Trajectory planning and tracking of ball and plate system using hierarchical fuzzy control scheme

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Received 14 February 2001; received in revised form 8 July 2002; accepted 24 March 2003

Abstract

Ball and plate system is the extension of traditional ball and beam system. In this paper, a trajectory planning and tracking problem of ball and plate system is put forward to proof-test diverse control schemes. Firstly, we derive the mathematical model of the ball and plate system in detail. Then a hierarchical fuzzy control scheme is proposed to deal with this problem. Our scheme is composed of three levels. The lowest level is a TS type fuzzy tracking controller; the middle level is a fuzzy supervision controller that takes actions in extreme situations; and the top level is a fuzzy planning controller that determines the desired trajectory. In order to optimize the ball’s moving trajectory, the genetic algorithm is introduced to adjust the output membership functions of the fuzzy planning controller. Simulation results have shown that the hierarchical fuzzy control scheme can control the ball from a point to another without hitting the obstacles and in the least time.

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Keywords: Trajectory tracking; Trajectory planning; Fuzzy control; Genetic algorithm; Ball and plate system

1. Introduction

A ball moving on a beam is a typical nonlinear dynamic system, which is often adopted to proof-test diverse control schemes [1,3,5,6,7]. Ball and plate system is the extension of the traditional ball and beam problem that moves a metal ball on a rigid plate as shown in Fig. 1 [4]. The slope of the plate can be manipulated in two perpendicular directions, so that the tilting of the plate will make the ball move on the plate. In this paper, a trajectory planning and tracking problem is
proposed for ball and plate system, which is to control the ball from point A through the route A → B → C → D → E → F → G to point G without hitting the obstacles and as fast as possible, as shown in Fig. 2. This problem adds difficulty on the control and may be a very interesting example for studying and testing various control methods. Due to the complexity of this problem, we decompose the control task into three subtasks and distribute them over three different levels. The lowest level controls the ball to track the desired trajectory, where a TS type fuzzy controller is designed based on local linearized models in different state space regions. The middle level performs supervisory operations to take actions in extreme situations. While the top level is a planning controller that determines the desired trajectory. The last two levels both adopt Mamdani type fuzzy controllers, which are designed based on experience. The proposed hierarchical fuzzy control scheme realizes the trajectory planning and tracking task quite well. Furthermore, in order to make the ball pass the route as fast as possible, we also introduce the genetic algorithm to optimize the parameters of the fuzzy planning controller.

This paper is organized as follows. Section 2 gives the control requirements, system parameters, and mathematical model of the ball and plate system in detail. Section 3 introduces the hierarchical fuzzy control scheme and the detailed design steps of the fuzzy controllers in planning, supervision and tracking levels, respectively. Section 4 presents how to use the genetic algorithm to optimize the parameters of the fuzzy planning controller and shows the simulation result. Finally, Section 5 concludes the paper.

2. Control requirements and mathematical model

The ball and plate system is shown in Fig. 1, where a metal ball stays on a rigid square plate with each side length of 1 m. The slope of the plate can be manipulated by two perpendicularly installed step motors, so that the tilting of the plate will make the ball moving. The ball’s position
is measured using a CCD camera. It is required to control the ball from point A (0, 0.33) to point G (0.84, 1) without hitting the obstacles and take as little time as possible. The path must be A → B → C → D → E → F → G as shown in Fig. 2. In order to compare different control schemes easily, all concerned parameter values are listed in Table 1. It should be noticed that the time needed to finish the tour is the performance index of various control schemes and the max velocity of the ball should be equal to or lower than 4 mm/s.

Next, we will derive the differential equations of the ball and plate system. Suppose the ball remains in contact with the plate and the rolling occurs without slipping at any time. Using the Lagrange method, the kinetic energy $T$ of the whole system is:

$$T = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} J_P (\dot{\theta}_x^2 + \dot{\theta}_y^2) + \frac{1}{2} J (\dot{\theta}_x^2 + \dot{\theta}_y^2) + \frac{1}{2} J (\dot{\theta}_x^2 + \dot{\theta}_y^2) + m(x\dot{\theta}_x + y\dot{\theta}_y)^2).$$  \hspace{1cm} (1)

