

Untrusted Predictions Improve Trustable Query Policies

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Abstract

We study how to utilize (possibly machine-learned) predictions in a model for optimization under uncertainty that allows an algorithm to query unknown data. The goal is to minimize the number of queries needed to solve the problem. Considering fundamental problems such as finding the minima of intersecting sets of elements or sorting them, as well as the minimum spanning tree problem, we discuss different measures for the prediction accuracy and design algorithms with performance guarantees that improve with the accuracy of predictions and that are robust with respect to very poor prediction quality. We also provide new structural insights for the minimum spanning tree problem that might be useful in the context of explorable uncertainty regardless of predictions. Our results prove that untrusted predictions can circumvent known lower bounds in the model of explorable uncertainty. We complement our results by experiments that empirically confirm the performance of our algorithms.

1 Introduction

Dealing with uncertainty is a common challenge in many real-world settings. The research area of *explorable uncertainty* [22, 39] considers such scenarios assuming that, for any uncertain input element, a *query* can be used to obtain the exact value of that element. The uncertain input value is often represented by an interval that contains the exact value, and a query returns that exact value. Queries are costly, and hence the goal is to make as few queries as possible until sufficient information has been obtained to solve the given problem. The major challenge is to balance the resulting exploration-exploitation tradeoff. For all problems that we consider, there exist input instances that are impossible to solve without querying the entire input. Therefore, instead of aiming to derive absolute bounds on the number of queries required for an input of size n in the worst case, we use competitive analysis to compare the number of queries made by an algorithm with the minimum number of queries among all feasible solutions, i.e., we aim for query-competitive algorithms.

In this query model, we consider very fundamental problems that underlie numerous applications: sorting, computing the minimum element, and computing a minimum spanning tree in a graph with uncertain edge weights. These problems are well understood in the setting of explorable uncertainty: The best known deterministic algorithms are 2-competitive and no deterministic algorithm can be better [24, 37, 39, 49]. For the sorting and minimum problems, we consider the setting where we want to solve the problem for a number of different, possibly overlapping subsets of a given ground set of uncertain elements. Such settings can be motivated e.g. by distributed database caches [52] where one wants to answer database queries using cached local data and a minimum number of queries to the master database. The minimum spanning tree (MST) problem is one of the most fundamental combinatorial problems. It is among the most widely

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studied problems in computing with explorable uncertainty and has been a cornerstone in the development of algorithmic approaches as well as lower bound techniques [22, 24, 49].

Instead of assuming that no information about an uncertain value is available except for the interval in which the value is contained, we study for the first time the setting of computing with uncertainty where predictions for the uncertain values are available. For example, machine learning could be used to predict the value of an interval. There has been tremendous progress in artificial intelligence and machine learning (ML) in recent decades. ML methods advance at rapid speed and can be expected to predict often, but not always, the exact values of uncertain intervals with good accuracy. This lack of provable performance guarantees for ML often causes concerns regarding how confident one can be that an ML algorithm will perform sufficiently well in all circumstances. In many settings, e.g., safety-critical applications, having provable performance guarantees is highly desirable or even obligatory. It is a very natural question whether the availability of such (ML) predictions can be exploited by query algorithms for computing with explorable uncertainty. Ideally, an algorithm should perform very well if predictions are accurate, but even if they are arbitrarily wrong, the algorithm should not perform worse than an algorithm that handles the problem without access to predictions. To emphasize that the predictions can be wrong, we refer to them as *untrusted predictions*.

We study for the first time the combination of explorable uncertainty and untrusted predictions. Our work draws inspiration from recent work that has considered untrusted predictions in the context of online algorithms, where the input is revealed to an algorithm incrementally and the algorithm must make decisions without knowledge of future inputs. We adopt the notions of α -consistency and β -robustness [41, 45]: An algorithm is α -consistent if it is α -competitive when the predictions are correct, and it is β -robust if it is β -competitive no matter how wrong the predictions are. Furthermore, we are interested in a smooth transition between the case with correct predictions and the case with arbitrarily incorrect predictions: We aim for performance guarantees that degrade gracefully with the *amount* of prediction error. This raises interesting questions regarding appropriate ways of measuring prediction errors, and we explore several such measures.

