POSITIVE SOLUTIONS FOR NONLINEAR FIFTH-ORDER DIFFERENTIAL EQUATIONS WITH NONLOCAL BOUNDARY CONDITIONS

Amir Elhaffaf and Mostepha Naceri
Department of Mathematics
Faculty of Science
Oran University
BP1524, Es-senia, Algeria
e-mail: Elhaffaf1@yahoo.com

Department of Economics, Commercial and Management Sciences
Preparatory School of Oran, Algeria
e-mail: nacerimostepha@yahoo.fr

Abstract

In this paper, some conditions are derived on the nonlinear function \( f(u) \) so that two boundary value problems associated with the fifth-order differential equation \( u^{(5)}(t) = f(u) \) have one or more positive solutions.

1. Introduction

Nonlinear fifth-order boundary value problems are one of the most important problems arising in the mathematical modeling of viscoelastic
fluid flows viz-a-viz in various branches of physical sciences and engineering [4, 8, 10]. One of the most frequently used tools for proving the existence of positive solutions to the boundary value problems Krasnoeslskii’s theorem on cone expansion and compression and its norm-type version due to Guo [7]. There are many applications for fifth-order boundary value problems [2, 3, 4, 5, 6, 9]. The purpose of this paper is to establish the existence of positive solution to nonlinear fifth-order boundary value problems:

\[
\begin{align*}
\begin{cases}
    u^{(5)}(t) = f(u), & 0 < t < 1, \\
    u(0) = u'(0) = u'(1) = u''(0) = 0, \\
    a_1 u'(\alpha) + b_1 u''(\alpha) = 0,
\end{cases}
\end{align*}
\]

where \(a_1 \geq 0, b_1 > 0, a_1 + b_1 > 0, 0 < \alpha < 1\).

Hereafter called Problem 1.

\[
\begin{align*}
\begin{cases}
    u^{(5)}(t) = -f(u), & 0 < t < 1, \\
    u(0) = u'(0) = u'(1) = u''(0) = 0, \\
    a_2 u''(\beta) + b_2 u'''(\beta) = 0,
\end{cases}
\end{align*}
\]

where \(a_2 \geq 0, b_2 > 0, a_2 + b_2 > 0, 0 < \beta < 1\).

Hereafter called Problem 2.

We shall assume that \(f(u) \geq 0\), so the sign change from 1 to 2 is significant. We seek conditions on the nonlinear function \(f(u)\) for which the above problems have one or more positive solutions.

For comparison with our results below, we include a statement of the basic theorem [8].

**Theorem 1.** Suppose that \(0 < a < b < \frac{c}{2}\) and \(f : R \to [0, \infty)\) is continuous and satisfies

\[
\text{...}
\]
Positive Solutions for Nonlinear Fifth-order Differential Equations … 223

1. \( f(u) < 8a \) for \( 0 \leq u \leq a \),

2. \( f(u) > 16a \) for \( b \leq u \leq 2b \),

3. \( f(u) \leq 8c \) for \( 0 \leq u \leq c \).

Then the following problem:

\[
\begin{cases}
  u^{(5)}(t) = f(u), & 0 < t < 1, \\
  u(0) = u'(0) = u''(1) = u'''(0) = u''''(1) = 0
\end{cases}
\]

has three symmetric positive solutions \( u_1, u_2, u_3 \) with \( \max_{0 \leq t \leq 1} u_1(t) < a, u_2\left(\frac{1}{2}\right) > a, u_2\left(\frac{1}{4}\right) < b, u_3\left(\frac{1}{4}\right) > b \).

A number of different methods have been used on these problems see [8]. Leggett and Williams proved an abstract fixed point theorem implying the existence of three solutions and obtained Parter’s result as corollary. Graef, Qian, and Yang and others have used Krasnoeslskii’s fixed point theorem. In 2005, Baxley et al. [3] considered the same problems with local boundary conditions: \( y(0) = y'(0) = y''(0) = y'''(0) = y''''(1) = 0 \) and \( y(0) = y'(0) = y''(0) = y''''(0) = y''''(1) = 0 \).

