A new method for solving linear ill-posed problems

Jianjun Zhang\textsuperscript{a,},* Musa Mammadov\textsuperscript{b,c}

\textsuperscript{a} Department of Mathematics, Shanghai University, Shanghai 200444, China
\textsuperscript{b} School of SITE, University of Ballarat, Vic 3350, Australia
\textsuperscript{c} National ICT Australia, VRL, Melbourne, Vic 3010, Australia

\textbf{A B S T R A C T}

In this paper, we propose a new method for solving large-scale ill-posed problems. This method is based on the Karush–Kuhn–Tucker conditions, Fisher–Burmeister function and the discrepancy principle. The main difference from the majority of existing methods for solving ill-posed problems is that, we do not need to choose a regularization parameter in advance. Experimental results show that the proposed method is effective and promising for many practical problems.

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1. Introduction

In this paper we consider linear systems that related to large-scale inverse problems in the form:

\[ Ax = b, \quad b = b_{\text{true}} + N, \quad b_{\text{true}} = A x_{\text{true}}, \]

where \( A \in \mathbb{R}^{m \times n} (m \geq n) \), \( b_{\text{true}} \in \mathbb{R}^m \) and \( x_{\text{true}} \in \mathbb{R}^n \). \( N \in \mathbb{R}^m \) represents unknown noise due to measurement interference and other errors in the recorded signal, as well as noise and inaccuracies in the measuring device.

Inverse problem of the form (1.1) arises in a variety of important applications in science and industry, including image reconstruction, image deblurring, geophysics, parameter identification and inverse scattering. See, for example [1–3] and the references therein. In these applications the goal is to estimate some unknown attributes of interest, given measurements that are only indirectly related to these attributes. Typically these problems are ill-posed, meaning that noise in the data may give rise to significant errors in computed approximations of \( x_{\text{true}} \). So regularization is necessary to deal with the ill-posedness.

Many regularization methods have been developed. Probably the most well known is Tikhonov regularization [1–5], which is equivalent to solving the least squares problem

\[ \min_x (\| Ax - b \|^2 + \lambda^2 \| L x \|^2), \]

where \( L \) is a regularization operator, often chosen as the identity matrix or a discretization of a differentiation operator. The regularization parameter \( \lambda \) is a scalar, usually satisfying \( \sigma_n \leq \lambda \leq \sigma_1 \), where \( \sigma_n \) is the smallest singular value of \( A \) and \( \sigma_1 \) is the largest singular value of \( A \). Here and throughout the paper the notation \( \| \cdot \| \) will stand for the Euclidian norm.

In Tikhonov regularization, parameter selection is very important. An optimal regularization parameter should fairly balance the perturbation error and the regularization error in the regularized solution. There are several possible strategies that depend on additional information referring to the analyzed problem and its solution, e.g., the discrepancy principle, the
L-curve and generalized cross validation (GCV) [2,3,6–9]. There are advantages and disadvantages to each of these approaches, especially for large-scale problems.

An alternative to Tikhonov regularization for large-scale problems is iterative regularization. In this case, an iterative method such as LSQR [10,11], is applied to the least squares problem,

$$\min_{x} ||b - Ax||,$$  \hspace{1cm} (1.3)

However, when applied to ill-posed problems, these iterative methods exhibit an interesting “semi-convergence” behavior [12,13], in that the quantity of the relative solution error first decreases and then increases.

The semi-convergence behavior of LSQR can be stabilized by using a hybrid method that combines an iterative Lanczos bidiagonalization algorithm with a direct regularization scheme, such as Tikhonov or truncated SVD. The basic idea of this approach is to project the large-scale problem onto Krylov subspaces of small (but increasing) dimension. The projected problem can be solved cheaply using any direct regularization method. In [12], a very effective regularization method called weighted-GCV (W-GCV) was developed.

