}{\partial \eta^2}
\]
\[
-2 \left( \frac{\partial \tilde{f}(\eta, p)}{\partial \eta} \right)^2 - N_2 \left( \frac{\partial^2 \tilde{f}(\eta, p)}{\partial \eta^2} \right)^2 \frac{\partial \tilde{f}(\eta, p)}{\partial \eta}
\]
\[
N_2(\tilde{\theta}(\eta, p), \tilde{\eta}(\eta, p)) = \left( 1 + \frac{4}{3} R \right) \frac{\partial^2 \tilde{\theta}(\eta, p)}{\partial \eta^2}
\]
\[
+ Pr \left( \frac{\partial \tilde{\theta}(\eta, p)}{\partial \eta} - \tilde{\eta}(\eta, p) \frac{\partial \tilde{\theta}(\eta, p)}{\partial \eta} \right)
\]
Here, \( p \) is an embedding parameter, \( h \) is the non-zero auxiliary parameter. For \( p = 0 \) and \( p = 1 \), we have
\[
\tilde{f}(\eta, 0) = f_0(\eta), \tilde{f}(\eta, 1) = f(\eta)
\]
\[
\tilde{\theta}(\eta, 0) = \theta_0(\eta), \tilde{\theta}(\eta, 1) = \theta(\eta)
\]
Further, when \( p \) increases from 0 to 1, \( \tilde{f}(\eta, p), \tilde{\theta}(\eta, p) \) vary from \( f_0(\eta), \theta_0(\eta), \theta(\eta) \) to \( f(\eta), \theta(\eta) \). By using Taylor’s series expansion, one can write
\[
f(\eta, p) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta) p^m, f_m(\eta) = \left. \frac{\partial^m f(\eta, p)}{\partial p^m} \right|_{p=0}
\]
\[
\theta(\eta, p) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta) p^m, \theta_m(\eta) = \left. \frac{\partial^m \theta(\eta, p)}{\partial p^m} \right|_{p=0}
\]
and the convergence of Eqs. (26) and (27) strictly depend upon \( h \). The values of \( h \) are selected in such a manner that Eqs. (26) and (27) are convergent at Therefore, \( mth \) order deformation problems are given by
where

\[ L_2[f_m(\eta) - \eta f_{m-1}(\eta)] = hR_m^p(\eta) \]  \( (36) \)

and

\[ L_2[\theta_m(\eta) - \eta \theta_{m-1}(\eta)] = hR_m^\theta(\eta) \]  \( (37) \)

with

\[ f_m(0) = S, \quad f_m'(0) = 1, \quad f_m'(\infty) = 0, \quad \theta_m(0) = 0, \quad \theta_m(\infty) = 0 \]

The general solution can be written as

\[ f_m(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta) \]  \( (40) \)

and

\[ \theta_m(\eta) = \theta_0(\eta) + \sum_{m=1}^{\infty} \theta_m(\eta) \]  \( (41) \)

The curves have been sketched at the 7th-order of approximations to determine the suitable ranges for \( h \) and \( S \). Figs. 2 and 3 show the acceptable range for \( h \) for \( \lambda = 0.1 \) and \( \lambda = 0.5 \). It is noticed that Eqs. (36) and (37) have the auxiliary parameters \( \lambda \) and \( S \). It is observed that the increase in \( \lambda \) enhances the velocity profiles. The behavior of \( f(\eta) \) by changing the Eyre-Powell parameter \( S \) is observed in Fig. 5. It is noticed that the increase in \( S \) reduces the velocity profiles. In Fig. 7 the temperature profiles \( \theta(\eta) \) have been shown for the different values of \( S \). The temperature at a point decreases for an increase of \( S \). Finally, the effects of the Prandtl number \( Pr \) on the dimensionless temperature profile are offered in Fig. 8. It is observed that temperature is decreasing with an increase of \( Pr \). Also, it is worth mentioning that the thermal boundary layer thickness reduces considerably due to increase in \( Pr \), since the Prandtl number is inversely proportional to the thermal conductivity. Thus, the fluid with higher Prandtl number has lower thermal conductivities which led the heat possible. The shrinking rate in the exponential case is much faster than of the linear case. Thus, the amount of velocity generated due to exponential shrinking is greater than that of linear shrinking.

**Fig. 4** Velocity profiles of \( f(\eta) \) for various values of \( S \) keeping \( \lambda = 0.1 \) and \( N = 0.1 \) fixed.
diffusion slows down. On the other hand, for the case of lower \( Pr \) fluids the heat diffusion gets faster, with higher thermal conductivities. The influence of the thermal radiation on temperature is depicted in Fig. 9. It is interesting to note that thermal radiation has a major influence on the temperature distribution in the fluid. We observed that the fluid temperature increases by increasing thermal radiation. This is due to the fact that increase in the values of the thermal radiation parameter increases radiation in the boundary layer, and hence increases the values of the temperature profiles in the thermal boundary layer.

In Table 1 the dimensionless velocity gradient on the sheet is approximated for various values of \( N \) and \( k \). We observed that skin friction coefficient is reduced by sufficiently large values of \( N \) and \( k \).

<table>
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<tr>
<th>( \lambda/N )</th>
<th>0.0</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1.0</th>
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<td>0.31007</td>
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<td>0.50186</td>
<td>0.56254</td>
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<td>0.29554</td>
<td>0.41241</td>
<td>0.49543</td>
<td>0.55876</td>
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<tr>
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<td>0.10112</td>
<td>0.27968</td>
<td>0.40119</td>
<td>0.48835</td>
<td>0.55460</td>
</tr>
<tr>
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<td>0.08330</td>
<td>0.26230</td>
<td>0.38777</td>
<td>0.48053</td>
<td>0.55000</td>
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</table>

Table 1 Numerical values of skin-friction coefficient \( Re \nu C_f \) for different values of \( N \) and \( \lambda \).
of surface heat transfer $-\theta'(0)$ have been computed for different values of $Pr$, $R$, $\tilde{\lambda}$, $S$ and $N$. It is apparent that the magnitude of surface heat transfer is an increasing function of $Pr$ and $\tilde{\lambda}$, while decreasing one for the different values of $S$, $N$ and $R$.

5. Conclusions

The boundary layer flow and heat transfer over an exponentially shrinking sheet have been studied. Series solutions are obtained for the velocity and temperature profiles. It is observed that velocity increases with an increase in mass suction $S$. In addition, for the temperature profiles opposite behavior is seen for increment in suction. Moreover, the thermal boundary layer thickness decreases due to increase in Prandtl number $Pr$ and thermal radiation $R$.

References