Then the system’s nonlinear dynamics equations are obtained in the following form:

$$\tau_x = (J_P + J + m\dot{x})\ddot{\theta}_x + 2m\dot{x}\dot{\theta}_x + m\dot{x}\dot{\theta}_y + m\dot{y}\dot{\theta}_x + mx\dot{y}\dot{\theta}_x + mgy \cos \theta_x,$$  \hspace{1cm} (2)

$$\tau_y = (J_P + J + m\dot{y})\ddot{\theta}_y + 2m\dot{y}\dot{\theta}_y + m\dot{y}\dot{\theta}_x + m\dot{x}\dot{\theta}_x + m\dot{x}\dot{\theta}_x + mg y \cos \theta_y,$$  \hspace{1cm} (3)

$$(m + J/R^2)\ddot{x} + mg \sin \theta_x - mx\dot{\theta}_x^2 - m\dot{y}\dot{\theta}_x \dot{\theta}_y = 0,$$  \hspace{1cm} (4)

$$(m + J/R^2)\ddot{y} + mg \sin \theta_y - my\dot{\theta}_y^2 - m\dot{x}\dot{\theta}_x \dot{\theta}_y = 0.$$  \hspace{1cm} (5)
Table 1
Parameters of the system

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Parameter value and unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>m</td>
<td>Mass of the ball</td>
<td>0.11 Kg</td>
</tr>
<tr>
<td>R</td>
<td>Radius of the ball</td>
<td>0.02 m</td>
</tr>
<tr>
<td>S</td>
<td>Dimension of the plate</td>
<td>1.0 × 1.0 m^2</td>
</tr>
<tr>
<td>x</td>
<td>Position of the ball in the x-axis</td>
<td>m</td>
</tr>
<tr>
<td>y</td>
<td>Position of the ball in the y-axis</td>
<td>m</td>
</tr>
<tr>
<td>\dot{x}</td>
<td>Velocity of the ball in the x-axis</td>
<td>m/s</td>
</tr>
<tr>
<td>\dot{y}</td>
<td>Velocity of the ball in the y-axis</td>
<td>m/s</td>
</tr>
<tr>
<td>w</td>
<td>Rolling angular velocity of the ball</td>
<td>Arc/s</td>
</tr>
<tr>
<td>\dot{r}</td>
<td>Velocity of the ball, \dot{r}^2 = \dot{x}^2 + \dot{y}^2</td>
<td>m/s</td>
</tr>
<tr>
<td>v_{max}</td>
<td>Maximum velocity of the ball</td>
<td>4 mm/s</td>
</tr>
<tr>
<td>\tau_x</td>
<td>Torque applied to the plate in the x-axis</td>
<td>Kg m^2/s^2</td>
</tr>
<tr>
<td>\tau_y</td>
<td>Torque applied to the plate in the y-axis</td>
<td>Kg m^2/s^2</td>
</tr>
<tr>
<td>\theta_x</td>
<td>Angle of the plate from x-axis</td>
<td>Arc</td>
</tr>
<tr>
<td>\theta_y</td>
<td>Angle of the plate from y-axis</td>
<td>Arc</td>
</tr>
<tr>
<td>\theta_{\dot{x}}</td>
<td>Angle velocity of the plate from the x-axis</td>
<td>Arc/s</td>
</tr>
<tr>
<td>\theta_{\dot{y}}</td>
<td>Angle velocity of the plate from the y-axis</td>
<td>Arc/s</td>
</tr>
<tr>
<td>u_x</td>
<td>Angle acceleration velocity of the plate from x-axis</td>
<td>Arc/s^2</td>
</tr>
<tr>
<td>u_y</td>
<td>Angle acceleration velocity of the plate from y-axis</td>
<td>Arc/s^2</td>
</tr>
<tr>
<td>J_p</td>
<td>Mass moment of inertia of the plate</td>
<td>0.5 Kg m^2</td>
</tr>
<tr>
<td>J</td>
<td>Mass moment of inertia of the ball</td>
<td>1.76e − 5 Kg m^2</td>
</tr>
<tr>
<td>g</td>
<td>Acceleration due to gravity</td>
<td>9.8 m/s^2</td>
</tr>
</tbody>
</table>