In [14] regularization problem is formulated as a noise constrained minimization problem assuming that the noise level is explicitly known. Such formulation does not require to know a good estimate of the regularization parameter, however it needs an estimate of the noise level that is achievable in most applications. Then [14] applies iterative Lagrangian methods to the constrained minimization problem to compute both the regularization parameter and the corresponding regularized solution.

In this paper, we propose a new method for solving large-scale ill-posed problems. This method uses a noise constrained minimization formulation and is based on the Karush–Kuhn–Tucker conditions, Fisher–Burmeister function and the discrepancy principle. Similar to [14], the method proposed does not require to know a good estimate of the regularization parameter. Experimental results show that the proposed method is effective and promising for many practical problems.

The rest of the paper is organized as follows. In Section 2, we introduce a formulation of regularization problem as a noise constrained minimization problem, and present some preliminary results. In Section 3, we describe our proposed method. Experimental results are provided in Section 4, and some concluding remarks are given in Section 5.

2. Lagrangian method for ill-posed problems

There are different regularization methods studied in the literature. For example, Tikhonov’s regularization. In [2], Hansen suggested two methods that are formulated as the following least squares problems with constraints:

$$\min_{x} ||Ax - b|| \text{ subject to } ||L(x - x^*)|| \leq \alpha,$$ \hspace{1cm} (2.1)

$$\min_{x} ||L(x - x^*)|| \text{ subject to } ||Ax - b|| \leq \delta,$$ \hspace{1cm} (2.2)

where $x^*$ is a priori estimate of the desired regularized solution, and $\alpha$ and $\delta$ are nonzero parameters each playing the role of regularization parameter in (2.1) and (2.2), respectively.

Clearly, regularized solutions to (2.1) and (2.2) are tends to be closer to $x^*$. In many applications it is impossible to provide some “good” estimate for $x^*$. Considering $x^* = 0$ is the most common way dealing with such cases. In this paper, we will consider the following regularization method:

$$\min_{x} \frac{1}{2} ||Lx||^2 \text{ subject to } \frac{1}{2} ||Ax - b||^2 \leq \frac{1}{2} \delta^2,$$ \hspace{1cm} (2.3)

where $\delta$ is the noise level assumed to be explicitly known. This is a particular case of (2.2) corresponding to $x^* = 0$.

We note that, in [14] a more general form of $\phi(x)$ for the objective function in the statement (2.3) is considered. However, in the applications to linear ill-posed problems the objective function is often taken in the form $\phi(x) = \frac{1}{2} ||Lx||^2$, that is in (2.3).

The following theorems, proved in [14] for a more general case of convex objective functions $\phi(x)$, describe the behavior of the solutions to (2.3).

**Theorem 2.1.** Problem (2.3) has a solution.

**Theorem 2.2.** Let $L^T L x \neq 0$ for all $x$ satisfying $\frac{1}{2} ||Ax - b||^2 \leq \frac{1}{2} \delta^2$. If $x^*$ is a solution of (2.3), then $\frac{1}{2} ||Ax^* - b||^2 = \frac{1}{2} \delta^2$.

**Theorem 2.3.** Let $\mathrm{rank}(A) = n$ and all the assumptions of Theorem 2.2 hold. Then the solution $x^*$ of (2.3) is unique.

**Theorem 2.4.** Let all the assumptions of Theorem 2.3 hold. Then, (2.3) has a unique solution $x^*$ with positive Lagrange multiplier $\lambda^* > 0$.