Inspired and motivated greatly by the above mentioned works, the present work may be viewed as a direct attempt to extend to a variety of other boundary value problems of other arrangements of boundary conditions.

2. Preliminaries

For the convenience of the reader, we present here some notations and lemmas that will be used in the proof our main results.

Definition 2. Let \( E \) be a real Banach space. A nonempty closed convex set \( K \subset E \) is called cone if it satisfies the following two conditions:

1. \((\alpha u + \beta v) \in K \) for all \( u, v \in K \) and all \( \alpha, \beta \geq 0 \)
and

2. $u \in K$ and $-u \in K \Rightarrow u = 0$.

**Theorem 3** (Krasnoesliskii’s). Let $(\mathbb{E}, \| \cdot \|)$ be a Banach space and $K \subset \mathbb{E}$ be a cone in $\mathbb{E}$. Assume that $\Omega_1$ and $\Omega_2$ are open with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that either:

1. $\| Tu \| \leq \| u \|$, $\forall u \in K \cap \partial \Omega_1$ and $\| Tu \| \geq \| u \|$, $\forall u \in K \cap \partial \Omega_2$

or

2. $\| Tu \| \geq \| u \|$, $\forall u \in K \cap \partial \Omega_1$ and $\| Tu \| \leq \| u \|$, $\forall u \in K \cap \partial \Omega_2$.

Then $T$ has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Notation and Auxiliary Results

To apply our results successfully, we must design a Banach space, describe a cooperative cone and two bounded open sets and then verify some inequalities. For our Banach space $E$, we take $C[0, 1]$. Then we have to transform each of our problems into a fixed point problem for some operator. We prepare for this task by letting $l$ denote the differential operator defined on the set $C^1_0[0, 1]$ of continuously differentiable functions $u$ defined on $[0, 1]$ satisfying $u(0) = 0$ by $lu = u'$ and let $\tau$ denote the differential operator defined on the set $C^2_{(0,1)}[0, 1]$ of twice continuously differentiable functions $u$ defined on $[0, 1]$ satisfying $u(0) = u(1) = 0$ by $\tau u = -u''$, then Problem 1 takes form

$$(\tau l o t o l)u = f(u)$$

and Problem 2 takes the form

$$(l o t o l)u = f(u).$$
Positive Solutions for Nonlinear Fifth-order Differential Equations … 225

Note that the negative sign has now disappeared in the second problem. It is easy to see that \( l \) and \( \tau \) are inverted by the following integral operators:

\[
(Lu)(t) = \int_0^t u(s)\,ds, \tag{5}
\]

\[
(Tu)(t) = \int_0^1 G(t, s)u(s)\,ds, \tag{6}
\]

where

\[
G(t, s) = \begin{cases} 
  s(1-t), & 0 \leq s \leq t, \\
  t(1-s), & t \leq s \leq 1
\end{cases}
\]

(7)

is the Green’s function for \( \tau \).

Note that for \( u \) in the domain of \( l \), we have \( L(u') = u \) and for \( u \) in the domain of \( \tau \), we have \( T(u'') = -u \).

Now our two problems can be re-written in the form

\[
A_1(u) \equiv (LoToTof)u = u \tag{8}
\]

and

\[
A_2(u) \equiv (LoLoToLo)u = u \tag{9}
\]

each a fixed point problem. So we need to find fixed point of each of the operators \( A_i \) for \( i = 1, 2 \) on \( C[0, 1] \).

It is straightforward to verify that each of these operators is completely continuous on \( C[0, 1] \).

**Lemma 4.** If \( u \in E \) satisfies the boundary conditions of Problem 1 and let \( z = A_1(u) \). Then the following holds: \( z'(\alpha)\left(\frac{1}{\alpha} + \frac{a_1}{b_1}\right) > 0 \) and \( z(1) \leq q_1z(\alpha) \), where \( q_1 = \frac{b_1(2-\alpha) - a_1(1-\alpha)^2}{\alpha b_1} \).
Proof. Let \( z = A_1(u) \). Among other assumptions, we will always assume that \( f(u) \geq 0 \). Then \(-z'' = (Tof')(u) \geq 0\). Thus, \( z'' \leq 0 \) on \((0, 1)\) and \( z'' \) is non-increasing whence

\[
z'(t) = \int_0^t z''(s) \, ds \geq tz''(t).
\]

For \( t = \alpha \), we get \( z''(\alpha) \leq \frac{z'(')(\alpha)}{\alpha} \) and by the relation \( a_1 z'(\alpha) + b_1 z''(\alpha) = 0 \), we obtain \( z'(\alpha) \left( \frac{1}{\alpha} + \frac{a_1}{b_1} \right) > 0 \).