Define \( B := m/(m + J/R^2) \). Define two new inputs \( u_x \) and \( u_y \) using the invertible nonlinear transformations (6) and (7):

\[
\tau_x = (J_p + J + mx^2)u_x + 2mx\ddot{x} + mx\dot{y}\ddot{y} + mx\dot{y}\ddot{y} + mx\dot{y}\ddot{y} + mgx \cos \theta_x, \tag{6}
\]

\[
\tau_y = (J_p + J + my^2)u_y + 2my\ddot{y} + my\dot{x}\ddot{x} + my\dot{x}\ddot{x} + my\dot{x}\ddot{x} + mgy \cos \theta_y. \tag{7}
\]

Then the system can be modeled in the state-space form as follows:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4 \\
\dot{x}_5 \\
\dot{x}_6 \\
\dot{x}_7 \\
\dot{x}_8
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
x_2 \\
x_4 \\
x_6 \\
x_8 \\
B(x_5x_8^2 + x_1x_4x_8 - g \sin x_3) \\
x_5x_8^2 + x_1x_4x_8 - g \sin x_7 \\
x_8 \\
0
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
u_x \\
u_y
\end{bmatrix},
\]

\[Y = h(X) = (x_1, x_5)^T \tag{8}\]

where \( X = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)^T = (x, \dot{x}, \theta_x, \ddot{x}, y, \dot{y}, \theta_y, \ddot{y})^T \). Please refer to Appendix A for detailed derivation of the ball and plate system’s mathematical model.
3. Hierarchical fuzzy control scheme

From above paragraph we have learnt that there are two control problems, i.e., trajectory planning and trajectory tracking. For the trajectory tracking problem, because the ball and plate system is a multi-variable, nonlinear process, the traditional control theory has not common methods to solve. While the fuzzy control theory can deal with the nonlinear control problems very easily. For the trajectory planning problem, the common optimal algorithms need more complicated operations, while the fuzzy control theory can adopt the human experience to overcome the difficulty of many restraint conditions. Therefore, we would like to apply the fuzzy control scheme to solve the trajectory planning and tracking problems of the ball and plate system.

In this paper, we propose a hierarchical fuzzy control scheme to fulfill the control requirements, which includes fuzzy tracking controller, fuzzy supervision controller and fuzzy planning controller, as shown in Fig. 3. The lowest level is a TS type fuzzy tracking controller, which controls the ball to track the desired trajectory. Both supervision and planning levels are Mamdani type fuzzy controllers. The fuzzy supervision controller consists of some safeguard control rules that take actions in extreme situations, while the fuzzy planning controller determines the desired trajectory that the ball should follow. In the following paragraphs we will explain the design details of each fuzzy controller successively.

3.1. Fuzzy tracking controller

In the ball and plate system, it is supposed that the ball remains in contact with the plate and the rolling occurs without slipping, which imposes a constraint on the rotation acceleration of the plate. Because of the low velocity and acceleration of the plate rotation, the mutual interactions of both coordinates can be negligible. So the model of the ball–plate system can be approximately
decomposed as follows:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4
\end{bmatrix} = \begin{bmatrix} x_2 \\ B(x_1x_4^2 - g \sin x_3) \\ x_4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_x,
\]

\[
\begin{bmatrix}
\dot{x}_5 \\
\dot{x}_6 \\
\dot{x}_7 \\
\dot{x}_8
\end{bmatrix} = \begin{bmatrix} x_6 \\ B(x_5x_8^2 - g \sin x_7) \\ x_8 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_y.
\]

(9)

Namely, the ball–plate system can be regarded as two individual sub-systems and we can control both coordinates independently. Due to the symmetry of \(x\) direction and \(y\) direction, hereafter we will discuss \(x\) direction only. The total control action \(u_x\) applied to the ball–plate system in \(x\) direction is the sum of a TS type fuzzy tracking control \(u_{t1}\) and a fuzzy supervisory control \(u_{s1}\), that is

\[ u_x = u_{t1} + u_{s1}. \]

(10)

In the TS type fuzzy tracking controller, the \(j\)th fuzzy control rule is expressed in the following form [8,10]:

\[ \text{If } e_x \text{ is } X^j \text{ and } \dot{x} \text{ is } R^j \text{ and } \theta_x \text{ is } \Theta^j \text{ and } \dot{\theta}_x \text{ is } \Omega^j, \]

\[ \text{then } u_{t1} = k_{1j} e_x + k_{2j} \dot{x} + k_{3j} \theta_x + k_{4j} \dot{\theta}_x, \]