Based on the above theorems, the inequality constrained minimization problem (2.3) is equivalent to the following equality constrained minimization problem.

$$\min_{x} \frac{1}{2} ||Lx||^2 \text{ subject to } \frac{1}{2} ||Ax - b||^2 = \frac{1}{2} \delta^2.$$ \hspace{1cm} (2.4)
Let \( h(x) = \frac{1}{2} \| Ax - b \|^2 \), and \( L(x, \lambda) = \frac{1}{2} \| Lx \|^2 + \lambda (h(x) - \frac{1}{2} \delta^2) \) be the Lagrangian function related to (2.4). Then the first-order necessary conditions for (2.4) are expressed as,

\[
\begin{align*}
\nabla_x L(x, \lambda) &= 0, \\
h(x) - \frac{1}{2} \delta^2 &= 0.
\end{align*}
\]

(2.5)

Let the merit function \( m : \mathbb{R}^{n+1} \rightarrow \mathbb{R} \) be defined as

\[
m(x, \lambda) = \frac{1}{2} \left( \| \nabla_x L(x, \lambda) \|^2 + \omega \left( h(x) - \frac{1}{2} \delta^2 \right)^2 \right),
\]

with \( \omega \in \mathbb{R} \) a positive parameter.

Using the Newton method to solve Lagrange equation (2.5), and also using Armijo’s condition [15] for line search, G. Landi presented the following algorithm in [14].

**Algorithm 2.1.** Second-order Lagrangian method

1. Computation of the search direction.
   Compute \( (\Delta x_k, \Delta \lambda_k) \) by solving the following linear equation.
   \[
   \begin{pmatrix}
   \nabla^2_x L(x_k, \lambda_k) & \nabla_x h(x_k) \\
   \nabla_x h(x_k)^T & 0
   \end{pmatrix}
   \begin{pmatrix}
   \Delta x_k \\
   \Delta \lambda_k
   \end{pmatrix}
   = - \begin{pmatrix}
   \nabla_x L(x_k, \lambda_k) \\
   h(x_k) - \frac{1}{2} \delta^2
   \end{pmatrix}.
   \]

2. Line search.
   Find the first number \( z_k \) of the sequence \( \left\{ 1, \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^n}, \ldots \right\} \) satisfying:
   (i) \( M(x_k + z\Delta x_k, \lambda_k + z\Delta \lambda_k) \leq m(x_k, \lambda_k) + \mu z_k (\Delta x_k, \Delta \lambda_k)^T \nabla_{(x, \lambda)} m(x_k, \lambda_k) \) with \( \mu = 10^{-4} \).
   (ii) \( \lambda_k + z_k \Delta \lambda_k > 0 \).
   (iii) \( A(x_k + z\Delta x_k) - b \neq 0 \).

3. Updates. Set
   \[
   x_{k+1} = x_k + z\Delta x_k, \\
   \lambda_{k+1} = \lambda_k + z_k \Delta \lambda_k.
   \]

Set \( k = k + 1 \) and go to step 1 until some stopping criteria is satisfied.

In [14], the following stopping criteria was used.

\[
\begin{align*}
\text{(i)} & \quad k \geq k_{\text{max}}, \\
\text{(ii)} & \quad \left\| \begin{pmatrix}
\nabla_x L(x_k, \lambda_k) \\
\nabla_x h(x_k) - \frac{1}{2} \delta^2
\end{pmatrix} \right\| \leq T_1 \left\| \begin{pmatrix}
\nabla_x L(x_0, \lambda_0) \\
\nabla_x h(x_0) - \frac{1}{2} \delta^2
\end{pmatrix} \right\|, \\
\text{(iii)} & \quad z_k \left\| \Delta x_k \lambda_k \right\| \leq T_2.
\end{align*}
\]

(2.8) (2.9) (2.10)

### 3. Our proposed method

In the last section, we show that, the inequality constrained minimization problem (2.3) is equivalent to the equality constrained minimization problem (2.4) under the conditions of Theorem 2.2. However in numerical implementations, the problem formulation in the form (2.3) is more preferable than (2.4) because of the difficulties related to equality constraints (see numerical results in Section 4). On the other hand, if the conditions of Theorem 2.2 are not satisfied, then the proposed algorithm in the last section cannot apply.

