The Dynamics of Time-Varying Threshold Learning

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A mathematical theory of the dynamics of a class of trainable signal detectors is described. Among the constructs yielded by the theory are learning curves and variance curves. A learning curve is a curve of correct-decision probability versus training length. A variance curve is a curve of the variance of correct-decision probability versus training length. The class of trainable signal detectors to which the theory is applicable consists of all those in which the training procedure (a) raises the threshold in response to a false alarm, lowers the threshold in response to a false rest, and keeps the threshold unchanged in response to a correct decision, and (b) adjusts the size of the threshold increment by an amount that depends only on the trial number, and such that the threshold can eventually reach any real number.

The “learning shape theorem” is derived. By this theorem the learning curves of all members of the above class of trainable detectors are either monotonic or single peaked.

The theory is based on an approximation of the difference equation of the Markov chain of the threshold’s motion by a differential equation.

The theory provides an over-all view of the trade-offs between performance (the learning curves) and performance variability (the variance curves) as functions of the training length of time-varying training procedures.

Several illustrative numerical examples are given.

1. INTRODUCTION

We present a mathematical theory of the dynamic behavior of trainable threshold detectors under the influence of time-varying training procedures. The training process is assumed to extend over a “training phase” of duration $N$, followed by a “working phase” of indefinite length during which training is absent. Examples of such detectors occur in certain types of pattern classifiers [2], in counting processes in psychophysics [3], and in mathematical statistics [4]. The mathematical model of these detectors is referred to here as the threshold learning process, or...
TLP for short. The proposed theory is concerned with the dynamic behavior of the TLP.

The theory gives us an overall view of the learning dynamics of a large class of time-varying training procedures. The theory permits us to study this class of training procedures in terms of both the expected performance of the detector and the confidence with which this performance can be achieved. As a result, the theory is likely to lead to improved techniques for designing stopping rules and training procedures.

The theory may also be of help to the psychophysicist in his search for a cohesive set of models of trainable perception mechanisms.

Earlier theories have been able to estimate only bounds on the asymptotic behavior of TLPs [2, 9]. The theory presented here can estimate the full learning curves and variance curves of TLPs.

1.1. The Threshold Learning Process

The TLP has been described elsewhere [1], but we describe it again for convenience. It consists of a noisy channel linking a binary source to a receiver or “observer,” as shown in Fig. 1. The source emits a stationary independent sequence, \( u_n \), of 0's and 1's. The channel mixes \( u_n \) with noise to form a received or “observed” quantity \( v \). The effect of the noise is described by two “constituent densities” \( f_0(v) \) and \( f_1(v) \). The quantity \( f_i(v) \, dv \) is defined as the joint probability of transmitting \( i \) and receiving a signal in the interval \( (v, v + \, dv) \).

The observer compares \( v \) with a threshold \( k \). If \( v - k \geq 0 \) then the observer's guess (or “decision”) is \( w = 1 \), which represents the guess that the source emitted a 1. If \( v - k < 0 \), then \( w = 0 \), which represents the

![Fig. 1. Threshold Learning Process](image-url)
guess that the source emitted a 0. The observer’s guesses are called “trials,” and numbered in sequence by the natural numbers 0, 1, 2, \ldots.

In engineering applications of the TLP, the threshold is an accessible quantity that can be controlled directly by the mechanical observer. In psychophysical applications the threshold is an unconscious quantity, perhaps having no physiological corporeality, that can be only indirectly affected by the human or animal observer.

During the training phase, the observer learns by moving the threshold to a new value whenever the observer receives a “reinforcement” signal indicating whether or not the observer’s guess was correct. The strategy or rule by which the observer chooses the new threshold is called a “training procedure.” After the training phase is over, the TLP enters a working phase, during which the threshold remains fixed at the value attained at the end of the training phase.

The TLP is an elementary form of pattern classifier. The quantity \(v\) represents the observed “feature.” The 0’s and 1’s are the categories into which the observations are classified.

1.2. Discrete and Continuous TLPs

Let \(\Delta k_n\) denote the size of the increment by which \(k\) is adjusted at the \(n\)-th trial. By replacing \(n\) by \(t/\Delta t\), and letting \(\Delta k_n\) and \(\Delta t \to 0\) such that \(\Delta k_0/\Delta t\) remains fixed, the TLP approaches a continuous TLP in the limit. The continuous TLP is to be distinguished from the discrete TLP, in which \(\Delta k_n > 0\) for all finite \(n\).

We shall show that a single canonical training procedure—the so-called “fixed-increment” training procedure—in a continuous TLP provides a convenient approximation and over-all view of a large class of time-varying training procedures in discrete TLPs.

1.3. Dynamic Behavior of TLPs

During the training phase the performance of a TLP is measured by the probability \(z_n\) of a correct guess at trial \(n\). The quantity \(z_n\) may also be defined as the expected fraction of correct guesses at time \(n\) in an infinite set of identical TLPs embedded in statistically identical environments. Specifically, let \(b_n\) denote the random sequence of 0’s and 1’s derived from \(u_n\) and \(w_n\) as follows:

\[b_n \triangleq 1 - (u_n - w_n)^2.\]
Note that \( b_n = 1 \) only when \( w_n \) is correct; otherwise, \( b_n = 0 \). We define the "success probability" \( z_n \) as the expected value of \( b_n \) over the ensemble of identical TLPs:

\[
z_n \triangleq E(b_n)
\]

It follows from this definition that \( z_n \) is the expected value of the conditional success probability, \( S(k) \):

\[
z_n = E[S(k)] = \sum_k S(k)p_n(k),
\]

where \( E \) is the expectation operator. Hereafter, \( z_n \) will be defined by the above equation.

The learning curve is the curve of \( z_n \) versus \( n \). During a working phase following a training phase of length \( N \), the threshold and the success probability remain constant at \( k_N \) and \( z_N \), respectively. Since the threshold \( k \) remains constant during the working phase, a useful performance measure of the working phase is the conditional success probability \( S(k) \), which is the fraction of correct guesses over an infinitely long sequence of trials in a TLP having its threshold fixed at \( k \).

Since \( k \) is a random variable during the training phase, \( S(k) \) is also a random variable during the training phase. The dynamic behavior of the TLP may be described by the mean and variance of \( S(k) \) as functions of the trial number \( n \). The mean of \( S(k) \), which we have noted to be \( z_n \), is a measure of the learning performance. The variance of \( S(k) \) is an inverse measure of the confidence with which \( z_n \) may be achieved. The mean and variance of \( S(k) \), viewed as functions of \( n \), are the learning curve and variance curve, respectively.

