SOME REMARKS ABOUT ARGUMENTATION AND MATHEMATICAL PROOF AND THEIR EDUCATIONAL IMPLICATIONS

Nadia Douek
I.U.F.M. de Creteil, Centre de Livry Gargan, 45 avenue J. Zay, 93190, Livry Gargan, France
ndouek@worldnet.fr

Abstract: *In spite of the undeniable epistemological and cognitive distance between argumentation and formal mathematical proof, argumentation and ordinary mathematical proof have many aspects in common, as processes and also as products. This paper aims at pointing out these aspects and sketching some educational implications of the fact of taking them into account.*
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1. Introduction

The general motivation for this study comes from the need to call into question the idea, widely shared among teachers and mathematics educators, that there are profound differences between mathematical thinking and thinking in other domains, and that these differences produce many difficulties in learning mathematics. In particular, some mathematics educators think that one of the main difficulties students face in approaching mathematical proof (one of the most characteristic and important mathematics subject) lies in their inability to grasp the differences between ordinary argumentation and mathematical proof. This position has been clearly presented by Duval (1991), and we will refer systematically to him in this article:

> "Deductive thinking does not work like argumentation. However these two kinds of reasoning use very similar linguistic forms and propositional connectives. This is one of the main reasons why most of the students do not understand the requirements of mathematical proof".

Duval's analysis (see 2.2.) offers a precise cognitive perspective for "formal proof" (i.e. proof reduced to a logical calculation). It suggests the following questions:

- What are the relationships between formal proof and proofs really performed in mathematics, in school mathematics as well as in the history of mathematics and the mathematics of modern-day mathematicians?
- What are the relationships between mathematical proof (as a written communication product), and the working mathematician's process of proving?
- In spite of the superficial analogies and profound differences between argumentation and formal proof, aren't there some deep connections between argumentation and mathematical proof (as products and as processes)?
- If those connections do exist, how can we take them into account, in order to manage the approach of students to mathematical proof?

In this paper I will try to explore only some aspects of these questions and show their relevance for the "culture of mathematical proof", which should be developed in teacher training and also for some direct educational implications. I will try to show how proving and arguing, as processes, have many common aspects from the cognitive and epistemological points of view, though significant differences exist between them as socially situated products. To provide evidence of the common aspects, it should be necessary:

- to compare the processes of producing an argumentation and producing a proof;
- to analyse the nature and limits of the reference corpus backing an argumentation and that backing specifically a mathematical proof;
- to differenciate the process of creating a proof and the product which is a written communicable object;
- to analyse the process of creation of proof and the process of writing a proof within social constraints;
- to analyse the structure of proof texts as particular argumentative texts.

This paper concerns mainly the first three points. I will focus only on "grounding connections" between argumentation and mathematical proof (see 3. and 4.) - although differences should be taken into account as well! I will also try to sketch some educational implications of my analysis (see 5.).

My approach will be phenomenological, i.e. I will consider how argumentation and mathematical proof "live" in different settings, today and in the past. My analysis will be mainly inspired by Thurston (1994) as concerns modern-day mathematicians' proofs. I will also refer to Lakatos (1985) as concerns definitions and proofs in the history of mathematics; Balacheff (1988), Bartolini et al. (1997) Arzarello et al. (1998); Simon (1996), Boero et al. (1996) and Harel & Sowder (1998) as concerns some epistemological, cognitive and educational aspects of proving; Lakoff and Nunez (1997) as concerns the idea of everyday experience as "grounding metaphor" for mathematics concepts; and Granger (1992) as concerns the relationships between formal proof and verification in mathematics.

2. About argumentation and proof

This section is intended to provide the reader with reference definitions and basic ideas for the following sections.