In this section we present a new approach for solving problem (2.3) for large-scale ill-posed problems. This approach involves Karush–Kuhn–Tucker conditions, Fisher–Burmeister function and the discrepancy principle. Similar to the method introduced in [14] (see Algorithm 2.1), the presented approach does not require any prior good estimate of the regularization parameter. The regularization parameter is updated iteratively.

Consider the minimization problem (2.3). By the Karush–Kuhn–Tucker conditions, there exists Lagrange multiplier \( \lambda > 0 \), such that

\[
\begin{align*}
L^T Lx + \lambda A^T (Ax - b) &= 0, \\
\lambda &> 0, \| Ax - b \|^2 \leq \delta^2.
\end{align*}
\]

(3.1)
Denote by $\varphi : R^2 \rightarrow R$ the Fisher–Burmeister function defined by [16]

$$\varphi(a, b) = \sqrt{a^2 + b^2} - a - b.$$ 

It possesses the following property:

$$\varphi(a, b) = 0 \iff a \geq 0, \quad b \geq 0 \quad \text{and} \quad ab = 0.$$ 

By using this property, it is easy to deduce that system (3.1) is equivalent to the following system of nonlinear equations

$$F(z) = 0,$$

where $F : R^{n+1} \rightarrow R^{n+1}$ is defined by

$$F(z) = \left( L^T L + \lambda_k A^T (Ax - b), \varphi \left( \lambda_k, \gamma \left( \delta^2 - \|Ax - b\|^2 \right) \right) \right).$$

Here $\gamma \in R$ is a positive parameter and $z = (x^T, \lambda)^T$. From Theorems 2.2 and 2.4 it follows that the Lagrange multiplier $\lambda$ corresponding to the solution of the problem (2.3) is positive: $\lambda > 0$. Then, we obtain that function $F$ is differentiable at the solution of the regularization problem (2.3). Therefore, the Newton method can naturally be applied to solve nonlinear Eqs. (3.2) and (3.3).

The application of the Newton method to solve nonlinear Eqs. (3.2) and (3.3) leads to the iterations

$$z_{k+1} = z_k + \Delta z_k, \quad k = 0, 1, 2 \ldots,$$

where $z_k = (x_k^T, \lambda_k)^T$, and $\Delta z_k = (\Delta x_k, \Delta \lambda_k)^T$ satisfies

$$\begin{pmatrix}
L^T L + \lambda_k A^T A & \zeta_k v_k^T \\
\zeta_k v_k & \eta_k
\end{pmatrix}
\begin{pmatrix}
\Delta x_k \\
\Delta \lambda_k
\end{pmatrix}
= -
\begin{pmatrix}
L^T L x_k + \lambda_k A^T (Ax_k - b) \\
\varphi \left( \lambda_k, \gamma \left( \delta^2 - \|Ax_k - b\|^2 \right) \right)
\end{pmatrix}.$$  

Here we use the following notations:

$$\zeta_k = \frac{-2\gamma^2 \left( \delta^2 - \|Ax_k - b\|^2 \right)}{\sqrt{\zeta_k^2 + \gamma^2 \left( \delta^2 - \|Ax_k - b\|^2 \right)^2}} + 2\gamma, \quad \eta_k = \frac{\lambda_k}{\sqrt{\zeta_k^2 + \gamma^2 \left( \delta^2 - \|Ax_k - b\|^2 \right)^2}} - 1$$

and $v_k = A^T (Ax_k - b)$.

We have the following theorem.

**Theorem 3.1.** Assume that $\lambda_k > 0$ and $\|Ax_k - b\|^2 > \delta^2$. Then the linear system (3.5) is well defined.