2. BASIC EQUATIONS

2.1. DIFFERENCE EQUATION OF A DISCRETE TLP

In the following derivation we assume that at the \( n \)-th trial the training procedure moves \( k_n \) to \( k_n - e_n\Delta k_n \), where

\[
e_n \triangleq u_n - w_n \triangleq "error" \text{ at trial } n,
\]

and \( \Delta k_n \) is a specified positive function of \( n \), i.e.,

\[
k_{n+1} = k_n - e_n\Delta k_n,
\]

where \( \Delta k_n > 0 \). Note that \( \Delta k_n \geq |k_{n+1} - k_n| \). Let \( f_0(v), f_1(v) \) denote the
two constituent densities, let
\[ \rho \triangleq \int_{-\infty}^{\infty} f_0(v) \, dv, \]
and let
\[ F_i(k) \triangleq \int_{-\infty}^{k} f_i(v) \, dv \quad (i = 0, 1). \quad (4) \]
Using this notation, the probabilities that \( e = -1, 0, 1 \), given that the threshold is \( k \), are, respectively,
\[ R(k) = \rho - F_0(k), \]
\[ S(k) = 1 - \rho + F_0(k) - F_1(k) \]
conditional probability of success given \( k \), (5)
and
\[ L(k) = F_1(k), \]

The motion of \( k \) during the training phase of a discrete TLP is a random walk \([1]\). Hence, the probability \( p_n(k) \) of occupying threshold \( k \) at time \( n + 1 \) is
\[ p_{n+1}(k) = p_n(k - \Delta k_n) R(k - \Delta k_n) \]
\[ + p_n(k) S(k) + p_n(k + \Delta k_n) L(k + \Delta k_n). \quad (6) \]
Putting (5) into (6), and subtracting \( p_n(k) \) from both members, we obtain the following difference equation.
\[ p_{n+1}(k) - p_n(k) \]
\[ = p_n(k - \Delta k_n)[\rho - F_0(k - \Delta k_n)] + p_n(k + \Delta k_n)F_1(k + \Delta k_n) \]
\[ - p_n(k)[\rho - F_0(k) + F_1(k)]. \quad (7) \]
Equation (7) together with the following two auxiliary conditions, form the basic equations of the dynamic behavior of a TLP.

**Initial condition:** \( p_0(k) = \delta(k, k_0) \triangleq \begin{cases} 1 & \text{for } k = k_0 \\ 0 & \text{for } k \neq k_0 \end{cases}. \quad (8) \]

**Sum-to-unity condition:** \( \sum_k p_n(k) = 1 \) for all \( n \geq n_0 \), (9)

where \( k_0, n_0 \) are the initial values of \( k \) and \( n \), respectively.
2.2. Differential Equation (or "Wave Equation") of a Continuous TLP

By defining a new time scale, \( t = n \Delta t \), and letting both \( \Delta t \) and \( \Delta k_n \) approach 0 in a properly chosen manner, we obtain a differential equation as the limit of (7). This differential equation describes the performance of a limiting form of TLP, which we refer to as a continuous TLP.

We are able to find a general solution for this differential equation but not for the difference equation, (7). Hence, we propose using the solution of the differential equation as an approximation for the solution of the difference equation.

To obtain the differential equation we define the limiting process as follows:

In (7), replace \( n \) by \( t/\Delta t \), and let

\[ \Delta k(t) \equiv \Delta k_n \quad \text{for} \quad n \Delta t \leq t < (n + 1) \Delta t. \]

Usually we shall use \( \Delta k \) as an abbreviated form of \( \Delta k(t) \). Let

\[ p_\Delta(k, t) \equiv p_n(k) \quad \text{for} \quad n \Delta t \leq t < (n + 1) \Delta t. \]

Then (7) becomes

\[
p_\Delta(k, t + \Delta t) - p_\Delta(k, t) = p_\Delta(k - \Delta k, t)[\rho - F_0(k - \Delta k)] - p_\Delta(k, t)F_1(k) - \{p_\Delta(k, t)[\rho - F_0(k)] - p_\Delta(k + \Delta k, t)F_1(k + \Delta k)\}. \tag{10}
\]

Let \( \Delta t \to 0, \Delta k \to 0 \). Let \( \ddot{\Delta}k \) be the infinitesimal versions of \( \Delta t, \Delta k \), respectively. (Note that \( \Delta k \) and \( \ddot{\Delta}k \) are never negative.) In the limiting process, let

\[
\frac{\Delta k}{\Delta t} = \frac{\ddot{\Delta}k}{dt} \triangleq g(t)
\]

(11)

= a function of \( t \) that describes the training procedure.

Constraints on \( g(t) \):

We assume that \( 0 < \Delta k_n < \infty \) for all \( n \). It follows from (11) that \( 0 < g(t) < \infty \). We also note that if \( \int_{t_0}^{\infty} g(t) \, dt < \infty \), where \( t_0 \) is the initial value of \( t \) during the training phase, then

\[
\infty > \int_{t_0}^{\infty} \frac{\ddot{\Delta}k}{dt} \, dt \geq \int_{t_0}^{\infty} \left| \frac{\ddot{\Delta}k}{dt} \right| \, dt.
\]
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since $\Delta k \geq |dk|$, by Eq. 3. Hence,

$$\infty \geq k_{\text{max}} - k_{\text{min}},$$

(12)

where $k_{\text{max}}$ and $k_{\text{min}}$ are the upper and lower extremes of the range of motion of $k$ during the above integration.

Thus certain values of $k$ are inaccessible if $\int_{t_0}^{\infty} g(t) \, dt < \infty$. Hence, we require that $\int_{t_0}^{\infty} g(t) \, dt = \infty$.

These constraints on $g(t)$ are summarized as follows:

$$0 < g(t) < \infty \quad \text{for} \quad t \geq t_0,$$

(13)

$$\int_{t_0}^{\infty} g(t) \, dt = \infty.$$

We define

$$F(k) \triangleq F_0(k) + F_1(k) - \rho,$$

(14)

Note that $F_0(k)$, $F_1(k)$ and $F(k)$ are differentiable by (4), assuming $f_0(v)$ and $f_1(v)$ are integrable.

We define $p(k, t)$ and $\frac{\partial p}{\partial t}$ as the following limits:

$$p(k, t) \triangleq \lim_{\Delta t \to 0, \Delta k = 0, \Delta t = g(t)} [p_\Delta(k, t)],$$

(15)

$$\frac{\partial p}{\partial t} \triangleq \lim_{\Delta t \to 0, \Delta k = 0, \Delta k = g(t)} \left[ \frac{p_\Delta(k, t + \Delta t) - p_\Delta(k, t)}{\Delta t} \right].$$

(16)

We define $\frac{\partial p}{\partial k}$ and $\frac{\partial (pF)}{\partial k}$ in a similar manner.

By these definitions, (10) becomes

$$(\partial p/\partial t) \, dt = (\partial /\partial k)(pF) \, dk.$$

(17)

Hence,

$$\frac{\partial p}{\partial t} = \frac{\Delta k}{\Delta t} \frac{\partial (pF)}{\partial k}$$

(18)

$$= g(t) \frac{\partial (pF)}{\partial k} \text{ by Eq. (11).}$$

Constraints (8) and (9) become

$$p(k, t_0) = \delta (k - k_0) = \text{Dirac delta function},$$

(19)
and

\[
\int_{-\infty}^{\infty} p(k, t) \, dk = 1 \quad \text{for all } t \geq t_0.
\]  

(20)

(The Dirac delta function must satisfy \( \delta(x) = 0 \) for \( x \neq 0 \) and \( \int_{-\infty}^{\infty} \delta(x) \, dx = 1 \). Discussions of the Dirac delta function or "Dirac distribution" are available at various levels of mathematical rigor [10, 11, and 12].)