We also obtain for \( \alpha \leq t \leq 1 \)

\[
z'(t) - z'(\alpha) = \int_\alpha^t z''(s) \, ds \leq z''(\alpha)(t - \alpha)
\]

\[
= -\frac{a_1}{b_1} z'(\alpha)(t - \alpha) \leq z'(\alpha)(\alpha - t) \frac{a_1}{b_1}.
\]

Then \( z(t) = \int_0^t z''(s) \, ds \geq \int_0^t sz''(s) \, ds \).

Integrating by parts leads to

\[
z(t) \geq tz'(t) - z(t)
\]

or

\[
2z(t) \geq tz'(t).
\]

Thus

\[
z'(\alpha) \leq \frac{2}{\alpha} z(\alpha). \quad (11)
\]

Finally

\[
z(1) - z(\alpha) = \int_\alpha^1 z'(s) \, ds \leq \frac{z'(\alpha)}{b_1} \int_\alpha^1 (a_1(\alpha - s) + b_1) \, ds
\]

\[
\leq \frac{z'(\alpha)}{b_1} \left( a_1 - b_1 - \frac{a_1\alpha^2}{2} + b_1 - \frac{a_1}{2} \right).
\]
And using the relation (11), we have
\[
z(1) - z(\alpha) \leq \frac{2z(\alpha)}{\alpha b_1} \left[ \alpha(a_1 - b_1) - \frac{a_1 \alpha^2}{2} + b_1 - \frac{a_1}{2} \right]
\]
so that
\[
z(1) \leq z(\alpha) \left[ \frac{b_1(2 - \alpha) - a_1(1 - \alpha)^2}{\alpha b_1} \right].
\]

**Lemma 5.** If \( u \in E \) satisfies the boundary conditions of Problem 2 and let \( z = A_2(u) \). Then the following holds: \( z^*(\beta) \left( \frac{1}{\beta} + \frac{a_2}{b_2^2} \right) > 0 \) and \( z(1) \leq q_2 z(\alpha) \), where \( q_2 = \frac{b_2(3 - 3 \beta + \beta^2) - a_2(1 - \beta)^3}{\beta^2 b_2} \).

**Proof.** Let \( z = A_2(u) \). We see that \( -z^{(4)} = (Lof)(u) \geq 0 \). Thus, \( z^{(4)} \leq 0 \) on \((0, 1)\) and \( z^{(n)}\) is non-increasing, whereas for \( A_1 \) it was the second derivative which was non-increasing.

\[
z^{(n)}(t) = \int_0^t z^{(n)}(s) ds \geq tz^{(n)}(t).
\]

For \( t = \beta \), we get \( z^{(n)}(\beta) \leq \frac{z^{(n)}(\beta)}{\beta} \) and by the relation \( a_2 z^*(\beta) + b_2 z^{(n)}(\beta) = 0 \), we obtain
\[
z^*(\beta) \left( \frac{1}{\beta} + \frac{a_2}{b_2^2} \right) > 0. \quad (12)
\]

We also obtain for \( \beta \leq t \leq 1 \)
\[
z^{(n)}(t) - z^*(\beta) = \int_{\beta}^t z^{(n)}(s) ds \leq z^*(\beta)(t - \beta) = -\frac{a_2}{b_2} z^*(\beta)(t - \beta) \quad (13)
\]
from which
\[ z''(t) \leq z''(\beta) \left( \frac{a_2(\beta - t) + b_2}{b_2} \right). \quad (14) \]

Then \( z'(t) = \int_0^t z''(s) \, ds \geq \int_0^t sz''(s) \, ds. \)

Integrating by parts leads to

\[ z'(t) \geq tz''(t) - z'(t) \]

or

\[ 2z'(t) \geq tz''(t). \]