(11)

where \(e_x = x_d - x\), \(x_d\) is the desired trajectory in \(x\) direction. The universe of discourse of \((e_x, \dot{x}, \theta_x, \dot{\theta}_x)\) is defined as a hypercube \([-0.1, 0.1 m] \times [-0.1, 0.1 m/s] \times [-0.1, 0.1 \text{ arc}] \times [-0.2, 0.2 \text{ arc/s}]\). For convenience, we introduce four scaling factors \(G_x = 10, G_{\dot{x}} = 10, G_{\theta_x} = 10, G_{\dot{\theta}_x} = 5\) for \(e_x, \dot{x}, \theta_x\) and \(\dot{\theta}_x\), respectively. So that the normalized inputs, \(e_x, \dot{x}, \theta_x\) and \(\dot{\theta}_x\) of the fuzzy controller are all varied in a normalized universe of discourse, i.e., \([-1, 1]\). For each input, two fuzzy sets PO and NE are defined, whose membership functions are chosen as \(1/(1 + |1 - x|^4)\) and shown in Fig. 4.

Obviously there are 16 fuzzy rules in all. In order to determine the parameters \(k_{1j}, k_{2j}, k_{3j}\) and \(k_{4j}\), \(j = 1, 2, \ldots, 16\), of the fuzzy rules, we adopt the linear quadratic optimal control method based on system’s local linearized models \(\dot{\mathbf{x}} = A_j \mathbf{x} + B_j u_{t1}\) corresponding to each fuzzy rule, where

![Fig. 4. Membership functions of normalized input variables.](image-url)
\( \dot{x} = (x_1, x_2, x_3, x_4)^T \), such that the feedback control law \( u_t = -k_j \hat{x} \) minimizes the quadratic cost function

\[
J = \int_{t=0}^{\infty} (\hat{x}^T Q \hat{x} + u_t^T R u_t) dt.
\]

Here we set both \( Q \) and \( R \) equal to identity matrices. The obtained control gain parameters are given in Appendix B. Using the Sum–Product fuzzy inference method and the Center of Gravity defuzzification, the total control is a fuzzy combination of 16 local linearized control laws, expressed in the following formulas:

\[
u_t = -\frac{\sum_{j=1}^{16} w_j K^T_j \hat{x}}{\sum_{j=1}^{16} w_j},
\]

\[
w_j = X^j(e_x) \times R^j(\dot{x}) \times \Theta^j(\theta_x) \times \Omega^j(\dot{\theta}_x)
\]

where \( w_j \) is the activation degree of the \( j \)th fuzzy rule.

However the global closed control system is not always stable even if all local subsystems are stable. One approach to check the global stability is to transfer the problem as a linear matrix inequality (LMI) problem [11]. Another approach uses the \( H^\infty \) control theory for uncertain linear systems [9]. In this case the stability of the whole control system can easily be checked using the second method and the procedures are described as follows:

Let \( \Delta G_{ij} = A_i - B_i F_j - G, \Delta T_{ij} = \Delta G_{ij} + \Delta G_{ji}, \quad i, j = 1, 2, \ldots, r, i < j, \)

where \( r = 16 \) and \( G = (1/r) \sum_{i=1}^{r} (A_i - B_i F_i). \)

Then the singular value decomposition is carried out:

\[
\Delta G_{ii} = U_{ii} S_{ii} V_{ii}^T, \quad \Delta T_{ij} = U_{ij} S_{ij} V_{ij}^T.
\]

Let \( D = [U_1 \quad U_2 \quad \cdots \quad U_r], \ E = [V_1 \quad V_2 \quad \cdots \quad V_r]^T \)

where \( U_i = [U_{ii} \quad U_{i+1} \quad \cdots \quad U_{ir}], \ V_i = [V_{ii} \quad V_{i+1} \quad \cdots \quad V_{ir}]. \)

Let \( M = N = block \_diag[S_1 \quad S_2 \quad \cdots \quad S_r] \)

where

\[
S_i = block \_diag \left[ \frac{d_{ii}}{2} S_1 \quad \frac{d_{ii+1}}{2} S_2 \quad \cdots \quad \frac{d_{ir}}{2} S_r \right], \quad d_{ij} = \max \left\{ \frac{w_i w_j}{(\sum_{k=1}^{r} w_k)^2} \right\}.
\]

**Theorem 1** (Tanaka et al. [9]). The nonlinear closed-loop control system is quadratically stable if

\[
\text{Re} \ \lambda_i(\Psi) \neq 0, \quad \text{where} \quad \Psi = \begin{bmatrix} G + DME & -DNN^T D^T \\ E^T E & -(G + DME)^T \end{bmatrix}.
\]