Our results show that, in the setting of explorable uncertainty, it is in fact possible to exploit ML predictions of the uncertain values in such a way that the performance of an algorithm is improved when the predictions are good, while at the same time a strong bound on the worst-case performance can be guaranteed even when the predictions are arbitrarily bad. Following this approach, ML can thus be embedded within a system to improve the typical performance while still maintaining provable worst-case guarantees. In this way users can be shielded from occasional failures of the ML algorithms. Therefore, our approach contributes methods and analysis techniques that can help to address the challenge of building trustable AI systems.

Main results We show how to utilize (possibly machine-learned) predictions in a query-based model for optimization under uncertainty and prove worst-case performance guarantees. Our major contribution is twofold: (a) we prove that untrusted predictions can circumvent lower bounds in the context of explorable uncertainty, and (b) we provide new structural insights for previously studied, fundamental problems that might be useful regardless of predictions. Finally, we conduct experiments that show that our algorithms are practical and that confirm the theoretical improvement.

For the problems of sorting or identifying minima in intersecting sets and finding an MST, we give algorithms that are 1.5-consistent and 2-robust, and show that this is the best possible consistency when aiming for optimal robustness. We also give a parameterized robustness-consistency tradeoff. Our major focus lies on a more fine-grained performance analysis with guarantees that improve with the accuracy of the predictions. We compare three different measures $k_{\#}$, k_M , k_h for the prediction accuracy. The number of inaccurate predictions $k_{\#}$ is too crude to allow for a performance improving on the lower bounds of 2 for the setting without predictions [24, 39]. We propose two measures that take structural insights about uncertainty intervals into account, the hop distance k_h and the mandatory query distance k_M . The latter can be proven to be more restrictive, i.e., $k_M \leq k_h$.

For the problem of identifying the minima in intersecting sets, we provide an algorithm with competitive ratio $\min\{(1 + \frac{1}{\gamma-1})(1 + \frac{k_M}{\text{opt}}), \gamma\}$, for any integral $\gamma \geq 2$. This is best possible for $k_M = 0$ and large k_M . With respect to the hop distance, we achieve the stronger bound $\min\{(1 + \frac{1}{\gamma})(1 + \frac{k_h}{\text{opt}}), \gamma\}$, for any integral $\gamma \geq 2$, which is also best possible for $k_h = 0$ and large k_h . It is not difficult to see that the sorting problem can be reduced to the minimum problem by creating a set for each pair of elements that are in the same set of the sorting instance, so these bounds also apply to the sorting problem. For the special case of sorting a single set, we obtain an algorithm competitive ratio $\min\{1 + k/\text{opt}, 2\}$ for any of the considered accuracy measures, which is best possible. Finding the MST under uncertainty is the combinatorially hardest problem. As our main result, we give an algorithm with competitive ratio $\min\{1 + \frac{1}{\gamma} + (5 + \frac{1}{\gamma}) \cdot \frac{k_h}{\text{opt}}, \gamma + \frac{1}{\text{opt}}\}$, for any integral $\gamma \geq 2$. Proofs omitted due to space restrictions are provided in the appendix.

Further related work There is a long history of research on the tradeoff between exploration and exploitation when coping with uncertainty in the input data. Stochastic models are often assumed, e.g., in work on multi-armed bandits [16, 28, 58], Weitzman’s Pandora’s box problem [60], and more recently query-variants of combinatorial optimization problems; see, e.g., [32, 57], and specific problems such as stochastic knapsack [19, 46], orienteering [10, 33], matching [8, 9, 12, 14, 17], and probing problems [1, 34, 35].

In our work, we assume no knowledge of stochastic information and aim for robust algorithms that perform well even in a worst case. This line of research on (adversarial) explorable uncertainty has been initiated by Kahan [39] in the context of selection problems. In particular, he showed for the problem of identifying all maximum elements of a set of uncertain values that querying the intervals in order of non-increasing right endpoints requires at most one more query than the optimal query set. Subsequent work addressed finding the k -th smallest value in a set of uncertainty intervals [26, 36], caching problems [52], computing a function value [40], sorting [37], and combinatorial optimization problems, such as shortest path [25], the knapsack problem [29], scheduling problems [3, 6, 20], the MST problem and matroids [21, 24, 27, 49, 50].