2.1 What argumentation are we talking about?

We cannot accept any discourse as an argumentation. In this paper, the word "argumentation" will indicate both the process which produces a logically connected (but not necessarily deductive) discourse about a given subject (from Webster Dictionary: *"1. The act of forming reasons, making inductions, drawing conclusions, and applying them to the case under discussion")* and the text produced by that process (Webster: *"3. Writing or speaking that argues"*). On each occasion, the linguistic context will allow the reader to select the appropriate meaning.

The word "argument" will be used as *"A reason or reasons offered for or against a proposition, opinion or measure"* (Webster), and may include verbal arguments, numerical data, drawings, etc. In brief, an "argumentation" consists of one or more logically connected "arguments". We may state that the discoursive nature of argumentation does not exclude the reference to non-discoursive (for instance, visual or gestural) arguments.

2.2 Formal proof

In this paper we will consider "formal proof" as proof reduced to a logical calculation. Duval (1991) offers a precise cognitive perspective for "formal proof".

Duval performs *"a cognitive analysis of deductive organisation versus argumentative organisation of reasoning"*. I will quote some points here:

- as concerns "*inference steps*": in argumentative reasoning, "*semantic content of propositions is crucial*", while in deductive reasoning "*propositions do not intervene directly by their content, but by their operational status*" (defined as "*their role in the functioning of inference*");
- as concerns "*enchaining steps*": argumentative reasoning works "*by reinforcement or opposition of arguments*". "*Propositions assumed as conclusions of preceding phases or as shared propositions are continuously reinterpreted*". "*The transition from an argument to another is performed by extrinsic connection*". On the contrary, in deductive reasoning "*the conclusion of a given step is the condition of application of the inference rule of the following step*". The proposition obtained as the conclusion of a given step is "*recycled*" as the entrance proposition of the following step. Enchaining makes deductive reasoning similar to a chain of calculations.
- as concerns the "*epistemic value*" (defined as the "*degree of certainty or convinction attributed to a proposition*"): in argumentative reasoning "*true propositions have not the same epistemic value*", while in mathematics "*true propositions have only one, specific epistemic value [...] - that is, certainty deriving from necessary conclusion"; and "proof modifies the epistemic value of the proved proposition: it becomes true and necessary*". This modification constitutes the "productivity of proof".

2.3 Mathematical proof

We could start by saying that mathematical proof is what in the past and today is recognized as such by people working in the mathematical field. This approach covers Euclid's proof as well as the proofs published in high school mathematics textbooks, and current modern-day mathematicians' proofs, as communicated in specialized workshops or published in mathematical journals (for the differences between these two forms of communication, see Thurston, 1994). We could try to go further and recognize some common features, in particular: a common function, i.e. the validation of a statement; the reference to an established knowledge (see the definition of "theorem" as "*statement, proof and reference theory*" in Bartolini et al., 1997); and some common requirements, like the enchaining of propositions.

We must distinguish between the process of proof construction (i.e. "proving") and the result (as a socially acceptable mathematical text): for a discussion, see 4.

This distinction and preceding considerations point out the fact that mathematical proof can be considered as a particular case of argumentation (according to the preceding definition). However, in this paper "argumentation" will exclude "proof" when we compare them.

Concerning the relationships between formal proof and proofs currently produced by mathematicians, we may quote Thurston:

> "We should recognize that the humanly understandable and humanly checkable proofs that we actually do are what is most important to us, and that they are quite different from formal proof. For the present, formal proofs are out of reach and mostly irrelevant: we have good human processes for checking mathematical validity".

We may also consider some examples of theorems in mathematical analysis (e.g. Rolle's Theorem, Bolzano-Weierstrass' Theorem, etc.) whose usual proofs in current university textbooks are formally incomplete: completion would bring students far from understanding; for this reason semantic (and visual) arguments are frequently exploited in order to fill the gaps existing at the formal level.

2.4 Argumentation in mathematics

Argumentation can be performed in pure and applied mathematical situations, as in any other area. Argumentations are usually held informally between mathematicians to develop, discuss or communicate mathematical problems and results, but are not recognised socially in a research paper presenting new results: in that case proofs and “rigorous” constructions (or counter-examples) are needed.