We refer to Eq. (18) as the "wave equation" of the TLP, since the solution of (18) under constraints (19) and (20) is a traveling wave on the \( k \) axis. This wave is an approximation to the solution of Eqs. (7)-(9) in the following sense. Let

\[
z(\Delta t) \triangleq z_n \triangleq \sum_k p_\Delta (k, t) S(k).
\]

(20a)

Let \( z(t) \triangleq \lim [z_\Delta (t)] \) as \( \Delta t \to 0, \Delta k \to 0 \) for all \( t, \frac{\Delta k}{\Delta t} = g(t) \). It follows that

\[
z(t) = \int_{-\infty}^{\infty} p(k, t) S(k) \, dk,
\]

(20b)

and

\[
z_\Delta (t) \cong z(t)
\]

(21)

for all \( t \) such that

\[
t = n\Delta t = n \frac{\Delta t}{\Delta k_0} \Delta k_0 = n \Delta k_0 \frac{dt}{dk_0} = n \frac{\Delta k_0}{g(t_0)}.
\]

(22)

Hence,

\[
z_\Delta (t) \cong z(t) = z(n \Delta k_0 / g(t_0)).
\]

(23)

Thus, \( p(k, t) \) is an approximation of \( p_n (k) \) in the sense that \( z_\Delta (t) \) differs little from \( z(t) \) for all \( t \), where \( t = n \Delta k_0 / g(t_0) \). In the sequel we show that \( p(k, t) \) is a traveling delta function, which is very different from \( p_\Delta (k, t) \). In particular, the variance of \( k \) for \( p(k, t) \) is zero for all \( t \), while the variance of \( k \) for \( p_\Delta (k, t) \) is positive for \( 0 < t < \infty \) if \( \Delta k_0 > 0 \) for \( n \geq 0 \). Hence, the variance curve cannot be estimated directly from \( p(k, t) \). Such an estimate has been obtained, however, from the limit of the difference equation for the variance of \( k \) as the continuous TLP is approached. We describe this estimation technique in Section 4.4.
3. SOLUTION OF THE WAVE EQUATION

To solve (18)–(20), we first consider the case where \( g(t) \) is a constant, say \( c \).

Case 1. \( g(t) = c \).

For this case, (18) becomes

\[
\frac{\partial p}{\partial t} = c \left( \frac{\partial}{\partial k} \right) (pF).
\]

(24)

Approximation (23) becomes

\[
z_{\Delta}(t) \cong z[n(\Delta k/c)],
\]

(25)

since \( \Delta k_0 = \Delta k \), \( g(t_0) = g(t) = c \). Often it is convenient to let \( c = 1 \), in which case approximation (25) becomes

\[
z_{\Delta}(t) \cong z(n\Delta k).
\]

(26)

Kae [5] has shown that the solution to Eq. (24) under constraints (19), (20) is

\[
p(k, t) = - \frac{1}{cF(k)} \delta \left[ t - t_0 + \frac{1}{c} \int_{k_0}^{k} \frac{dv}{F(v)} \right].
\]

(27)

One may also express (27) as follows:

\[
p(k, t) = \delta[k - V[c(t - t_0)]],
\]

(28)

where \( V(y) \) is the solution of

\[
y = - \int_{k_0}^{v(y)} \frac{dv}{F(v)}
\]

(29)

or, equivalently, of the pair of equations

\[
dV/dy + F(V) = 0,
\]

(30)

\[
V(0) = k_0.
\]

(31)

The reader may verify that (28) satisfies (24) by using the auxiliary variable \( \phi = k - V[c(t - t_0)] \), and expressing (24) in terms of \( d / d\phi \).

Case 2. \( g(t) \) is any function satisfying (13).

\(^1\) An alternative technique of solving (24) is the “characteristic curve” method for solving linear partial differential equations in three variables (page 245 of [8]). In the present case the three variables are \( k \), \( t \), and \( p \). This method is also applicable to (18).
The solution to this case can be obtained from the solution to Case 1 by a change in the variable $t$.

Let

$$y(t) = \int_{t_0}^{t} g(u) \, du + y_0,$$

(32)

We show in Appendix 1 that $y(t)$ is a monotonic differentiable function in $t_0 \leq t < \infty$ and $y(\infty) = \infty$ if, and only if, $g(t)$ satisfies (13). Hence, if $g(t)$ satisfies (13), there exists $T(y)$ such that $T[y(t)] = t$ for all $t \in [t_0, \infty)$.

Replace $t$ by $T(y)$ everywhere in $p(k, t)$. Thus,

$$p(k, t) = p[k, T(y)] \triangleq \hat{p}(k, y).$$

(33)

Then (18) may be written as follows:

$$\frac{\partial \hat{p}}{\partial y} \frac{dy}{dt} = \frac{\partial k}{\partial t} \frac{\partial}{\partial k} (\hat{p}F) = g(t) \frac{\partial}{\partial k} (\hat{p}F).$$

(34)

By (32), we have

$$\frac{dy}{dt} = g(t).$$

(35)

Hence, (34) simplifies to

$$\frac{\partial \hat{p}}{\partial y} = \frac{\partial k}{\partial \hat{p}} (\hat{p}F),$$

(36)

where $\hat{p}$ is the function of $k$ and $y$ defined in (33).

Thus, by choosing an appropriate new variable $y$ in place of $t$, we reduce the general wave equation, (18), to the fixed-increment wave equation, (36), which has the same form as (24), with $c = 1$.

Recalling (28), we see that

$$p(k, t) = \delta[k - V(y - y_0)],$$

(37)

where $V(y)$ is the solution of (29), and $y(t)$ is given by (32).

We illustrate the presented theory by three running examples. In these examples we give special attention to a TLP having the triangular constituent densities shown in Fig. 2. We refer to this TLP as TLP 1.