Most related to our work are previous results on the MST problem and sorting with explorable uncertainty. For the MST problem with uncertain edge weights represented by open intervals, a 2-competitive deterministic algorithm was presented and shown to be best possible [24]. The algorithm is based on the concept of *witness sets*, i.e., sets of uncertain elements with the property that any feasible query set must query at least one element of the set. The algorithm from [24] repeatedly identifies a witness set of size 2 that corresponds to two candidates for the maximum-weight edge in a cycle of the given graph, and queries both its elements. It is also known that randomization admits an improved competitive ratio of 1.707 for the MST problem with uncertainty [49]. Both, a deterministic 2-competitive algorithm and a randomized 1.707-competitive algorithm, are known for the more general problem of finding the minimum base in a matroid [23, 49], even for the case with non-uniform query costs [49]. For sorting a single set, a 2-competitive algorithm exists (even with arbitrary query costs) and is best possible [37]. In the case of uniform query costs, the algorithm simply queries witness sets of size 2; in the case of arbitrary costs, it first queries a minimum-weight vertex cover of the interval graph corresponding to the instance and then executes any remaining queries that are still necessary. For uniform query cost, the competitive ratio can be improved to 1.5 using randomization [37].

Our work is the first to consider explorable uncertainty in the recently proposed framework of online algorithms using (machine-learned) predictions [45, 48, 54]. Kumar et al. [54] studied online algorithms with respect to consistency and robustness in the context of classical online problems, ski-rental and non-clairvoyant scheduling. They also studied the performance as a function of the prediction error. This work initiated a vast growing line of research. Studied problems include rent-or-buy problems [31, 54], revenue optimization [48], scheduling and bin packing [4, 42, 51, 54], caching [5, 45, 55] and matching [41]. Very recently and in a similar spirit as our work, Lu et al. [44] studied a generalized sorting problem with additional predictions. Their model strictly differs from ours, as they focus on bounds for the absolute number of pair-wise comparisons whereas we aim for query-competitive algorithms. Overall, learning-augmented online

optimization is a highly topical concept which has not yet been studied in the explorable uncertainty model.

There is a significant body of work on computing in models where information about a hidden object can be accessed only via queries. The hidden object can for example be a function, a matrix, or a graph; we focus on the graph case in the following. In property testing [30], which has been studied extensively since the early 1990s, a typical problem is to decide whether a given graph has a certain property or is “far” from having that property using a small (sublinear or even constant) number of queries that look up entries of the adjacency matrix of the graph. Many other types of queries have also been studied (see e.g. [7, 11, 18, 47, 56] and many more): Linear queries (that return the scalar product of the query vector with a vectorized adjacency matrix), OR queries (given a subset of positions in the adjacency matrix, return if there is at least one 1 in those positions), Cross queries (given two disjoint subsets A and B of the vertex set, return the number of edges between A and B), BIS queries (given two disjoint subsets A and B of the vertex set, return whether there is at least one edge between A and B), and additive queries (given a subset S of the vertex set, return the number of edges in the subgraph induced by S), to name but a few. Work in these models has often considered graph reconstruction problems or parameter estimation problems (e.g., estimating the number of edges). The bounds on the number of queries made by an algorithm that have been shown in these problems are usually absolute, i.e., given as a function of the input size, but independent of the input graph itself, and the resulting correctness guarantees are often probabilistic.

In contrast to much of the work on algorithms with query access to a hidden object, we evaluate our algorithms in an instance-dependent manner: For each input, we compare the number of queries made by an algorithm with the best possible number of queries *for that input*, using competitive analysis. In computing with uncertainty, there are typically inputs where even an optimal query set has to query essentially the whole input in order to be able to solve the problem, hence absolute bounds on the number of queries depending only on the size of the input would often be trivial. The goal is hence to devise algorithms that use, on each input, a number of queries that is not much larger than the optimal query set for that input.

2 Definitions, accuracy of predictions, and lower bounds

Problem definitions In the *minimum problem under uncertainty*, we are given a set \mathcal{I} of n uncertainty intervals with a predicted value $\bar{w}_i \in I_i$ for each $I_i \in \mathcal{I}$, and a family \mathcal{S} of m subsets of \mathcal{I} . The *true* value of interval I_i is denoted by w_i and can be revealed by a *query*. The task is to identify for each $S \in \mathcal{S}$ the element with the minimum true value.