In this example the eigenvalues of \( \Psi \) are \([0.4765 \pm 1.0138i, 1.0023 \pm 0.2441i, -0.4765 \pm 1.0138i, -1.0023 \pm 0.2441i] \), which satisfy the condition of Theorem 1. So the global closed-loop control system is quadratically stable. Fig. 5 shows the responses of the fuzzy tracking control system under the initial condition \((x, \dot{x}, \theta_x, y, \dot{y}, \theta_y) = (0.1, 0, 0, 0, 0, 0)\). It indicates that the TS
3.2. Fuzzy supervision controller

In order to guarantee that the ball does not hitting the boundaries when it travels, we add a fuzzy supervision controller that takes action in extreme situations. It is easy to get fuzzy control rules from the common experience, for example,

\[
\begin{align*}
\text{IF } e \text{ is PB1 (positive big), } & \text{ THEN } u_{s1} \text{ is PB2 (positive big);} \\
\text{IF } e \text{ is PS1 (positive small), } & \text{ THEN } u_{s1} \text{ is PS2 (positive small);} \\
\text{IF } e \text{ is ZR1 (zero), } & \text{ THEN } u_{s1} \text{ is ZR2 (zero);} \\
\text{IF } e \text{ is NS1 (negative small), } & \text{ THEN } u_{s1} \text{ is NS2 (negative small);} \\
\text{IF } e \text{ is NB1 (positive big), } & \text{ THEN } u_{s1} \text{ is NB2 (positive big).}
\end{align*}
\]

(15)

Where \( e = d_1 - d_2 \), \( d_1 \) and \( d_2 \) are the distances from the ball to the left and right boundaries, respectively. The universes of discourse of \( e \) and \( u_{s1} \) are \([-0.07, 0.07 \text{ m}]\) and \([-0.03, 0.03 \text{ arc/s}^2]\), respectively. For convenience, we introduce two scaling factors \( G_e = 14.3 \) and \( G_{u_s} = 30 \), such that the normalized input and output of the fuzzy supervision controller are both varied in \([-1, 1]\). The
normalized input $e$ and output $u_{s1}$ both adopt triangular-shaped, full-overlapped and equally spaced membership functions as shown in Fig. 6, where NB1, NS1, ZR1, PS1, PB1 are five fuzzy sets for input, while NB2, NS2, ZR2, PS2, PB2 are five fuzzy sets for output $u_{s1}$. Here we still adopt the Sum–Product method for fuzzy inference and the Center of Gravity method for defuzzification.

### 3.3. Fuzzy planning controller

Fuzzy planning controller should determine a desired trajectory in order to control the ball from point A through the path $A \rightarrow B \rightarrow C \rightarrow D \rightarrow E \rightarrow F \rightarrow G$ to point G without hitting the obstacles and in the least time. Now we will explain how to translate the human experience into the fuzzy control rules. Define the coordinates of B, C, D, E, and F are $[0.33, 0.33]$, $[0.33, 0.77]$, $[0.63, 0.77]$, $[0.63, 0.11]$ and $[0.84, 0.11]$, respectively. Then the general experience tell us (refer to Fig. 2):

- IF the distance to point B is small, THEN adjust the desired trajectory to left greatly;
- IF the distance to point C is small, THEN adjust the desired trajectory to right greatly;
- IF the distance to point D is small, THEN adjust the desired trajectory to right greatly;
- IF the distance to point E is small, THEN adjust the desired trajectory to left greatly;
- IF the distance to point F is small, THEN adjust the desired trajectory to left greatly.

(16)

Define:

$$\dot{x}_d = g_1,$$

$$\dot{y}_d = g_2.$$  

(17)

(18)

Notice that $g_1$ and $g_2$ are the derivatives of the desired trajectory in $x$ and $y$ directions, respectively. We can turn the ball right, only if we make $g_1$ positive and large, and $g_2$ near zero. So the rule set (16) is equivalent to:

- IF $d_B$ is small, THEN $g_2$ is positive large and $g_1$ is near zero;
- IF $d_C$ is small, THEN $g_1$ is positive large and $g_2$ is near zero;
- IF $d_D$ is small, THEN $g_2$ is negative large and $g_1$ is near zero;
- IF $d_E$ is small, THEN $g_1$ is positive large and $g_2$ is near zero;
- IF $d_F$ is small, THEN $g_2$ is positive large and $g_1$ is near zero.