**Proof.** It is sufficient to show that the coefficient matrix in system (3.5) is nonsingular.

Suppose $y_1 \in R^a$ and $y_2 \in R$ such that the following condition holds

$$\begin{pmatrix}
L^T L + \lambda_k A^T A & \zeta_k v_k^T \\
\zeta_k v_k & \eta_k
\end{pmatrix}
\begin{pmatrix}
y_1 \\
y_2
\end{pmatrix}
= 0,$$

that is,

$$\begin{pmatrix}
(L^T L + \lambda_k A^T A)y_1 + y_2 v_k = 0, \\
\zeta_k v_k^T y_1 + \eta_k y_2 = 0.
\end{pmatrix}$$

From the first equation of (3.7), we have

$$y_1 = -y_2 (L^T L + \lambda_k A^T A)^{-1} v_k.$$ 

Substituting this in the second equation of (3.7), we obtain

$$-\zeta_k v_k^T (L^T L + \lambda_k A^T A)^{-1} v_k + \eta_k y_2 = 0.$$ 

Since

$$\zeta_k = \frac{-2\gamma^2 \left( \delta^2 - \|Ax_k - b\|^2 \right)}{\sqrt{\zeta_k^2 + \gamma^2 \left( \delta^2 - \|Ax_k - b\|^2 \right)^2}} + 2\gamma = 2\gamma \left( 1 - \frac{\gamma \left( \delta^2 - \|Ax_k - b\|^2 \right)}{\sqrt{\zeta_k^2 + \gamma^2 \left( \delta^2 - \|Ax_k - b\|^2 \right)^2}} \right) > 0$$

\(\Box\)
and
\[ \eta_k = \frac{\lambda_k}{\sqrt{\lambda_k^2 + \gamma^2 (\delta^2 - \|A\xi_k - b\|^2)^2}} - 1 < 0, \]
we have
\[ \left(-\lambda_k v_k^T (L^T L + \lambda_k A^T A)^{-1} v_k + \eta_k \right) \neq 0. \]
Therefore \( y_2 = 0 \). From the positive definiteness of matrix \( L^T L + \lambda_k A^T A \) and from the first equation in (3.7), we obtain \( y_1 = 0 \).
The above discussion shows that, linear system (3.6) has only zero solution. The theorem is proved. \( \Box \)

In order to globalize the Newton method, a line search technique is used to achieve a sufficient decrease in the natural merit function
\[ M(z) = \frac{1}{2} \|F(z)\|^2. \]

Now we describe the main algorithm of this paper.

**Algorithm 3.1** (Global Newton method for ill-posed problems).

0. Given initial point \( z_0 = (\xi_0^T, \lambda_0)^T \in R^{n+1} \). Set \( k = 0 \).
1. Computation of the search direction.
   Compute \((\Delta\xi_k, \Delta\lambda_k)\) by solving the following linear equation
   \[
   \begin{pmatrix}
   (L^T L + \lambda_k A^T A) & v_k \\
   \xi_k v_k^T & \eta_k
   \end{pmatrix}
   \begin{pmatrix}
   \Delta\xi_k \\
   \Delta\lambda_k
   \end{pmatrix}
   = - \begin{pmatrix}
   (L^T L + \lambda_k A^T A)(\Delta\xi_k - b) \\
   \eta_k (\Delta\lambda_k - \delta^2 - \|A\xi_k - b\|^2)
   \end{pmatrix}.
   \]
2. Line search.
   Find the first number \( a_k \) of the sequence \( \left\{ 1, \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{k}, \ldots \right\} \) satisfying:
   (i) \( m(x_k + a\Delta\xi_k, \lambda_k + a\Delta\lambda_k) \leq M(x_k, \lambda_k) + \mu a (\Delta\xi_k, \Delta\lambda_k)^T \nabla_{x, \lambda} M(x_k, \lambda_k) \) with \( \mu = 10^{-3} \).
   (ii) \( \lambda_k + a\Delta\lambda_k > 0 \).
3. Updates. Set
   \[
   x_{k+1} = x_k + a\Delta\xi_k,
   \]
   \[
   \lambda_{k+1} = \lambda_k + a\Delta\lambda_k.
   \]
Set \( k = k + 1 \) and goto step 1 until the following stopping criteria are satisfied.
   (i) \( k \geq k_{\text{max}} \).
   (ii) \( \frac{|M(x_{k+1}, \lambda_{k+1}) - M(x_k, \lambda_k)|}{M(x_k, \lambda_k)} \leq \tau_1 \).
   (iii) \( \left\| x_k \left( \begin{array}{c} \Delta\xi_k \\ \Delta\lambda_k \end{array} \right) \right\| \leq \tau_2 \).
   (iv) \( \|A\xi_k - b\|^2 \leq \delta^2 \).