The *sorting problem under uncertainty* is closely related to the minimum problem. For the same input, the task is to sort, for each set $S \in \mathcal{S}$, the intervals in non-decreasing order of their true values.

In the *minimum spanning tree (MST) problem under uncertainty*, we are given a graph $G = (V, E)$, with uncertainty intervals I_e and predicted values $\bar{w}_e \in I_e$ for the weight of each edge $e \in E$. A minimum spanning tree (MST) is a tree that connects all vertices of G at a minimum total edge weight. The task is to find an MST with respect to the true values of the edge weights.

In all three problems, the goal is to solve the task using a minimum number of queries. Note that the exact value of a solution, i.e., the minimum value or the weight of the MST, does not need to be determined. We further assume that each uncertainty interval is either trivial or open, i.e., $I_i = (L_i, U_i)$ or $I_i = \{w_i\}$, as otherwise a simple lower bound of n on the competitive ratio exists for the minimum and MST problems [36]. Further, we study *adaptive* strategies that make queries sequentially and utilize the results of previous steps to decide upon the next query. We impose no time/space complexity constraints on the algorithms, as we are interested in understanding the competitive ratio of the problems. We assume the algorithms never query intervals that are trivial or that were previously queried. A set W of queries is called a *witness set* [15, 24] if every feasible solution (i.e., every set of queries that solves the problem) contains at least one query in W .

Competitive analysis We employ competitive analysis and compare the outcome of our algorithms with the best offline solution, i.e., the minimum number of queries needed to verify a solution when all values are known in advance. We call the offline variants of our problems *verification problems*. By OPT we denote an arbitrary optimal query set for the verification problem, and by opt its cardinality. For an algorithm for the online problem, we denote by ALG the set of queries it makes and by $|\text{ALG}|$ the cardinality of that set. An algorithm is ρ -competitive if it executes, for any problem instance, at most $\rho \cdot \text{opt}$ queries. Further, we quantify the performance of our algorithms depending on the quality of predictions. For the extreme cases, we say that an algorithm is α -consistent if it is α -competitive if the predictions are correct, and β -robust if it is β -competitive if the predictions are inaccurate.

Clearly, an algorithm that assumes the predicted values to be correct and solves the verification problem is 1-consistent. However, such an algorithm may have an arbitrarily bad performance if the predictions are incorrect. Similarly, the known deterministic 2-competitive algorithms for the online problems without predictions [24, 39] are 2-robust and 2-consistent. The known lower bounds of 2 rule out any robustness factor less than 2 for our model. We give a bound on the best achievable tradeoff between consistency and robustness. Later, we will provide algorithms with matching performance guarantees.

Theorem 2.1. Let $\beta \geq 2$ be a fixed integer. For the minimum (even in a single set), sorting and MST problems under uncertainty, there is no deterministic β -robust algorithm that is α -consistent for $\alpha < 1 + \frac{1}{\beta}$. And vice versa, no deterministic α -consistent algorithm, with $\alpha > 1$, is β -robust for $\beta < \max\{\frac{1}{\alpha-1}, 2\}$.

Accuracy of predictions We aim for a more fine-grained performance analysis giving guarantees that depend on the quality of predictions. A very natural, simple error measure is the number of inaccurate predictions, i.e., $k_{\#} = |\{I_i \in \mathcal{I} \mid w_i \neq \bar{w}_i\}|$. However, we show that for $k_{\#} \geq 1$ the competitive ratio cannot be better than the known lower bounds of 2. The reason for the weakness of this measure is that it completely ignores the interleaving structure of the intervals. To address this, we consider two measures for the predictor quality, the *hop distance* and the *mandatory query distance*.