(19)
Where \(d_B, d_C, d_D, d_E\) and \(d_F\) are the distances from the ball to point B, C, D, E, and F, respectively. The universes of discourse of \(d_B, d_C, d_D, d_E\) and \(d_F\) are all designed as [0, 0.14 m]. The universes of discourse of outputs \(g_1\) and \(g_2\) are \([-4, 4]\) m/s. For convenience, we introduce scaling factors \(G_d = 7.14\) and \(G_g = 0.25\), such that the inputs of the fuzzy planning controller are all varied in [0, 1], while the outputs in \([-1, 1]\). Suppose that normalized inputs \(d_B, d_C, d_D, d_E,\) and \(d_F\) all adopt five fuzzy sets for each, namely, Z, PSS, PS, PM, PL, and triangular-shaped, full-overlapped and equally spaced membership functions as shown in Fig. 7. Normalized outputs \(g_1\) and \(g_2\) adopt nine fuzzy sets named by NL, NM, NS, NSS, Z, PSS, PS, PM, PL with central values \(\delta_k = k/4\), where \(k = -4, -3, \ldots, 0, \ldots, 3, 4\), and triangular-shaped, full-overlapped and equally spaced membership functions as shown in Fig. 8. The final fuzzy control rules are summarized as follows:

\[
\begin{align*}
\text{IF } d_B \text{ is } (Z, \text{ PSS}, \text{ PS}, \text{ PM}, \text{ PL}), \\
\text{THEN } g_2 \text{ is } (\text{PL}, \text{ PM}, \text{ PS, PSS, Z}) \text{ and } g_1 \text{ is } (\text{Z, PSS, PS, PM, PL}); \\
\text{IF } d_C \text{ is } (Z, \text{ PSS}, \text{ PS, PM, PL}), \\
\text{THEN } g_1 \text{ is } (\text{PL, PM, PS, PSS, Z}) \text{ and } g_2 \text{ is } (\text{Z, PSS, PS, PM, PL}); \\
\text{IF } d_D \text{ is } (Z, \text{ PSS, PS, PM, PL}), \\
\text{THEN } g_2 \text{ is } (\text{NL, NM, NS, NSS, Z}) \text{ and } g_1 \text{ is } (\text{Z, PSS, PS, PM, PL}); \\
\text{IF } d_E \text{ is } (Z, \text{ PSS, PS, PM, PL}), \\
\text{THEN } g_1 \text{ is } (\text{PL, PM, PS, PSS, Z}) \text{ and } g_2 \text{ is } (\text{Z, NSS, NS, NM, NL}); \\
\text{IF } d_F \text{ is } (Z, \text{ PSS, PS, PM, PL}), \\
\text{THEN } g_2 \text{ is } (\text{Z, PSS, PS, PM, PL}) \text{ and } g_1 \text{ is } (\text{Z, PSS, PS, PM, PL}).
\end{align*}
\]

(20)

Here we still use Sum–Product method for fuzzy inference and Center of Gravity method for defuzzification.
In summary, the whole trajectory planning and tracking control system consists of
(a) The ball and plate system’s mathematical model (2)–(5);
(b) The fuzzy tracking and supervision controllers given by (10), (13), (15);
(c) The desired trajectory \((x_d(t), y_d(t))\) characterized by (17) and (18), where \(g_1\) and \(g_2\) are given by fuzzy planning controller (20).

The simulation result of the controlled Ball and Plate system is shown in Fig. 9. The traveling velocity of the ball remains 4 mm/s. When the ball closes to the points B, C, D, E, and F, the velocity in the former traveling direction decreases to zero and the velocity vertical to the former direction increases to 4 mm/s. At the same time, the supervision controller guarantees the ball does not touch the obstacles. Simulation results have shown that the hierarchical fuzzy control scheme is successful to control the ball from point A to point G without hitting the obstacles. But the trajectory is not optimal in any sense. The whole time used to finish this tour is 681.3 s.

4. Optimization of the moving trajectory

From the above simulation result we noticed that the trajectory at each turn is not satisfactory. This is mainly because that the fuzzy planning controller is designed just using human experience and its parameters are not optimized. Therefore, in this section we will show how to adjust the parameters \(\delta_k\) of fuzzy planning controller to shorten the ball’s moving trajectory.