3.1. Derivation of $V(y)$ of TLP 1

Suppose $f_0(v)$ and $f_1(v)$ have the triangular shapes shown in Fig. 2. Then $F(v)$ is the curve shown in Fig. 3, by (4) and (14). In mathematical
terms, this curve is described as follows:

\[
F(v) = \begin{cases} 
  v/b & \text{for } |v| \leq b/2 \\
  1/2 & \text{for } v > b/2 \\
  -1/2 & \text{for } v < -b/2.
\end{cases}
\] (38)

Suppose \( k_0 = b/2 \). Then, by (29),

\[
y = -\int_{-\infty}^{v(y)} \frac{dv}{F(v)} = -b \int_{-\infty}^{v(y)} \frac{dv}{v} = -b \log \left[ \frac{2V(y)}{b} \right].
\] (39)

Hence,

\[
V(y) = \frac{b}{2} e^{-y/b} \text{ for } y \geq 0.
\] (40)

**Example 1.** Suppose \( g(t) = c \). Then \( y(t) = ct \). For TLP 1, (40) gives us

\[
V[y(t)] = \frac{b}{2} e^{-ct/b}.
\] (41)

By (28), we obtain

\[
p(k, t) = \delta[k - (b/2) \exp\{-c/b(t - t_0)\}].
\] (42)

**Example 2.** Suppose \( g(t) = c/t \). Then \( y = c \log t \). Then (28) be-
comes

\[ p(k, t) = \delta \left[ k - V \left( c \log \frac{t}{t_0} \right) \right] \]

\[ = \delta \left[ k - \frac{b}{2} \left( \frac{t}{t_0} \right)^{-\alpha/b} \right] \quad \text{by Eq. (40).} \]  

(43)

**Example 3.** Suppose \( g(t) = ct^{-\gamma} \). Then, by (32),

\[ y = \frac{c}{1 - \gamma} \quad \text{for} \quad 0 \leq \gamma < 1. \]  

(44)

Note that if \( \gamma < 0 \) the increments become indefinitely large as \( t \) approaches infinity. We show in a subsequent section that under this condition the variance of \( k \) grows indefinitely large as \( t \to \infty \). Hence, \( \gamma < 0 \) may be viewed as an "unstable" condition. Also note that \( \gamma > 1 \) is not realizable, because \( \gamma > 1 \) implies that

\[ \int_{t_0}^{\infty} g(t) \, dt = \int_{t_0}^{\infty} t^{-\gamma} \, dt < \infty, \]  

(45)

which violates constraint (13). (This result supports the observations of
Robbins and Monro, Dvoretzky, and others who studied the problems associated with choosing the size of the increment, $\Delta k_n$, so that one may obtain convergence of $\varepsilon(t)$ to its asymptote without excessive variance in $k$ as $t$ grows large. Their results, which form a part of the theory of stochastic approximation \[4\], are applicable only to $\frac{1}{2} \leq \gamma \leq 1$.

Using Eq. (42), (37) becomes

$$p(k, t) = \delta \left[ k - V \left( \frac{c}{1 - \gamma} t^{1-\gamma} - \frac{c}{1 - \gamma} t_0^{1-\gamma} \right) \right]$$

$$= \delta \left\{ k - \frac{b}{2} \exp \left[ - \frac{c}{b(1 - \gamma)} (t^{1-\gamma} - t_0^{1-\gamma}) \right] \right\} \text{ by Eq. (40)}$$

The relationships among the $y(t)$'s in Examples 1 to 3 are shown in Fig. 4. In this figure we see that to reach a given value of $y$, the longest time is consumed in the case $\gamma = 1$, and the shortest time is consumed in the case $\gamma = 0$. Hence, $\gamma = 0$ yields the fastest learning.

\[ \text{Fig. 4. } y(t) \text{ in Examples 1-3, with } t_0 = 1 \]
We shall see below, however, that the following two properties of 
TLP 1 persuade the designer to choose $\gamma > 0$:

(a) Raising $\gamma$ tends to raise $z_\Delta(\infty).

(b) The variance of $k_N$, where $N$ is the training length, tends to 
decrease as $\gamma$ is increased, provided $N$ is sufficiently large.

4. LEARNING CURVES, CENTROID CURVES, AND VARIANCE CURVES

The learning curve and the variance curve of a TLP serve as principal 
means for describing a TLP's dynamic behavior. In the following we 
show how a discrete TLP's learning curve and variance curve may be 
estimated from the associated continuous TLP.

In the analysis of the learning curve and variance curve, a third curve 
is also of significance: the "centroid curve," to be defined below.

4.1. DEFINITIONS

For convenience we repeat the definition of the learning curve:

$$z_n \triangleq E[S(k)] = \sum_k S(k)p_n(k). \quad (48)$$

The variance curve $q_n$ is defined as the variance of $S(k)$ at trial $n$:

$$q_n \triangleq \sum_k [S(k) - z_n]^2 p_n(k). \quad (49)$$

The centroid curve (or "mean learning curve" [13]) $\mu_n$ is defined as the 
expected value of $k$ at trial $n$:

$$\mu_n \triangleq \sum_k k p_n(k). \quad (50)$$

The threshold variance curve is defined as the variance of $k$ at trial $n$:

$$r_n \triangleq \sum_k (k - \mu_n)^2 p_n(k). \quad (51)$$

By plotting the learning curve and the variance curve on the same 
graph, both performance and performance variability may be displayed 
simultaneously as functions of $n$.

In the sequel we derive equations that estimate the learning curve and 
centroid curve of a TLP having any time-varying training procedure 
satisfying constraints (13).

We note from (37) that in a continuous TLP the variance of $k$ and 

hence the variance of $S(k)$ are zero for all $t \geq t_0$. This, of course, is not
so in a discrete TLP. Hence, an equation for the relative variance curve of a continuous TLP will be derived. The relative variance of a continuous TLP is obtained by dividing the variance of a discrete TLP by $\Delta k$, and then letting $\Delta k$ approach zero.

Since the centroid curve appears in the equations determining the learning curve and the relative variance curve, we first derive the equation for the centroid curve.

### 4.2. CENTROID CURVES

The centroid curve of a discrete TLP is defined as the expected value of $k_n$, plotted versus $n$. Thus,

$$\mu_n \triangleq E[k_n] = \sum_k kp_n(k).$$

In the proposed theory, this curve is approximated by the centroid curve $\mu(t)$ of the associated continuous TLP.

It is clear from (28) that the centroid of $p(k, t)$ is given by

$$\mu(t) = V[y(t) - y(t_0)], \quad (52)$$

where $y(t)$ is given by (32), and $V(y)$ is given by (29).

Any $g(t)$ satisfying (13) can be accounted for in (52) by replacing $y(t)$ by the right member of (32).

Equations 22 and 52 lead to following approximation of $\mu_n$:

$$\mu_n \approx \mu(\Delta k\Delta t)/g(0), \quad (53)$$

where $\mu(t)$ is given by (52). This approach to estimating $\mu_n$ provides insight into the effect of the choice of the increment-adjusting function $g(t)$ on the dynamics of stochastic approximation [4].

In Appendix 2, we show that $V(\infty) = 0$ if $F(v)$ satisfies the conditions $F(0) = 0, F'(0) > 0$ and $|F^{(m)}(0)| < \infty$ for $m \geq 0$. These conditions hold in almost all nontrivial physical situations. [Recall that $F'(0) = f_0(0) + f_1(0)$.] We show in the sequel that $y = \infty$ implies $t = \infty$ and vice versa whenever constraints (13) on $g(t)$ are satisfied. Thus for these cases, which again are representative of almost all nontrivial time-varying TLPs, we have $\mu(\infty) = \theta$, by (52), i.e., for these TLPs, the mean of $k$ approaches $\theta$, the equal-error point, as $t \to \infty$. This implies, by Eq. (53) that in discrete TLPs satisfying these conditions, the mean of $k$ approaches a value near $\theta$ as $t \to \infty$. 