As concerns “communication”, Thurston writes:

> “Mathematical knowledge can be transmitted amazingly fast within a subfield of mathematics. When a significant theorem is proved, it often (but not always) happens that the solution can be communicated in a matter of minutes from one person to another within the subfield. The same proof would be communicated and generally understood in an hour’s talk to members of the subfield. It would be the subject of a 15- or 20- page paper, which could be read and understood in a few hours or perhaps days by the members of the subfield. Why is there such a big expansion from the informal discussion to the talk, to the paper? One-to-one, people use wide channels of communication that go far beyond formal mathematical language. They use gestures, they draw pictures and diagrams, they make sound effects and use body language”

As concerns “rigour”, it is considered here because it appears as a requirement of mathematical texts although it needs to be defined or rather to be questioned and historicised - see Lakatos (1985). The problem of rigour will be reconsidered later with the question of the epistemic value of statements.

2.5 Reference corpus

The expression “reference corpus” will include not only reference statements but also visual and, more generally, experimental evidence, physical constraints, etc. assumed to be unquestionable (i. e. “reference arguments”, or, briefly, “references”, in general). In Section 4.1. I will discuss the social determination of the fact that a “reference” is not questioned, as well as the necessary existence of references which are not made explicit.

2.6 Tools of analysis and comparison of argumentation and proof

Aren’t there some criteria (even implicit ones) that enable us to accept or refuse an argumentation, as it happens for a proof? And are they not finally related to logical constraints and to the validity of the references, even if entangled with complex implicit knowledge? If we follow Duval's analysis, for argumentation it seems as if there is no recognised reference corpus for argumentation, whereas for proof it exists systematically. I do not think that this distinction is correct. The following criteria of comparison, inspired by Duval's analysis, will help me to argue this point in the next section: the existence of a "reference corpus" for developing reasoning; the means by which doubts about the "epistemic value" of a given statement can be dispelled; and the form of reasoning.

3. Analysis and comparison of argumentation and proof as products

3.1 About the reference corpus

No argumentation (individual or between two or more protagonists) would be possible in everyday life if there were no reference corpus to support the steps of reasoning. The reference corpus for everyday argumentation is socially and historically determined, and is largely implicit. Mathematical proof also needs a "reference corpus". We could think that this "reference corpus" is completely explicit and not socially determined, but we will see that this is not true.

A) Social and historical determination of the "reference corpus" for proof

In this subsection I will try to support the idea that the "reference corpus" for mathematical proof is socially and historically determined. In order to do so, I will exploit arguments of different nature (historical and epistemological considerations as well as reflections on ordinary school practices) that are not easy to separate.

The reference corpus used in mathematics depends strongly on the users and their listeners/readers. For example, in secondary school some detailed references can be expected to support a proof, but in communication between higher level mathematicians those may be considered evident and as such disregarded. As Yackel and Cobb (1998) pointed out, the existence of jumps related to "obvious" arguments in the presentation of a proof can be considered as a sign of familiarity with knowledge involved in that proof. On the other hand, some statements accepted as references in secondary school are questioned and problematised at higher levels; questions of "decidability" may surface. We may remark that today problems of "decidability" are dealt with by few mathematicians and seldom encountered in mathematics teaching (although in my opinion simple examples concerning euclidean geometry vs non euclidean geometries could be of great pedagogical value). Let us quote Thurston:

> "On the most fundamental level, the foundations of mathematics are much shakier than the mathematics that we do. Most mathematicians adhere to foundational principles that are known to be polite fictions".

Thus for almost all the users of mathematics in a given social context (high school, university, etc.) the problem of epistemic value does not exist (with the exception of the case: "true" after proving, or "not true" after counter-example) although it was and it still is an important question for mathematics as for any other field of knowledge. Mathematics concepts are the most stable, giving an experience of "truth" which should not be necessarily taken for truth.