Hop distance. For a non-trivial interval $I_j = (L_j, U_j)$, we say that the value of interval I_i passes over L_j if one of w_i, \bar{w}_i is $\leq L_j$ and the other is $> L_j$. Similarly, the value of I_i passes over U_j if one of w_i, \bar{w}_i is $< U_j$ and the other is $\geq U_j$. Intuitively, the value of I_i passes over one endpoint of I_j if it enters or leaves I_j , and it passes over both endpoints of I_j if it jumps over I_j when going from predicted to true values. For a trivial interval $I_j = \{w_j\}$, we say that the value of I_i jumps over I_j if one of w_i, \bar{w}_i is strictly smaller than w_j and the other is strictly larger than w_j . To avoid counting values passing over endpoints of irrelevant intervals, let A_i be the set of intervals that potentially interact with I_i (defined in a problem-specific way). For the minimum and the sorting problem, we let A_i be the union of all sets that contain I_i . When considering the MST problem, we consider the maximal biconnected component containing I_i , i.e., A_i is the union of all intervals on cycles containing I_i . Now define $h_i = h_i(A_i)$ to be the number of non-trivial intervals $I_j \in A_i$ such that the value of I_i passes over L_j plus the number of non-trivial intervals $I_j \in A_i$ such that the value of I_i passes over U_j , plus the number of trivial intervals $I_j = \{w_j\}$ in A_i such that the value of I_i jumps over I_j . The hop distance of a given instance is then $k_h = \sum_{i=1}^n h_i$. Note that $k_{\#} = 0$ implies $k_h = 0$, so Theorem 2.1 implies that no algorithm can simultaneously have competitive ratio better than $1 + \frac{1}{\beta}$ if $k_h = 0$ and β for arbitrary k_h .

Mandatory query distance. While the hop distance takes structural information regarding the interval structure into account, it does not distinguish whether a ‘‘hop’’ affects a feasible solution. We introduce a third and strongest measure for the prediction accuracy based on the following definition.

Definition 2.2 (mandatory). Given a problem instance with uncertainty intervals, an interval is *mandatory* if it is in each feasible query set of the verification problem. An interval is *prediction mandatory* if it is in each feasible query set assuming that the predictions \bar{w} are accurate.

Let \mathcal{I}_P be the set of prediction mandatory elements, and let \mathcal{I}_R be the set of really mandatory elements. The *mandatory query distance* is the size of the symmetric difference of \mathcal{I}_P and \mathcal{I}_R , i.e., $k_M = |\mathcal{I}_P \Delta \mathcal{I}_R| = |(\mathcal{I}_P \cup \mathcal{I}_R) \setminus (\mathcal{I}_P \cap \mathcal{I}_R)| = |(\mathcal{I}_P \setminus \mathcal{I}_R) \cup (\mathcal{I}_R \setminus \mathcal{I}_P)|$.

We can relate k_M to k_h in the following theorem.

Theorem 2.3. For any instance of the minimum, sorting and MST problems under uncertainty, the hop distance is at least as large as the mandatory query distance, $k_M \leq k_h$.

We provide a lower bound on the competitive ratio that is stronger than Theorem 2.1, and later we give matching algorithms for the minimum and sorting problems. The choice of $\gamma \geq 2$ is due to the lower bound of 2 in the robustness for all problems we consider.

Theorem 2.4. Let $\gamma \geq 2$ be a fixed rational value. If a deterministic algorithm for the minimum, sorting or MST problem is γ -robust, then it cannot have competitive ratio better than $1 + \frac{1}{\gamma-1}$ for $k_M = 0$. Furthermore, if an algorithm has competitive ratio $1 + \frac{1}{\gamma-1}$ for $k_M = 0$, then it cannot be better than γ -robust.

We conclude the definition and discussion of measures for the prediction accuracy with a simple lower bound on the competitive ratio regardless of any desired robustness.

Theorem 2.5. Any deterministic algorithm for minimum, sorting or MST under uncertainty has a competitive ratio $\rho \geq \min\{1 + \frac{k}{k_{\text{opt}}}, 2\}$, for any error measure $k \in \{k_{\#}, k_M, k_h\}$, even for disjoint sets.

3 General Methodology for Achieving Trustable Guarantees

Given predicted values for the uncertainty intervals, it is tempting to simply run an optimal offline algorithm (verification algorithm) under the assumption that the predictions are correct, and to then perform all the queries computed by that verification algorithm. This is obviously optimal with respect to consistency, but might give arbitrarily bad solutions in the case when the predictions are faulty. Instead of blindly following the offline algorithm, we need a strategy to be robust against prediction errors. Therefore, we carefully combine structural properties of the problem with the additional knowledge of untrusted predictions.