Thus

\[ z''(\beta) \leq \frac{2}{\beta} z'(\beta). \quad (15) \]

We also obtain for \( \beta \leq t \leq 1 \) by using relation (14)

\[ z'(t) - z'(\beta) = \int_{\beta}^{t} z''(s) \, ds \leq \frac{z''(\beta)}{b_2} \int_{\beta}^{t} (a_2(\beta - s) + b_2) \, ds. \]

And by relation (15), we have

\[ z'(t) - z'(\beta) \leq \frac{2z'(\beta)}{\beta b_2} \left[ t(a_2\beta + b_2 - \frac{a_2t}{2}) - \beta(b_2 + \frac{a_2\beta}{2}) \right]. \]

Thus

\[ z'(t) \leq z'(\beta) \left[ \frac{t(2a_2\beta + 2b_2 - a_2t) - \beta(b_2 + a_2\beta)}{b_2} \right]. \quad (16) \]

Then \( z(t) = \int_0^t z'(s) \, ds \geq \int_0^t \frac{sz''}{2}(s) \, ds. \)

Integrating by parts leads to

\[ z(t) \geq \frac{1}{2}(tz'(t) - z(t)) \]

or

\[ 3z(t) \geq tz'(t). \]
Thus
\[ z'(\beta) \leq \frac{3}{\beta} z(\beta). \] (17)

Finally
\[ z(1) - z(\beta) = \int_{\beta}^{1} z'(s) \, ds. \]

By using relation (16), we have
\[ z(1) - z(\beta) \leq \frac{z'(\beta)}{\beta b_2} \int_{\beta}^{1} \left[ t(2a_2 \beta + 2b_2 - a_2 t) - \beta(b_2 + a_2 \beta) \right] dt. \]

After computing this elementary integral, and using relation (17), we obtain
\[ z(1) - z(\beta) \leq \frac{z(\beta)}{\beta^2 b_2} \left[ 3b_2(1 - \beta) - a_2 (1 - \beta)^3 \right] \]
so that
\[ z(1) \leq \frac{q_2 z(\beta)}{\beta^2 b_2} \left[ b_2(3 - 3 \beta + \beta^2) - a_2 (1 - \beta)^3 \right]. \]

4. Main Results

We now need to choose a cone for each problem. For the operators \( A_i, i = 1, 2, \) we choose \( K_i \) to be all \( z \in E \) for which \( z(0) = 0, \) \( z(t) \) non-decreasing, \( z(1) \leq q_i z(\alpha_i), \) where \( \alpha_2 = \beta \) and \( q_1, q_2 \) defined as in Lemma 4 and Lemma 5. Also note that since \( z' = (ToTof)(u) \geq 0 \) on \((0, 1). \) Thus, \( z \) is non-decreasing and, since \( z \) is the range of \( L, \) \( z(0) = 0. \) So any \( z = A_1(u) \) will belong to the cone \( K_1. \)

In other words, the number \( q_1 \) is chosen so that \( A_1 \) maps every member of \( C[0, 1] \) into the cone \( K_1, \) so the mapping hypothesis in Theorem 3 is
satisfied. A similar argument, although longer, shows that \( q_2 \) has property that \( A_2 \) maps every member of \( C[0, 1] \) into the cone \( K_2 \).

**Theorem 6.** Let \( f(u) \geq 0 \) be continuous for \( u \geq 0 \). Then there exist positive constants \( M_i, N_i, q_i, i = 1, 2 \), so that if there exist \( a, b \) with \( 0 < a < b < q_i b \) and

1. \( f(u) < M_i a \) for \( 0 \leq u \leq a \),
2. \( f(u) \geq N_i b \) for \( b \leq u \leq q_i b \), where \( \alpha_1 = \alpha, \alpha_2 = \beta \).

Then for \( i = 1, 2 \); the Problem (i) has a positive non-decreasing solution \( u_i \) satisfying \( a < u_i(1) < q_i b, u_i(\alpha_i) < b \).