**Remark.** It is generally difficult to solve the Newton equation (3.5) because of a large number of variables for large-scale ill-posed problem. In this case, inexact Newton methods are useful candidates [17]. Combining with Krylov subspace methods, the nonlinear equation can be solved efficiently. Based on (3.2 and 3.3), any efficient method for solving nonlinear equations can be used to solve linear ill-posed problems.

4. Numerical results

In this section, we present four numerical examples taken from the “Regularization Tools” package [9]. In each case we generate a \( 256 \times 256 \) matrix \( A \), solution \( x_{\text{true}} \) and a noise free observation vector \( b_{\text{true}} = Ax_{\text{true}} \). The noise vector \( b \) was generated by \( b = b_{\text{true}} + N \), where \( N \) is a noise vector whose entries are chosen from a normal distribution with mean 0 and variance 1, and scaled so that
\[ \frac{\|N\|}{\|b_{\text{true}}\|} = 0.01. \]
As an evaluation measure, we use the relative error
\[ R_{err} = \frac{k \| \text{computed} \|}{k \| \text{true} \|} \]
between the computed solution and the exact one.

### Table 1
Relative error of Algorithms 3.1 and 2.1 and HyBR for the four examples with noise level \( \delta = (1 + \text{eps}) \times \| N \| \).

<table>
<thead>
<tr>
<th>Problem</th>
<th>Phillips</th>
<th>Heat</th>
<th>Deriv2</th>
<th>Baart</th>
</tr>
</thead>
<tbody>
<tr>
<td>HyBR</td>
<td>0.0226</td>
<td>0.1002</td>
<td>0.2518</td>
<td>0.1670</td>
</tr>
<tr>
<td>Algorithm 2.1</td>
<td>0.1253</td>
<td>0.0902</td>
<td>0.3762</td>
<td>0.1021</td>
</tr>
<tr>
<td>Algorithm 3.1</td>
<td>0.0225</td>
<td>0.0937</td>
<td>0.0475</td>
<td>0.0475</td>
</tr>
</tbody>
</table>

### Table 2
Relative error of Algorithm 3.1 and 2.1 for the four examples with noise level \( \delta = 1.1 \times \| N \| \).

<table>
<thead>
<tr>
<th>Problem</th>
<th>Phillips</th>
<th>Heat</th>
<th>Deriv2</th>
<th>Baart</th>
</tr>
</thead>
<tbody>
<tr>
<td>Algorithm 2.1</td>
<td>0.1254</td>
<td>0.1210</td>
<td>0.3764</td>
<td>0.1022</td>
</tr>
<tr>
<td>Algorithm 3.1</td>
<td>0.0284</td>
<td>0.1208</td>
<td>0.0839</td>
<td>0.0682</td>
</tr>
</tbody>
</table>

**Fig. 1.** The true solution and the computed solutions, corresponding to noise level \( \delta = (1 + \text{eps}) \times \| N \| \), for the four problems (a) Phillips; (b) Heat; (c) Deriv2; and (d) Baart.
Example 4.1. “Phillips”. This test problem is obtained by discretizing the first kind Fredholm integral equation
\[ b(s) = \int_0^s a(s, t)x(t)\,dt, \]
where
\[
\begin{align*}
a(s, t) &= \rho(s - t), \quad x(t) = \rho(t), \\
b(s) &= (6 - |s|) \left(1 + \frac{1}{2} \cos \frac{\pi s}{3} + \frac{9}{2\pi} \sin \frac{\pi |s|}{3}\right), \\
\rho(t) &= \begin{cases} 1 + \cos \frac{\pi t}{2}, & |t| < 3, \\
0, & |t| \geq 3. \end{cases}
\end{align*}
\]