Here we adopt the genetic algorithm (GA), which is a global optimization algorithm based on the mechanism of natural selection and heredity. It is easy to implement and efficient for multi-variable
optimization problems. In this case we only adjust the output variables’ membership functions of the fuzzy planning controller for convenience. The operation steps are briefly described as follows:

1. Define the “String of Genes” as \([\delta_1, \delta_2, \delta_3]\), see Fig. 8. It is required that \(0 < \delta_1 < \delta_2 < \delta_3 < 1\).
2. Define the “Fitness Function” as the total length of ball’s moving trajectory.
3. Randomly generate an “Initial Population” with the size equal to 20.
4. Generate the “Next Generation” using the operations of Reproduction, Crossover, and Mutation with the \(P_c = 0.1\) and \(P_m = 0.05\). When the highest fitness value remains the same over six generations, then the operation of GA will stop; otherwise, it will continue to generate the next generation.

In the simulation experiment, the optimal string of genes is found in the 24th generation, which is \([0.4002, 0.7293, 0.9713]\) as shown in Fig. 10. Fig. 11 shows the optimized moving trajectory, which is obviously improved by the genetic algorithm. The improvement can be more specified by the whole time 668.9 s used to finish the whole tour, compared with 681.3 s in the former one, that is 12.4 s are saved.

In this case, only the output variables’ membership functions of the fuzzy planning controller are optimized using the GA algorithm, it is quite reasonable to anticipate, if more parameters of 3 fuzzy controllers are optimized, the whole traveling time could be further shortened. About the GA algorithm, please refer [2] for details.

5. Conclusion

In this paper, we propose a hierarchical fuzzy control scheme to solve the trajectory planning and tracking problem of ball and plate system. This scheme is composed of three levels. The lowest level is a TS type fuzzy tacking controller; the middle level is a fuzzy supervision controller; and the top level is a fuzzy planning controller that determines the desired trajectory. In order to optimize the trajectory, the genetic algorithm is introduced to adjust the output membership function parameters of the fuzzy planning controller. Simulation results have shown that the hierarchical fuzzy system can control the ball from point A to point G without hitting the obstacles and in the quite short time.

Since the ball and plate system is the extension of the traditional ball and beam problem, in this paper we propose a trajectory planning and tracking problem for the ball and plate system.
This problem adds difficulty on the control and may be a very interesting example for studying and testing various control methods. Due to the time limitation, we could not fulfill more simulation experiments to compare the hierarchical fuzzy control scheme with alternative approaches. We would like to keep this problem open to all interested researchers. In addition, the simulation results can only demonstrate the proposed hierarchical fuzzy control scheme is feasible, for further test, the control experiments must be done on the real ball and plate system.

Appendix A. Derivation of the ball and plate system's mathematical model

Using the Lagrange method, the kinetic energy of the ball and plate system is:

\[
T = \frac{1}{2} m r^2 + \frac{1}{2} J_P (\dot{\theta_x}^2 + \dot{\theta_y}^2) + \frac{1}{2} J \left(\frac{\dot{r}}{R}\right)^2 \\
+ \frac{1}{2} \left(J + m r^2 \cos^2 \left(\arctan \frac{\dot{y}}{\dot{x}} - \arctan \frac{\dot{\theta_y}}{\dot{\theta_x}}\right)\right) (\dot{\theta_x}^2 + \dot{\theta_y}^2)
\]

(A.1)

in which

\[
r^2 = x^2 + y^2, \quad \dot{r}^2 = \dot{x}^2 + \dot{y}^2.
\]

\[
\therefore \cos^2 \left(\arctan \frac{\dot{y}}{\dot{x}} - \arctan \frac{\dot{\theta_y}}{\dot{\theta_x}}\right) = \frac{1}{1 + \tan^2 (\arctan \frac{\dot{y}}{\dot{x}} - \arctan \frac{\dot{\theta_y}}{\dot{\theta_x}})} = \frac{1}{1 + \left[\frac{y/x - \dot{\theta_y}/\dot{\theta_x}}{1 + (y/x)(\dot{\theta_y}/\dot{\theta_x})}\right]^2}
\]
\[
T = \frac{1}{2} m r^2 + \frac{1}{2} J_P (\dot{\theta}_x^2 + \dot{\theta}_y^2) + \frac{1}{2} J \left( \frac{\dot{r}}{R} \right)^2 + \frac{1}{2} \left( J (\dot{\theta}_x^2 + \dot{\theta}_y^2) + m (x \dot{\theta}_x + y \dot{\theta}_y)^2 \right),
\]
\[
= \frac{1}{2} (J_P + J) (\dot{\theta}_x^2 + \dot{\theta}_y^2) + \frac{1}{2} \left( m + \frac{J}{R^2} \right) (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} m (x \dot{\theta}_x + y \dot{\theta}_y)^2. \tag{A.2}
\]