Moreover, if there exists \( c \) with \( q_i b < c \) and

3. \( f(u) < M_i c, \) for \( 0 \leq u \leq c \). Then for \( i = 1, 2 \); the Problem (i) has a second positive non-decreasing solution \( v_i \) satisfying \( v_i(\alpha_i) > b, v_i(1) < c \); where \( \alpha_1 = \alpha, \alpha_2 = \beta \).

**Proof.** To get insight into how to set up the open sets, suppose \( f \) satisfies \( 0 \leq f(u) < M_1 a \) for \( 0 \leq u \leq a \). Then focusing attention on \( A_1 \) and letting \( \omega(t) = (Tof)(u) \), we have

\[
0 \leq \omega(t) = \int_0^1 G(t, s) f(u(s)) ds < M_1 a \int_0^1 G(t, s) ds = \frac{M_1 a t(1 - t)}{2}
\]

and with \( v = T(\omega) \), we have

\[
0 \leq v(t) = \int_0^1 G(t, s) \omega(s) ds < \frac{M_1 a}{2} \int_0^1 s(1 - s) G(t, s) ds.
\]

If we compute this elementary integral and then use this estimate in the integral for \( z = L(v) \) we will get

\[
0 \leq z(1) \leq \int_0^1 v(t) dt \leq \frac{M_1 a}{120}.
\]
Positive Solutions for Nonlinear Fifth-order Differential Equations ... 231

Thus, if $M_1 = 120$, we get $z(1) < a$. Thus, if we let $\Omega_1$ be all $u \in C[0, 1]$ so that $\|u\| \leq a$, then $u \in K \cap \delta \Omega_1$ implies that $u(1) = a$. Thus, $(A_1u)(1) = z(1) < a = u(1)$ implies $u \notin A_1u$ for $K \cap \delta \Omega_1$ and we have one of the required condition of Theorem 3. So $M_1 = 120$ is satisfactory.

For the operator $A_2$ we would let $\omega = (Lof)(u)$ and then

$$\omega(t) = \int_0^t f(u)(s)ds < M_2at.$$ 

Then with $v = T(\omega)$, we have

$$0 \leq v(t) = \int_0^1 G(t, s)\omega(s)ds < M_2a\int_0^1 sG(t, s)ds = \frac{M_2a(t - t^3)}{6}.$$

Applying $L$ in succession two times to this inequality, we conclude that $z = A_2(u)$ satisfies

$$z(t) < \frac{M_2a(10t^3 - 3t^5)}{360} \leq \frac{7M_2a}{360}.$$ 

It follows that $M_2 = \frac{360}{7}$ is satisfactory.

We now let $\Omega_2$ be all $u \in C[0, 1]$ for which $u(\alpha_i) < b$ and $\|u\| \leq q_i b$, $b$ is any number greater than $a$. If $u \in K \cap \delta \Omega_2$, then $u(\alpha_i) = b$, $u(0) = 0$, $u$ is non-decreasing and $u(1) \leq q_i b$, for if $u(\alpha_i) < b$.

Then the cone conditions forces $u(1) \leq q_i u(\alpha_i) < q_i b$ and $u$ is then inside the open set $\Omega_2$. Then $b \leq u(t) \leq q_i b$ for $\alpha \leq t \leq 1$.

Consider first the case of $A_1$.

If we assume that $f(u) \geq N_1 b$ for $b \leq u \leq q_i b$, we have for $\omega = T(u)$

$$\omega(t) = \int_0^1 G(t, s)f(u(s))ds \geq \int_0^1 G(t, s)f(u(s))ds > N_1b\int_0^1 G(t, s)ds.$$
Computing this integral for $0 \leq t \leq \alpha$ gives

$$\omega(t) > N_1 b t \frac{(1-\alpha)^2}{2}.$$  

Computing for $\alpha < x < 1$, we break the integral into two parts (from $\alpha$ to $t$ and from $t$ to 1) and find

$$\omega(t) > N_1 b \frac{(1-t)(t-\alpha^2)}{2}.$$  

Next, we get for $\nu = T(\omega)$,

$$v(t) = \int_0^1 G(t, s) \omega(s) ds.$$  

We only need a lower bound on this integral for $0 \leq t \leq \alpha$ because we only need a lower bound on $(A_1 u)(\alpha)$. We break this integral into three parts, from 0 to $t$, $t$ to $\alpha$ and $\alpha$ to 1 and omitting this computation, we get

$$v(t) > \frac{N_1 b (1-\alpha)^2}{2} \left[ \frac{t^3}{3} - \frac{t^3}{2} + \left( \frac{\alpha^2}{3} + \frac{\alpha^2}{4} \right) t \right].$$