The crucial structure and unifying concept in all problems under consideration are *witness sets*. Witness sets are the key to any comparison with an optimal solution. A “classical” witness set is a set of elements for which we can guarantee that any feasible solution must query at least *one* of these elements. Depending on the particular problem, witness sets can be structurally very different (e.g., simply pairs of overlapping intervals for one problem, or sets resulting from a complex consideration of cycles in a certain order for another problem). Nevertheless, in the classical setting without access to predictions, all our problems admit 2-competitive online algorithms that rely essentially on identifying and querying disjoint witness sets of size two. We refer to witness sets of size two also as *witness pairs*. While completely relying on querying witness pairs ensures 2-robustness, it does not lead to any improvements in terms of consistency. In order to obtain an improved consistency while maintaining optimal or near-optimal robustness, we need to carefully balance the usage of an offline algorithm with the usage of witness sets.

The general framework We begin by describing our general algorithmic framework. On a high level, it follows the structure of the offline algorithm: In a first stage, it queries elements that are mandatory under the assumption that the predictions are correct. In a second stage, when no elements are prediction mandatory any more, the algorithm has to decide for each witness pair which of the two elements to query. If the predictions are correct, the second phase reduces to a vertex cover type problem. We show the following result.

Theorem 3.1. For each of the problems, minimum, sorting and MST under uncertainty, there is an algorithm that is 1.5-consistent and 2-robust.

During the first phase, just querying prediction mandatory elements can be arbitrarily bad in terms of robustness (cf. Theorem 2.1). Thus, given a prediction mandatory element that we wish to query, we query further elements in such a way that they form witness sets. Note that it is not sufficient to augment the set to a “classical” size-2 witness set (which already might be non-trivial to do for some problems), as this would not yield a performance guarantee better than 2 even if the predictions are correct. Instead, we identify a set of elements for which we can guarantee that at least 2 out of 3 of them must be queried by any feasible solution. Since we cannot always find such elements based on structural properties alone (otherwise, there would be a 1.5-competitive algorithm for the problem without predictions), we identify such sets under the assumption that the predictions are correct. Even after identifying such elements, the algorithm needs to query them in a careful order: If the predictions are wrong, we lose the guarantee on the elements and querying all of them might violate the 2-robustness. The first phase of our framework repeatedly identifies such elements and queries them in a careful order while adjusting for potential errors, until no unqueried prediction mandatory elements remain. Identifying the three elements with the guarantee mentioned above is a major contribution which might be of independent interest regardless of predictions, especially in the context of the MST problem. While existing algorithms for MST under uncertainty [24, 49] essentially follow the algorithms of Kruskal or Prim and only identify witness sets in the cycle or cut that is currently under consideration, we derive criteria to identify additional witness sets outside the current cycle/cut. The first phase of our framework differs for the different problems only in the identification of witness sets and the order in which we query them.

In the second phase, there are no more prediction mandatory elements. Therefore, the algorithm cannot identify any more “safe” queries and, for each witness pair, has to decide which element to query. This phase boils down to finding a minimum vertex cover in an auxiliary graph representing the structure of the witness sets. In particular, the second phase for the minimum problem and the sorting problem is relatively straightforward and consists of finding and querying a minimum vertex cover. If the predictions are correct, querying the vertex cover solves the remaining problem with an optimal number of queries. Otherwise, additional queries might be necessary, but we can show that this does not violate 2-robustness.

In the MST problem, wrong predictions can change the vertex cover instance dynamically, and therefore it must be solved in a very careful and adaptive way, requiring substantial additional work. Nevertheless, the general framework is the same for all problems.

Parameterized error-sensitive guarantees We show how to refine our general framework to achieve error-sensitive and parameterized performance guarantees. We parameterize the first phase as follows: we repeatedly query $\gamma - 2$ prediction mandatory elements in addition to the set of three elements of the general framework. The second phase remains unchanged. For both, the minimum problem and sorting, we can bound the error-sensitivity by charging each queried prediction mandatory element that turns out to be not mandatory and each additional query in the second phase to a distinct prediction error in the k_h -metric.