Let us now consider other aspects of the social determination of the reference corpus which concern the nature of references. If we consider the "references" that can back an argumentation for validating a statement in primary school, we see that at this level of approach to mathematical work references can include experimental facts. And we cannot deny their "grounding" function for mathematics (see Lakoff and Nunez, 1997), both for the long term construction of mathematical concepts and for establishing some requirements of validation which prepare proving (e.g. making reference to acknowledged facts, deriving consequences from them, etc.). For instance, in primary school geometry we may consider the superposition of figures for validating the equality of segments or angles, and superposition by bending for validating the existence of an axial symmetry. Later on in secondary school, these references no longer have value in proving; they are replaced by definitions or theorems (see Balacheff, 1988). For instance, in order to prove that an axial symmetry exists, reference can be made to the definition of symmetry axis as the axis of segments joining corresponding points. For older students, similar examples can be found in the field of discrete mathematics, where a lot of familiar statements which are necessary to build a proof are not part of the elementary axiomatics.

In general, at a higher level it is a hypothesis or a partial result of the problem to solve that informs us of equalities, and not "experimental" validation (see Balacheff, 1988). At such a level the meaning of equality is not questioned as might (and should) happen at "lower" levels. We may note that, in the history of mathematics, visual evidence supports many steps of reasoning in Euclid's "Elements". This evidence was replaced by theoretical constructions (axioms, definitions and theorems) in later geometrical theories.

B) Implicit and explicit references

The reference corpus is generally larger than the set of explicit references. In mathematics, as in other areas, the knowledge used as reference is not always recognised explicitly (and thus appears in no statement): some references can be used and might be discovered, constructed, or reconstructed, and stated afterwards. The example of Euler's theorem discussed by Lakatos (1985) provides evidence about this phenomenon in the history of mathematics. The same phenomenon also occurs for argumentation concerning areas other than mathematics. Let us consider the interpretations made by a psychoanalyst: we cannot fathom his ability unless we believe that he bases his work on chains of reasoning that refer to a great deal of shared knowledge about mankind and society, this knowledge being obviously impossible to reduce to explicit knowledge. And, in general, we could hold no exchange of ideas, whatever area we are interested in, without exploiting implicit shared knowledge. Implicit knowledge, which we are generally not conscious of, is a source of important "limit problems" (especially in non mathematical fields, but also in mathematics): in the "fuzzy" border of implicit knowledge we can meet the challenge of formulating more and more precise statements and evaluating their epistemic value. Lakatos (1985) provides us with interesting historical examples about this issue.

3.2 How to dispel doubts about a statement and the form of reasoning

Thurston writes:

> "Mathematicians can and do fill in gaps, correct errors, and supply more detail and more careful scholarship when they are called on or motivated to do

so. Our system is quite good at producing reliable theorems that can be solidly backed up. It's just that the reliability does not primarily come from mathematicians formally checking formal arguments; it comes from mathematicians thinking carefully and critically about mathematical ideas".

And considering the example of Wiles's proof of Fermat's Last Theorem:

"The experts quickly came to believe that his proof was basically correct on the basis of high-level ideas, long before details could be checked".

These quotations raise some interesting questions concerning the ways by which doubts about mathematics statements are dispelled. Formal proof "produces" (according to Duval's analysis) the reliability of a statement (attributing to it the epistemic value of "truth") . But what Thurston argues is that "*reliability does not primarily come from mathematicians formally checking formal arguments*". In Thurston's view, the requirements of formal proof represent only guidelines for writing a proof - once its validity has been checked according to "substantial" and not "formal" arguments. The preceding considerations directly concern the form of reasoning: the model of formal proof as described by Duval and based on the "operational status" of propositions rather than on their "semantic content" does not seem to fit the description of the activities performed by many working mathematicians when they check the validity of a statement or a proof. Only in some cases (for instance, proofs based on chains of transformations of algebraic expressions) does Duval's model neatly fit proof as a product.