Example 4.2. “Heat” is an inverse heat equation using the Volterra integral equation of the first kind on [0, 1] with
\[ a(s, t) = k(s - t), \]
where
\[ k(t) = \frac{t^{\frac{3}{2}}}{2\sqrt{\pi}} \exp \left(-\frac{1}{4t}\right). \]

Example 4.3. “Deriv2” constructs A and \( b_{\text{true}} \) by discretizing a first kind Fredholm integral equation
\[ b(s) = \int_0^s a(s, t)x(t)\,dt, \]
\[ 0 \leq s \leq 1, \]
where the kernel \( a(s, t) \) is given by the Green’s function for the second derivative.
Example 4.4. “Baart” constructs \( A \) and \( b_{\text{true}} \) by discretizing a first kind Fredholm integral equation \( b(s) = \int_0^s a(s, t)x(t)dt \), \( 0 \leq s \leq \frac{1}{2} \), where

\[
\begin{align*}
a(s, t) &= \begin{cases} 
    s(t - 1), & s < t, \\
    t(s - 1), & s \geq t,
  \end{cases} \\
x(t) &= t, \quad b(s) = (s^2 - s)/6.
\end{align*}
\]

For the above four examples, we apply the algorithm presented in this paper (Algorithm 3.1), as well as the second-order Lagrangian method proposed in [14] (Algorithm 2.1). We also apply the hybrid method with W-GCV regularization (denoted as HyBR) proposed recently in [12], as it is proven to be a very efficient method for solving large-scale ill-posed problems.

In the experiments, we set \( x_0 \) to be the zero vector, \( \lambda_0 = 1 \) as in [14]. For both, Algorithm 3.1 and Algorithm 2.1, we set \( \tau_1 = 10^{-4}, \tau_2 = 10^{-8} \), \( w = \gamma = 10^{10} \). All the algorithms are stopped after 50 iterations. HyBR method is stopped when

\[
\frac{|\text{GCV}(k) - \text{GCV}(k - 1)|}{\text{GCV}(1)} \leq 10^{-6},
\]

or \( k > 50 \) (see [12] for details).

We set \( \delta = (1 + \epsilon) \cdot ||N|| \), where \( \epsilon \) is the machine precision and \( N \) represents the noise, for the noise level in Algorithms 3.1 and 2.1.

The relative errors of these three algorithms are listed in Table 1. The true solution \( x_{\text{true}} \), and the computed solution are depicted in Fig. 1.

According to Fig. 1 and Table 1, we see that, Algorithm 3.1, presented in this paper, generally provides more accurate results.

Algorithms 3.1 and 2.1 assume that, the noise level is explicitly known. But in applications, the noise level is usually not known in advance. In this case, when the noise level, estimated by some method, is not accurate, however is a “good” estimate, Algorithms 3.1 and 2.1 can still get satisfied results. It can be seen from the results presented in Table 2, where relative errors for these algorithms are obtained by setting the noise level as \( \delta = 1.1 \cdot ||N|| \). The true solution and the corresponding computed solution are depicted in Fig. 2.

5. Conclusions

In this paper, we present a new method for large-scale ill-posed problems based on the Karush–Kuhn–Tucker conditions, Fisher–Burmeister function and the discrepancy principle. This method does not require a prior good estimate for the regularization parameter. The regularization parameter is updated iteratively. We compare our proposed method with other two regularization methods used in the literature. The results of numerical experiments show the effectiveness of our proposed method.

Acknowledgments

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References