Theorem 3.2. There is an algorithm for the minimum problem under uncertainty that, given an integer parameter $\gamma \geq 2$, achieves a competitive ratio of $\min\{(1 + \frac{1}{\gamma})(1 + \frac{k_h}{\text{opt}}), \gamma\}$. Furthermore, if $\gamma = 2$, then the competitive ratio is $\min\{1.5 + k_h/\text{opt}, 2\}$.

The more adaptive and dynamic nature of the MST problem complicates the handling of Phase 2. We therefore employ again a more adaptive strategy to cover for potential prediction errors in Phase 2. By using a charging/counting scheme that builds on König-Egerváry’s famous theorem on the duality of minimum vertex covers and maximum matchings in bipartite graphs, we achieve the following result.

Theorem 3.3. There is an algorithm for the MST problem under uncertainty with competitive ratio $\min\{1 + \frac{1}{\gamma} + \left(5 + \frac{1}{\gamma}\right) \cdot \frac{k_h}{\text{opt}}, \gamma + \frac{1}{\text{opt}}\}$, for any $\gamma \in \mathbb{Z}$ with $\gamma \geq 2$, where k_h is the hop distance of an instance.

Further, we refine our framework to obtain guarantees dependent on the mandatory query distance k_M for the minimum and sorting problems. We observe that we can hope to obtain only a slightly worse consistency for the same robustness in comparison to the guarantees dependent on k_h (cf. Theorem 2.4). Based on this observation, we adjust the framework to be “less careful” in the first phase and repeatedly query $\gamma - 1$ prediction mandatory elements and one additional element that forms a witness set with one of the queried prediction mandatory elements. For sorting and minimum, we show that this adjustment with an unchanged second phase leads to the competitive ratio stated in the following theorem.

Theorem 3.4. There is an algorithm for the minimum problem under uncertainty that, given an integer parameter $\gamma \geq 2$, achieves a competitive ratio of $\min\{(1 + \frac{1}{\gamma-1}) \cdot (1 + \frac{k_M}{\text{opt}}), \gamma\}$.

The error-sensitivity can again be shown by charging each queried prediction mandatory element that turns out to not be mandatory and each additional query in the second phase to a distinct error in the k_M -metric. In contrast to the k_h -dependent error-sensitivity, only elements that are prediction mandatory based on the initially given information and that turn out not to be mandatory contribute to k_M . Thus, we fix the set of prediction mandatory elements at the beginning of the algorithm and show that querying only those elements in the first phase is sufficient. We remark that the integrality requirement in Theorem 3.4 can be removed by randomization at the cost of a slightly worse guarantee.

Finally, we observe that for sorting a single set, a substantially better algorithm is possible: a 1-consistent 2-robust algorithm with a competitive ratio that linearly degrades depending on the prediction error. Note that for achieving 1-consistency, an algorithm *must* follow the offline algorithm and cannot afford additional queries unless the predictions are wrong. To simultaneously guarantee 2-robustness and error dependency, the algorithm has to perform queries in a very carefully selected order, both for the prediction mandatory elements in Phase 1 and the vertex cover in Phase 2. By employing such a strategy, we achieve the following theorem.

Theorem 3.5. For sorting under uncertainty for a single set, there is a polynomial-time algorithm with competitive ratio $\min\{1 + k/\text{opt}, 2\}$, for any error measure $k \in \{k_{\#}, k_M, k_h\}$.

4 The minimum problem

We show how to implement our general framework for the minimum problem under uncertainty achieving best possible competitive ratios with respect to all three accuracy measures. To do so, we firstly give a characterization of (prediction) mandatory queries and secondly present a verification algorithm.

Lemma 4.1. An interval I_i is *mandatory* for the minimum problem if and only if (a) I_i is a true minimum of a set S and contains w_j of another interval $I_j \in S \setminus \{I_i\}$ (in particular, if $I_j \subseteq I_i$), or (b) I_i is not a true minimum of a set S but contains the value of the true minimum of S . *Prediction mandatory* intervals are characterized equivalently, replacing true values by predicted values.

Lemma 4.1 is now used to design a verification algorithm. It follows the same two-phase structure as our general framework: First, we query all mandatory intervals. After that, each unsolved set S has the following configuration: The leftmost interval I_i has true value outside all other intervals in S , and each other interval in S has true value outside I_i . Thus we can either query I_i or all other intervals that intersect I_i in S to solve it. The optimum solution is to query a minimum vertex cover in the graph with a vertex for each interval and, for each unsolved set, an edge between the leftmost interval and the intervals that intersect it. This algorithm may require exponential time, but this is not surprising as we can show that the verification version of the minimum problem is NP-hard (a proof is given in Appendix B).