Despite the distance between the ways of dispelling doubts (and the forms of reasoning) in mathematics and in other fields, the preceding analysis shows many points of contact - even between mathematical proof and argumentation in non-mathematical fields. Granger (1992) suggests the existence of deep analogies which might frame (from an epistemological point of view) these points of contact. Naturally, as concerns the form of reasoning visible in the final product, argumentation presents a wider range of possibilities than mathematical proof: not only deduction, but also analogy, metaphor, etc. Another significant difference lies in the fact that an argumentation can exploit arguments taken from different reference corpuses which may belong to different theories with no explicit, common frame ensuring coherence.

For instance, the argumentation developed in this paper derives its arguments from different disciplines (history of mathematics, epistemology, cognitive psychology); at present there is no mean to tackle the problem of coherence between reference theories belonging to these domains. On the contrary, mathematical proof refers to one or more reference theories explicitely related to a coherent system of axiomatics. But I would prefer to stress the importance of the points of contact (especially from an educational point of view: see 5.).

4. The processes of argumentation and construction of proof

In 2.3. I proposed distinguishing between the process of construction of proof ("proving") and the product ("proof"). Of what does the "proving" process consist? Experimental evidence has been provided about the hypothesis that "proving" a conjecture entails establishing a functional link with the argumentative activity needed to understand (or produce) the statement and recognizing its plausibility (see Bartolini et al., 1997). Proving itself needs an intensive argumentative activity, based on "transformations" of the situation represented by the statement. Experimental evidence about the importance of "transformational reasoning" in proving has been provided by various, recent studies (see Arzarello et al., 1998; Boero et al., 1996; Simon, 1996; Harel and Sowder, 1998). Simon defines "transformational reasoning" as follows:

> "the physical or mental enactment of an operation or set of operations on an object or set of objects that allows one to envision the transformations that these objects undergo and the set of results of these operations. Central to transformational reasoning is the ability to consider, not a static state, but a dynamic process by which a new state or a continuum of states are generated".

It is interesting to compare Simon's definition with Thurston (1994):

> "People have amazing facilities for sensing something without knowing where it comes from (intuition), for sensing that some phenomenon or situation or object is like something else (association); and for building and testing connections and comparisons, holding two things in mind at the same time (metaphor). These facilities are quite important for mathematics. Personally, I

> put a lot of effort into ”listening" to my intuitions and associations, and building them into metaphors and connections. This involves a kind of simultaneous quietening and focusing of my mind. Words, logic and detailed pictures rattling around can inhibit intuitions and associations". And then: “We have a facility for thinking about processes or sequences of actions that can often be used to good effect in mathematical reasoning”.

These quotations open some interesting research questions about metaphors:

- What are the relationships between metaphors and transformational reasoning in mathematical activities, especially in proving?
- What is the role of physical and body referents (and metaphors) in conjecturing and proving?

Metaphors can be considered as particular outcomes of transformational reasoning. For a metaphor we may consider two poles (a known object, an object to be known) and a link between them. In this case the “creativity” of transformational reasoning consists in the choice of the known object and the link - which allows us to know some aspects of the unknown object as suggested by the knowledge of the known object (“abduction”)(cf. Arzarello et al., 1998).