EXAMPLE 1.

\[ \mu(t) = V[c(t - t_0)]. \]  

For TLP 1, we have

\[ \mu(t) = (b/2)e^{(-c/b)(t-t_0)}. \]  

EXAMPLE 2.

\[ \mu(t) = V[c \log (t/t_0)]. \]

For TLP 1, we have

\[ \mu(t) = e^{(c/b)} \left( \frac{t}{t_0} \right)^{(c/b)}. \]

EXAMPLE 3.

\[ \mu(t) = V[c/(1 - \gamma)](t^{1-\gamma} - t_0^{1-\gamma}) \text{ for } 0 \leq \gamma \leq 1. \]

For TLP 1,

\[ \mu(t) = \frac{b}{2} \exp \left( -\frac{c}{b(1 - \gamma)} (t^{1-\gamma} - t_0^{1-\gamma}) \right) \text{ for } 0 \leq \gamma \leq 1. \]

4.3. **Learning Curves**

Recall from (20b) that

\[ z(t) = \int_{-\infty}^{\infty} p(k, t)S(k) \, dk. \]

Putting (37) into the above equation, we obtain

\[ z(t) = S[V[y(t) - y(t_0)]] \]

as the learning curve of a continuous TLP.

Since the proposed theory can estimate the centroid curves of the class of training procedures satisfying (13), Eq. (63) is a convenient approximation of the learning curves for that class of training procedures.

**EXAMPLE 1.**

For TLP 1, we have, using Eqs. (5) and Fig. 2.

\[ S(k) = \begin{cases} 
1/2 & \text{for } |k| > \frac{b}{2} \\
3 - k^2 \frac{b^2}{4} & \text{for } |k| \leq \frac{b}{2}
\end{cases} \]
Hence, by Eqs. (41) and (63),
\[ z(t) = S \left( \frac{b}{2} e^{-\frac{c}{b}(t-t_0)} \right) = \frac{3}{4} - \frac{1}{4} e^{-\frac{2c}{b}(t-t_0)}. \] (65)

**Example 2.**
For this example we note in Section 4.2 that we merely replace \( t - t_0 \) by \( \log t/t_0 \) in Example 1. Thus, we obtain
\[ z(t) = \frac{3}{4} - \frac{1}{4} \left( \frac{t}{t_0} \right)^{-2c/b}. \]

**Example 3.**
Here we replace \( t - t_0 \) by \( \left(1/(1 - \gamma)\right)(t^{1-\gamma} - t_0^{1-\gamma}) \), so that
\[ z(t) = \frac{3}{4} - \frac{1}{4} \exp \left[ -\frac{2c}{b(1 - \gamma)} (t^{1-\gamma} - t_0^{1-\gamma}) \right]. \]

The learning curves of Examples 1 to 3 are illustrated in Fig. 5 for the case \( t_0 = 1, b = 4, c = 1. \)
EXAMPLE 4.

This new example provides empirical evidence of the effectiveness of the presented theory.

Suppose \( f_0(v) \) and \( f_1(v) \) are the step functions shown in Fig. 6. On this figure a parameter \( \alpha \) is defined which determines the narrowness of the shapes of the \( f_i(v) \)'s. In Figs. 7 and 8 are shown the effect of changing \( \Delta k \) in the fixed-increment training procedure. All of the curves are plotted versus \( t \), where \( t = n\Delta k \). [In this case \( g(t_0) = c = 1 \). See Eq. (25).]

In Figs. 7 and 8, the largest fractional error—about 7%—is caused by the error in computing the asymptote when the learning curve has a peak. (A peak can occur in the case of Fig. 6 only when \( \rho < \frac{1}{2} \).) This error can be eliminated by computing the asymptote exactly, using any of several techniques. One such technique is the "delay method" of analyzing Markov chains [7]. Using this technique, the case \( \Delta k = 1 \) in Fig. 7 is approximated by a horizontal line adjoined to the associated continuous TLP's learning curve, as shown in Fig. 9.

4.4. VARIANCE CURVES

In the preceding section we showed how the learning curve of a discrete TLP may be estimated from the solution of the partial differential
equation of a continuous TLP. In this section we derive from the theory of the continuous TLP a method of estimating the variance curve $q_n$ of a discrete TLP. This curve serves as a measure of the variation of conditional success probability $S(k)$ caused by the random motion of $k$.

The proposed estimation technique provides an overall view of the variance curves of the class of training procedures satisfying constraints (13) and (81). Let

$$r_\Delta(t) \triangleq \sum_{k=-\infty}^{\infty} [k - \mu_\Delta(t)]^2 p_\Delta(k, t)$$

$$\triangleq \text{variance of } k \text{ at trial } n \text{ for } n = \frac{g(t)}{\Delta k_0} t.$$
Let
\[ r(t) \triangleq \text{relative threshold variance} \]
\[ \triangleq \lim [r_\Delta(t)/\Delta k] \text{ as } \Delta k \to 0, \Delta t \to 0, \text{ and } \Delta k/\Delta t = g(t). \]  

From the results of a recent paper by Norman [18], it follows that
\[ p_\Delta(k, t) \] is asymptotically normally distributed with mean \( \mu(t) \) and variance \( r(t)\Delta k \) as \( \Delta k \to 0, \Delta t \to 0, \Delta k/\Delta t = c \), provided \( F''(v) \) exists. Norman's results can be specialized to yield the following differential equation for \( r(t) \) for the case \( g(t) = c = \text{constant} \):
\[ r'(t) + 2cF'[\mu(t)]r(t) = -\frac{[\mu'(t)]^2}{c} + c[1 - z(t)], \]
provided \( F''(v) \) exists.

Now suppose \( g(t) \neq c \). Let \( y \) and \( p_\Delta(k, y) \) be defined by Eqs. (32) and (33). In (10), replace \( t \) by \( T(y) \) and \( t + \Delta t \) by \( T(y + \Delta y) \). Then we see that \( p_\Delta(k, y) \) satisfies the equation obtained by replacing \( t \) by \( y \) and \( p \) by \( \hat{p} \) in (10). Hence, \( p_\Delta(k, y) \) is asymptotically normal in the variable \( k \) as \( \Delta k \to 0, \Delta y \to 0, \Delta k/\Delta y = 1 \). Since \( \Delta y \to g(t) \Delta t \) as \( \Delta t \to 0 \), it follows that \( p_\Delta(k, t) \) is asymptotically normal in \( k \) as \( \Delta k \to 0, \Delta t \to 0, \Delta k/\Delta t = g(t) \).
Let
\[
\hat{r}(y) \triangleq \lim \{r_\Delta[T(y)]\} \quad \text{as} \quad \Delta k \to 0, \Delta y \to 0, \Delta k/\Delta y = 1. \tag{69}
\]
Since \( \hat{p}_\Delta(k, y) \) satisfies the equation obtained by replacing \( t \) by \( y \) and \( p \) by \( \hat{p} \) in \( (10) \), it follows that \( \hat{r}(y) \) satisfies the equation obtained by replacing \( t \) by \( y \), \( r \) by \( \hat{r} \), and \( c \) by \( 1 \) in \( (68) \). Thus, \( \hat{r}(y) \) satisfies
\[
\hat{r}(y) + 2F'[\mu(y)]\hat{r}(y) = -[\mu'(y)]^2 + 1 - \hat{z}(y), \tag{70}
\]
where
\[
\mu(y) \triangleq \mu[T(y)]
\]
and
\[
\hat{z}(y) \triangleq z[T(y)],
\]
provided \( F''(v) \) exists.
Since Ay → g(t) Δt as Δt → 0, it follows that \( \dot{r}(y) = r(t) \). It is easy to show that \( \ddot{\mu}'(y) = \mu'(y)/g(t) \) and \( \ddot{r}'(y) = r'(t)/g(t) \). Hence, (70) may be written

\[
\dot{r}'(t) + 2g(t)F'[\mu(t)]r(t) = -\frac{[\mu'(t)]^2}{g(t)} + g(t)[1 - z(t)],
\]

(71)

provided \( F''(v) \) exists.