Then for $z = L v$,

$$z(\alpha) = \int_0^\alpha v(s) ds > \frac{N_1 b (1-\alpha)^2 \alpha^4}{8}.$$  

Thus, $z(\alpha) > b$ if $N_1 = \frac{8}{(1-\alpha)^2 \alpha^4}$.

Thus, $(A_1 u)(\alpha) = z(\alpha) > b = u(\alpha)$ implies $A_1(u) \not\geq u$.

For $u \in K \cap \delta \Omega_2$ and we have the other required condition of Theorem 3 in the case of $A_1$ if we assume that there exists $b > a$ for which $f(u) \geq N_1 b$ for $b \leq u \leq q_1 b$.

Now consider the case of $A_2$. 
If we now assume that \( f(u) \geq N_2 b \) for \( b \leq u \leq q_2 b \), then we have for \( \omega = L(u) \), \( \omega = \int_0^t f(u(t)) \, ds \) and we can only conclude that \( \omega(t) \geq 0 \) for \( 0 \leq t \leq \alpha \) and \( \omega(t) = \int_\alpha^t N_2 bds = N_2 b(t - \alpha) \) for \( \alpha \leq t \leq 1 \). Letting \( v = T(\omega) \), we then get \( v(t) = \int_0^1 G(t, s) \omega(s) \, ds > N_2 b \int_\alpha^1 (s - \alpha) G(t, s) \, ds \).

Computing the integral gives

\[
v(t) > \frac{N_2 b t}{6} (1 - \alpha)^3 \quad \text{for} \quad 0 \leq t \leq \alpha.
\]

Applying \( L \) two more times, we find for \( z = A_2(u) \)

\[
z(t) > \frac{N_2 b t^3}{36} (1 - \alpha)^3 \quad \text{for} \quad 0 \leq t \leq \alpha.
\]

Thus, \( z(\alpha) > b \) if

\[
N_2 = \frac{36}{(1 - \alpha)^3 \alpha^3}.
\]

Again we obtain the second required condition for Theorem 3 in the case of \( A_2 \) if we assume that there exists \( b > 0 \) for which \( f(u) \geq N_2 b \) for \( b \leq u \leq q_2 b \).

It follows that there is fixed point of \( A_i \) in \( K \cap (\Omega_2 \setminus \Omega_1) \), giving us for \( i = 1, 2 \) a positive solution \( u_i \) of Problem (i) with these assumptions.

If \( f(u) \leq 120c \) for \( 0 \leq u \leq c \), where \( q_1 b < c \), then we can show that there is a fixed point of \( A_i \) in \( K \cap (\Omega_3 \setminus \Omega_2) \), where \( \Omega_3 \) consists of all \( u \in C[0, 1] \) for which \( \| u \| < c \), giving another positive solution of Problem 1. The verification of the needed hypothesis of Theorem 3 on the boundary of \( \Omega_3 \) is a repeat of the verification on the boundary of \( \Omega_1 \). Similar comments apply for Problem 2.
From the work above, it is seen that the numbers asserted to exist in the statement of Theorem 6 are:

\[ M_1 = 120, \quad N_1 = \frac{8}{(1 - \alpha)^2 \alpha^4}, \quad q_1 = \frac{b_1(2 - \alpha) - a_1(1 - \alpha)^2}{\alpha b_1}; \]

\[ M_2 = \frac{360}{7}, \quad N_2 = \frac{36}{(1 - \alpha)^3 \alpha^3}, \quad q_2 = \frac{b_2(3 - 3\beta + \beta^2) - a_2(1 - \beta)^3}{\beta^2 b_2}. \]

Acknowledgement

The authors would like to thank the anonymous referees for their valuable suggestions.

References


Positive Solutions for Nonlinear Fifth-order Differential Equations ...