Coming to the second question, we may remark that mathematics “officially” concerns only mathematical objects. Metaphors where the known pole is not mathematical are not acknowledged. But in many cases the process of proving needs these metaphors, with physical or even bodily referents (sometimes their traces can be detected when a mathematician produces an informal description of the ideas his proof is based on: see 2.3., quotation from Thurston). In general, Lakoff and Nunez (1997) suggest that these metaphors have a crucial role in the historical and personal development of mathematical knowledge (“*grounding metaphors*”). The example of continuity is illuminating. Simon (1996) discusses the importance of a physical enactment in order to check the results of a transformation in transformational reasoning. In some situations the mathematical object is the known object and the other pole concerns a non-mathematical situation: the aim is to validate some statements concerning this situation, exploiting the properties of the mathematical model. In other cases it happens to relate two non mathematical objects (one known, the other not) by a mathematical, metaphoric link which sheds light on the unknown object and/or on its relationships with the known object. By these means, argumentative activities concerning non-mathematical situations rely upon mathematical creations (metaphors), an observation that should be taken into account in mathematics education (see 5.).

The example of metaphors shows the "semantic" complexity of the process of proving - and suggests the existence of other links with other mathematical activities. It also shows the importance of transformational reasoning as a free activity (in particular, free from usual boundaries of knowledge). However, metaphors represent only one side of the process of proving. Induction in general is relevant - and the need to produce a deductive chain guides the search for arguments to "enchain" when coming to the writing process (see Boero et al., 1996).

5. Some educational implications

Let us come back to the processes of argumentation and proof construction as opposed to the final static results. In my opinion, an important part of the difficulties of proof in school mathematics comes from the confusion of proof as a process and proof as a product, and results in an authoritarian approach to both activities. Frequently, mathematics teaching is based on the presentation (by the teacher, and then by the student when asked to repeat definitions and theorems) of mathematical knowledge as a more or less formalized theory based on rigorous proofs. In this case, authority is exercised through the form of the presentation (see Hanna, 1989); in this way school imposes the form of the presentation over the thought, leads to the identification between them and demands a thinking process modelled by the form of the presentation (eliminating every "dynamism"). This analysis may explain the strength of the model of proof, which gives value to the idea of the "linearity" of mathematical thought as a necessity and a characteristic aspect of mathematics.

If a student (or a teacher) assumes such "linearity" as the model of mathematicians' thinking without taking the complexity of conjecturing and proving processes into account, it is natural to see "proof" and "argumentation" as extremely different. But it is also important to consider the consequence of such a conception in other fields: it can reinforce a style of “thinking” for which no “sacred” assumption is challenged, only “deductions” are allowed (obviously, also school practice of argumentation may suffer from authoritarian models!). On the contrary, giving importance to “transformational” reasoning (and, in general, to non-deductive aspects of argumentation needed in constructive mathematical activities - including proving) can develop different potentials of thinking. On the possibility of educating manners of thinking other than deduction, Simon considers “transformational reasoning” and hypothesizes:

> “[...] transformational reasoning is a natural inclination of the human learner who seeks to understand and to validate mathematical ideas. The inclination [...] must be nurtured and developed.[...]school mathematics has failed to encourage or develop transformational reasoning, causing the inclination to reason transformationally to be expressed less universally.”

I am convinced that Simon’s assumption is a valid working hypothesis, needing further investigation not only regarding “*the role of transformational reasoning in classroom discourse aimed at validation of mathematical ideas*” but also its functioning and its connections with other “creative” behaviours (in mathematics and in other fields).

As concerns possible educational developments, the analyses perfomed in this paper suggest some immediate consequences:

- classroom work should include (before any “institutionalisation”) systematic activities of argumentation about work that has been done;
- validation, in mathematical work and in other fields, should be demanded whenever it can be meaningful;
- the fact that validation has not been done or was unsatisfactory or impossible should be openly recognized;
- and, finally, references as such should be explicitly recognized, be they statements, experiments or axioms (this does not mean that references are fixed as true once and for all, but rather that for at least a certain time we have to consider them as “references” for our reasoning).

The passage from argumentation to proof about the validity of a mathematical statement should openly be constructed on the basis of limitation of the reference corpus (see 3.2., last paragraph). It could be supported by exploiting different texts, such as historical scientific and mathematical texts, and different modern mathematical proofs (see Boero et al., 1997, for a possible methodology of exploitation).

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