An intuitive, nonrigorous derivation of Eq. (71) can be obtained by multiplying both members of Eq. (10) by \( [k - \mu_0(t)]^2 \), and summing both members of the resulting equation over \(-\infty < k < \infty\). The resulting difference equation in \( r_\Delta(t) \) approaches (71) as \( \Delta k \to 0, \Delta t \to 0, \Delta k/\Delta t = g(t) \).}

Since the initial threshold is assumed to be known, it follows that \( r(t_0) = 0 \). Hence, the general solution of (71) is [6, 14]:

\[
r(t) = \left[\mu'(t)\right]^2 \left[\int_{t_0}^{t} \frac{1 - z(\xi)}{[\mu'(\xi)]^2} d\xi - (t - t_0)\right].
\]

(72)

We may obtain the following estimate of \( r_\Delta(t) \) from (67):

\[
r_\Delta(t) \approx r(t)\Delta k = r(t) (\Delta k_0/g(t_0))g(t).
\]

(77)

By Eq. (71) and Appendix 2,

\[
2F'(\theta)r(\infty) = -\lim_{t \to \infty} \left[\frac{\mu'(t)}{g(t)}\right]^2 + 1 - z(\infty),
\]

(78)

assuming \( r'(\infty) = 0 \).

By Eqs. (52), (32), and (30),

\[
\mu'(t) = V'(y)(dy/dt) = -F[V(y)]g(t).
\]

(79)

Putting Eq. (79) into Eq. (78), and putting the result into (77) we obtain

\[
r_\Delta(\infty) \approx \frac{1}{2F'(\theta)} \left\{-F^2[V(\infty)] + 1 - z(\infty)\right\} \Delta k_0/g(t_0) g(\infty).
\]

(80)

If \( F(k) \) satisfies the hypotheses of Theorem A2 in Appendix 2, then \( V(\infty) = \theta \), and \( F[V(\infty)] = 0 \). If, in addition, \( F'(\theta) \neq 0 \), then, by (80),

\[
r_\Delta(\infty) \approx \left[1 - z(\infty)/2F'(\theta)\right](\Delta k_0/g(t_0))g(\infty).
\]

(80a)

Hence, in order to insure that \( r_\Delta(\infty) < \infty \) when \( F(k) \) satisfies the hypotheses of Theorem A2, we must impose the following constraint on
$g(t)$, in addition to (13):

$$g(\infty) < \infty. \quad (81)$$

We now derive an expression for estimating the variance curve $q_n$ in terms of the relative threshold variance $r(t)$.

Let

$$q_\Delta(t) \overset{\Delta}{=} q_n \text{ for } n = (g(t_0)/\Delta k_0)t. \quad (86)$$

By Eqs. (48) and (49),

$$q_\Delta(t) = E[S^2(k)] - \{E[S(k)]\}^2$$

$$= \sum_k S^2(k)p_\Delta(k, t) - \{\sum_k S(k)p_\Delta(k, t)\}^2. \quad (87)$$

Expand $S(k)$ about $\mu_\Delta(t)$:

$$S(k) = S[\mu_\Delta(t)] + S'[\mu_\Delta(t)][k - \mu_\Delta(t)]$$

$$+ \frac{1}{2!} S''[\mu_\Delta(t)][k - \mu_\Delta(t)]^2, \quad (88)$$

where the primes indicate differentiation.

Substitute Eq. (88) into Eq. (87). Then recall that $p_\Delta(k, t)$ is asymptotically normal as $\Delta t \to 0$, $\Delta k/\Delta t = g(t)$, if $F''(v)$ exists. Hence, the ratio of the $m$-th moment of $k$ to the second moment of $k$ approaches zero as $\Delta t \to 0$, $\Delta k \to 0$, $\Delta k/\Delta t = g(t)$, for $m \geq 3$. Hence,

$$q_\Delta(t) \approx \{S'[\mu_\Delta(t)]\}^2r_\Delta(t), \quad (91)$$

from which we obtain

$$q_\Delta(t) \approx \{S'[\mu(t)]\}^2r(t)\Delta k \text{ for } S'[\mu(t)] \neq 0. \quad (92)$$

If there exists $t_m$ such that $S'[\mu(t_m)] = 0$, then $q_\Delta(t_m)$ will be very small but not necessarily zero. Hence, Eq. (92) will not give an indication of the size of $q_\Delta(t_m)$ other than that $q_\Delta(t_m) \ll q_\Delta(t)$ when $t$ is not in the vicinity of $t_m$. This may be a significant problem when $t_m = \infty$, for which case the "vicinity" of $t_m$ is infinite in extent. In this case computing the exact value of $r_\Delta(\infty)$ may be the most convenient recourse.

**Example 1.*** $y = ct$.

For TLP 1, we have, by Eq. (55),

$$\dot{\mu}(y) \overset{\Delta}{=} \mu[T(y)] = (b/2)e^{-(\varphi - \varphi_0)/b}. \quad (96)$$
Hence,
\[ \hat{\mu}'(y) = \frac{1}{2} e^{-(y-y_0)/b}. \] (97)

By Eq. (65),
\[ \hat{z}(y) \equiv z[T(y)] = \frac{3}{4} - \frac{1}{4} e^{-(y-y_0)/b}. \] (98)

By Eq. (85),
\[ \hat{r}'(y) + 2\{f_0(\hat{\mu}(y)] + f_1[\hat{\mu}(y)]\} \hat{r}(y) = \frac{3}{4}, \] (99)

where \( \hat{\mu}(y) \) is given by Eq. (96). In the region of motion of \( k \),
\[ f_0(k) + f_1(k) = 1/b. \] (100)

Hence, Eq. (99) becomes
\[ \hat{r}'(y) + (2/b) \hat{r}(y) = \frac{1}{4}. \] (101)

The boundary conditions of this equation are
\[ \hat{r}'(\infty) = 0 \text{ and } \hat{r}(0) = 0. \]

Thus, \( \hat{r}(\infty) = b/8 \). The solution of Eq. (101) is
\[ \hat{r}(y) = (b/8)[1 - e^{-(y-y_0)/b}]. \] (102)

Recalling Eq. (75), we see that if \( c = 1 \) and \( \Delta k = 1 \), we have
\[ r_\Delta(t) \equiv r(ct) \Delta k = r(t) = (b/8)[1 - e^{-(t-t_0)/b}]. \] (103)

By Eq. (64)
\[ S'(k) = \begin{cases} -\frac{2k}{i\sigma} & \text{for } |k| < \frac{b}{2} \\ 0 & \text{for } |k| > \frac{b}{2}. \end{cases} \] (104)

By Eq. (96),
\[ \mu(t) = (b/2)e^{-(t-t_0)/b}. \]

Hence,
\[ S'[\mu(t)] = -\frac{1}{b} e^{-(t-t_0)/b}. \] (105)

By Eqs. (89) and (103),
\[ q_\Delta(t) \equiv (1/8b)e^{-(3b/5)(t-t_0)}[1 - e^{-2(t-t_0)/b}]. \] (106)
This curve is plotted in Fig. 10 for the case $b = 4$, $t_0 = 1$, and labeled "$\gamma = 0$".