Then, we have
\[
\frac{\partial T}{\partial \dot{\theta}_x} = (J_P + J) \dot{\theta}_x + m x (x \dot{\theta}_x + y \dot{\theta}_y), \quad \frac{\partial T}{\partial \dot{\theta}_y} = 0,
\]
\[
\frac{\partial T}{\partial \dot{x}} = \left( m + \frac{J}{R^2} \right) \dot{x}, \quad \frac{\partial T}{\partial \dot{y}} = m (x \dot{\theta}_x + y \dot{\theta}_y) \dot{\theta}_y. \tag{A.3}
\]

According to the Lagrange equation, we obtain
\[
\frac{d}{dt} \frac{\partial T}{\partial \dot{\theta}_x} - \frac{\partial T}{\partial \theta_x} = (J_P + J) \ddot{\theta}_x + m x \dot{\theta}_x + 2 m x \dot{x} \dot{\theta}_x + m x y \dot{\theta}_y + m x y \dot{\theta}_y + m x \dot{\dot{x}}
\]
\[
= \tau_x - m g x \cos \theta_x,
\]
\[
\tau_x = (J_P + J) \ddot{x} + 2 m x \dot{x} \dot{\theta}_x + m x y \dot{\theta}_y + m x y \dot{\theta}_y + m x \dot{\dot{x}} + m g x \cos \theta_x, \tag{A.4}
\]
\[
\tau_y = (J_P + J + m y^2) \ddot{\theta}_y + 2 m y \dot{y} \dot{\theta}_y + m x \dot{y} \dot{\theta}_x + m x y \dot{\theta}_x + m x \dot{\dot{y}} + m g y \cos \theta_y, \tag{A.5}
\]
\[
\frac{d}{dt} \frac{\partial T}{\partial \dot{x}} - \frac{\partial T}{\partial x} = \left( m + \frac{J}{R^2} \right) \ddot{x} - m (x \dot{\theta}_x + y \dot{\theta}_y) \dot{x} = -m g \sin \theta_x, \tag{A.6}
\]
\[
\left( m + \frac{J}{R^2} \right) \ddot{x} + m g \sin \theta_x - m x \dot{\theta}_x^2 - m y \dot{\theta}_y - m x \dot{\dot{x}} = 0, \tag{A.7}
\]
\[
\left( m + \frac{J}{R^2} \right) \ddot{y} + m g \sin \theta_y - m y \dot{\theta}_y^2 - m x \dot{\theta}_y - m y \dot{\dot{y}} = 0. \tag{A.8}
\]
Define state variables as

\[
X = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)^T = (x, \dot{x}, \theta, \dot{\theta}, y, \dot{y}, \dot{\theta}_y)^T.
\] (A.9)

The output of the system is

\[
Y = h(X) := (x, y)^T.
\] (A.10)

From Eqs. (A.4), (A.5), (A.7) and (A.8), we can get the Ball and Plate System’s state equation as follows:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4 \\
\dot{x}_5 \\
\dot{x}_6 \\
\dot{x}_7 \\
\dot{x}_8 \\
\end{bmatrix} =
\begin{bmatrix}
B(x_1 x_2^2 + x_4 x_5 x_8 - g \sin x_3) & x_2 \\
0 & x_4 \\
0 & x_6 \\
B(x_5 x_6^2 + x_7 x_4 x_8 - g \sin x_7) & x_8 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
\end{bmatrix}
\] (A.11)

in which \(B := m/m + J/R^2\). (A.12)

**Appendix B. Control gain parameters of the fuzzy tracking controller**

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<th>(j)</th>
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<th>(x_{j1})</th>
<th>(x_{j2})</th>
<th>(x_{j3})</th>
<th>(x_{j4})</th>
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<th>(k_{j2})</th>
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