In the subsequent examples, we assume

$$\Delta k = [\Delta k_0/g(t_0)]g(t),$$

where $g(t) \neq \text{constant}$ and satisfies (13) and (8).  

**Example 2.**

$$g(t) = c/t. \quad (107)$$
$$y(t) = c \log t. \quad (108)$$
$$r(t) = \hat{r}(c \log t) = (b/8)[1 - (t/t_0)^{-2b}]. \quad (109)$$
$$\Delta k = (t_0/t)\Delta k_0. \quad (110)$$
$$r_\Delta(t) \approx r(t)\Delta k = \frac{b}{8}\left[1 - \left(\frac{t}{t_0}\right)^{-2b}\right]t_0\Delta k_0. \quad (111)$$

By Eq. (92)

$$q_\Delta(t) \equiv |S'[\mu(t)]|^2 r_\Delta(t). \quad (112)$$

By Eq. (96),

$$\mu(t) = \hat{\mu}[y(t)] = (b/2)(t/t_0)^{-c/b}. \quad (113)$$
By Eqs. (104) and (113),
\[ S'[\mu(t)] = -(1/b)(t/t_0)^{-c/b}. \tag{114} \]

By Eqs. (114), (112), and (111),
\[ q_\Delta(t) \approx \frac{1}{8b} \left( \frac{t}{t_0} \right)^{-\left(1+\frac{2c}{b}\right)} \left[ 1 - \left( \frac{t}{t_0} \right)^{-2c/b} \right] \Delta k_0. \tag{115} \]

This function is plotted in Fig. 10 for the case \( \Delta k_0 = 1, t_0 = 1, b = 4, \) and labeled "\( \gamma = 1. \)"

Example 3. \( g(t) = ct^{-\gamma}. \)
\[ y = \frac{c}{(1-\gamma)(t^{1-\gamma})} \text{ for } 0 \leq \gamma < 1. \tag{116} \]
\[ r(t) = \hat{r}[y(t)]. \tag{117} \]

By Eqs. (102), (111), and (117),
\[ r(t) = (b/8)[1 - e^{-2\Delta}], \tag{118} \]
where
\[ \Delta \triangleq \frac{y - y_0}{b} = \frac{c}{b(1-\gamma)}(t^{1-\gamma} - t_0^{1-\gamma}). \tag{119} \]

Hence,
\[ r_\Delta(t) \approx r(t) \frac{\Delta k_0}{g(t_0)} g(t) \tag{120} \]
\[ = \frac{b}{8} (1 - e^{-2\Delta}) \left( \frac{t}{t_0} \right)^{-\gamma} \Delta k_0. \tag{121} \]

By Eqs. (96), (116), and (119),
\[ \mu(t) = \hat{\mu}[y(t)] = (b/2)e^{-\Delta}. \]

By Eq. (104),
\[ S'[\mu(t)] = -(2/b^2)\mu(t) = -(2/b)e^{-\Delta}. \]

Hence, by (112),
\[ q_\Delta(t) \approx \frac{1}{8b} e^{-2\Delta}(1 - e^{-2\Delta}) \left( \frac{t}{t_0} \right)^{-\gamma} \Delta k_0. \tag{122} \]

The right member of Eq. (122) is plotted in Fig. 10 for the case \( c = 1, \Delta k_0 = 1, t_0 = 1, \gamma = \frac{1}{2}, \) and labeled "\( \gamma = \frac{1}{2}. \)"
5. ERROR ANALYSIS

In the preceding sections, the learning curve \( z(t) \) of a continuous TLP has been proposed as an estimate of the learning curve \( z_\Delta(t) \) of any of the associated discrete TLPs. In this section we derive an expression for estimating the approximation error \( z_\Delta(t) - z(t) \).

Let

\[
\eta(t) \triangleq z_\Delta(t) - z(t)
\]

by Eqs. (20a), (63), and (52). Expand \( S(k) \) in a Taylor expansion about \( k = \mu(t) \), and apply the identity

\[
\sum_k [k - \mu(t)^2] p_\Delta(k, t) = \sum_k \{k - \mu_\Delta(t) + \mu_\Delta(t) - \mu(t)^2\} p_\Delta(k, t)
\]

This yields

\[
\eta(t) \approx \frac{1}{2} S'[\mu(t)]\{r_\Delta(t) + [\mu_\Delta(t) - \mu(t)]^2\}
\]

To make effective use of Eq. (125), we need to estimate \( \mu_\Delta(t) - \mu(t) \) for all \( t \). We don’t know a convenient way of doing this. For a less informative but still useful error analysis, we suggest computing \( \mu_\Delta(t) - \mu_\Delta(t) \) as an estimate of \( \mu(t) - \mu(t) \).

6. THE LEARNING SHAPE THEOREM

Under a relatively weak restriction on the constituent densities and \( k_0 \), the learning curves of all the continuous TLP training procedures satisfying (13) are monotonic decreasing, monotonic increasing, or single-peaked. We refer to this observation as the “learning shape theorem.”

**Theorem.** If in a continuous TLP there exists \( b \) such that

\[
|b| < \infty, v < b \Rightarrow f_0(v) > f_1(v),
\]

\[
v > b \Rightarrow f_0(v) < f_1(v),
\]

\( F_0(k) \) and \( F_1(k) \) are continuous functions of \( k \),

\( g(t) \) satisfies (13),
there exists θ such that \( F(θ) = 0 \) and \( \{v \neq θ \Rightarrow F(v) \neq 0\} \), \( (129) \)
and \( k_0 \neq θ \), \( (130) \)
then \( z(t) \) is either monotonic increasing, monotonic decreasing, or single-peaked.

Proof. By hypothesis (126) and Eq. (5), it follows that
\[
\frac{dS(v)}{dv} < 0 \text{ for } v > b, \\
\frac{dS(v)}{dv} > 0 \text{ for } v < b.
\]
By hypothesis (127), \( S(v) \) is a continuous function of \( v \). Hence, \( S(v) \) has a unique maximum at \( v = b \).

Using (29) and a restriction on hypothesis (129), we show in Appendix 2 that \( V(y) \), when plotted versus \( y \), starts at \( k_0 \) and asymptotically approaches \( θ \). Hence, if \( b \) lies between \( k_0 \) and \( θ \), then \( V(y) \) must approach \( θ \) monotonically with respect to \( y \). Since \( S(v) \) is single-peaked, and \( y(t) \) is a monotonic function of \( t \), \( z(t) \) must be single-peaked.

In a similar manner, it follows that \( z(t) \) is monotonic decreasing when \( b \leq k_0 < θ \) or \( θ < k_0 \leq b \), and that \( z(t) \) is monotonic increasing when \( k_0 < θ \leq b \) or \( b \leq θ < k_0 \). Q.E.D.

Empirical evidence supports the conjecture that the above Theorem applies to \( z_A(t) \) as well as to \( z(t) \), i.e., the Theorem seems to apply to discrete TLPs as well as continuous TLPs.

7. CONCLUDING REMARKS AND SUMMARY

By the theory presented in this paper, the learning curve and variance curve of a continuous TLP with a fixed-increment training procedure serve as canonical curves from which a linear or nonlinear change of time scale yields the learning curve and variance curve of any time-varying training procedure satisfying (13) and (81).

Thus the presented theory provides an over-all view of a large class of time-varying training procedures. This class of training procedures consists of all those which (a) raise \( k \) in response to a false alarm, lower \( k \) in response to a false rest, and keep \( k \) unchanged in response to a correct guess, and (b) adjust the size of the increment by an amount that depends only on the trial number and such that any value of \( k \) can eventually be reached.

Although in a specific threshold detector the success probability \( z_n \) may be high over an ensemble of threshold detectors, there is usually a
nonzero probability that \( k_n \) may be far from \( \mu_n \) and consequently that the conditional success probability \( S(k) \) may be far from \( z_n \). The variance curve \( q_n \) provides a measure of the likely difference between the conditional success probability and its mean value \( z_n \) in the working phase of any given TLP. Choosing a good training procedure is therefore facilitated by displaying the variance curves and learning curves together.

The dependence of the asymptotic success probability of the discrete TLP on the choice of \( \Delta k \) when \( g(t) = \text{constant} \), is also of some significance. This dependence is often relatively small, however, as illustrated in Figs. 7 and 8.

The task of optimizing the training procedure is facilitated by displaying the variance curves and the learning curves together. This is done in Fig. 11 for TLP 1 and two values of \( \gamma \). We see in this figure that a large \( \gamma \) (say \( \gamma = 1 \)) provides a relatively rapidly decreasing variance curve at the expense of a relatively slow learning curve, while a small \( \gamma \) (say \( \gamma = 0 \)) provides a relatively fast learning curve at the expense of a slowly decreasing variance curve. The designer may, if he wishes, choose a value of \( \gamma \) that provides a compromise between the speed of learning and the rates of reduction of the variance of \( S(k) \).

![Graph showing learning curves and variance curves for TLP 1](image-url)
It is interesting to compare the threshold variance curve and the
variance curve obtained for the case $\gamma = 0$. Example 1 in Section 4.4
illustrates this situation in Eqs. (103) and (106). We see here that $r_A(t)$
rises exponentially, reaching an asymptote of $b/8$, while $q_A(t)$ eventually
becomes negligibly small. Thus, even though the threshold continues to
move significantly for large $t$, this motion is to a great extent masked
by the function $S(k)$. By this masking effect $\varepsilon(\infty)$ is hardly affected by
the fluctuations in $k$ if the duration of training is sufficiently long. This
phenomenon points out that a perception mechanism does not necessarily
"learn" a distinct value of $k$; rather it learns a range of $k$ within which all
values of $k$ yield roughly the same success probability.

The learning shape theorem demonstrates that under certain weak
restrictions which are met by practically all nontrivial physical threshold
detectors with time-varying training procedures $z(t)$ must be either
monotonic or single-peaked.

We see three areas of application of the proposed theory.

1. **Engineering**: choosing an optimum or near-optimum training
procedure and a stopping rule for a given signal detector with
unknown noise statistics in the channel.

2. **Statistics**: choosing an optimum sequence of sizes of increment ad-
justments in various methods of stochastic approximation.
(This is conceptually similar to the engineering application.)

3. **Psychology**: modeling the constituent densities and the training
procedure associated with a human being's or an animal's
mechanisms of perception.

*Two interesting open problems are:

1. Does the learning shape theorem extend to discrete TLPs?
2. How does one optimize the training procedure over a given
class of constituent densities?

**APPENDIX 1**

Let

$$y(t) \triangleq \int_{t_0}^{t} g(t) \, dt + y_0 \quad (A1-1)$$

**Theorem A1.** $y(t)$ is a monotonic differentiable function in the interval
t$_0 \leq t < \infty$ and $y(\infty) = \infty$ if, and only if, $g(t)$ satisfies

$$0 < g(t) < \infty \quad \text{for} \quad t \geq t_0 \quad (A1-2)$$
and
\[ \int_{t_0}^{\infty} g(t) \, dt = \infty. \]  

**Proof.**

(a) Suppose \( g(t) \) satisfies Eqs. (A1-2) and (A1-3). Then \( y(t) \) is monotonic increasing and differentiable, by Eqs. (A1-2) and (A1-1). \( y(\infty) = \infty \) by Eq. (A1-2).

(b) Suppose \( y(t) \) is monotonic increasing and differentiable in \( t_0 \leq t < \infty \), and suppose \( y(\infty) = \infty \). Then
\[ 0 < \frac{dy}{dt} < \infty \quad \text{for} \quad t \geq t_0. \]  

Eqs. (A1-1) and (A1-4) imply Eq. (A1-2). Since \( y(\infty) = \infty \), we obtain Eq. (A1-3). Q.E.D.

**APPENDIX 2**

**Theorem A2.** If \( F'(\theta) = 0, F''(\theta) > 0, |F^{(m)}(\theta)| < \infty \) for \( m \geq 0 \), then \( V(\infty) = \theta \).

**Proof.** Since all of the derivatives of \( F(v) \) exist at \( v = \theta \),
\[ F(v) \cong F'(\theta)(v - \theta) \]

in the vicinity of \( v = \theta \). Hence, by Eq. (29),
\[ y = - \int_{k_0}^{V(y)} \frac{dv}{F(v)} \cong \frac{-1}{F'(\theta)} \int_{k_0}^{V(y)} \frac{dv}{v - \theta} \quad \text{for} \quad V(y) < \theta, V(y) \cong \theta, \]

since \( F(v) \) and \( v - \theta \) have the same sign, and \( F''(\theta) > 0 \) by hypothesis.
\[ \lim_{V(y) \to \theta} \left[ \frac{-1}{F'(\theta)} \int_{k_0}^{V(y)} \frac{dv}{v - \theta} \right] = \infty. \]

Hence,
\[ \lim_{V(y) \to \theta} \left[ - \int_{k_0}^{V(y)} \frac{dv}{F(v)} \right] = \infty = y. \]
Hence, \( V(\infty) = \theta \). Q.E.D.

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