Theorem 4.2. The verification problem for the minimum problem under uncertainty is NP-hard. The above algorithm solves it with a minimum number of queries.

Algorithm 1: Algorithm for the minimum problem under uncertainty w.r.t. hop distance k_h

Input: Intervals I_1, \dots, I_n , prediction \bar{w}_i for each I_i , and family of sets \mathcal{S}

- 1 **repeat**
- 2 **while** *there is a known mandatory interval I_i* **do** query I_i ;
- 3 $Q \leftarrow \emptyset$; $P \leftarrow$ set of prediction mandatory intervals for current instance;
- 4 **while** $P \neq \emptyset$ **and** $|Q| < \gamma - 2$ **do**
- 5 pick and query some $I_j \in P$; $Q \leftarrow Q \cup \{I_j\}$;
- 6 **while** *there is a known mandatory interval I_i* **do** query I_i ;
- 7 $P \leftarrow$ set of prediction mandatory intervals for current instance;
- 8 **if** \exists distinct I_i, I_j, I_l such that \bar{w}_j enforces I_i and $\{I_j, I_l\}$ is a witness set **then**
- 9 query I_j, I_l ;
- 10 **if** $w_j \in I_i$ **then** query I_i ;
- 11 **else if** $\exists I_i, I_j$ such that \bar{w}_j enforces I_i **then** query I_i ;
- 12 **until** *no query was performed in this iteration* ;
- 13 **while** *there is a known mandatory interval I_i* **do** query I_i ;
- 14 compute and query a minimum vertex cover Q' on the current dependency graph;
- 15 **while** *there is a known mandatory interval I_i* **do** query I_i ;

Regarding hop distance We present an algorithm with a performance guarantee depending on the hop distance k_h as the measure for the prediction accuracy.

Theorem 3.2. There is an algorithm for the minimum problem under uncertainty that, given an integer parameter $\gamma \geq 2$, achieves a competitive ratio of $\min\{(1 + \frac{1}{\gamma})(1 + \frac{k_h}{\text{opt}}), \gamma\}$. Furthermore, if $\gamma = 2$, then the competitive ratio is $\min\{1.5 + k_h/\text{opt}, 2\}$.

This result is best possible for $k_h = 0$ and for large k_h , due to Theorem 2.1. Note that it also proves Theorem 3.1 for the minimum problem under uncertainty, since $k_{\#} = 0$ implies $k_h = 0$. A complete proof of Theorem 3.2 is given in Appendix C.1; here we only describe the algorithm and some high level arguments.

We call an interval *leftmost* in a set S if it is an interval with minimum lower limit in S . Our algorithm (pseudocode in Algorithm 1) repeatedly queries intervals that it learns to be mandatory from previous queries, using the the following condition, which follows from Lemma 4.1: An interval I_i that is leftmost in a set S and contains another (possibly trivial) interval in S is mandatory. We call such intervals *known mandatory*. It alternates this process with querying at most $\gamma - 2$ intervals that are prediction mandatory for the current instance, which are identified using Lemma 4.1.

After querying $\gamma - 2$ intervals, the algorithm tries to identify a set of three elements with at least two prediction mandatory intervals, or one prediction mandatory interval that is part of a witness set. For that purpose, we use the following concept: A predicted value \bar{w}_j *enforces* another interval I_i if $\bar{w}_j \in I_i$ and $I_i, I_j \in S$, where S is a set such that either I_i is leftmost in S , or I_j is leftmost in S and I_i is leftmost in $S \setminus \{I_j\}$. We also use the following fact to identify witness sets: A set $\{I_i, I_j\} \subseteq S$ with $I_i \cap I_j \neq \emptyset$, and I_i or I_j leftmost in S , is a witness set [39]. It is then easy to see that, if \bar{w}_j enforces I_i , then $\{I_i, I_j\}$ is a witness set and I_i is prediction mandatory; moreover, if $w_j \in I_i$ then I_i is mandatory. The algorithm first tries to identify three distinct intervals I_i, I_j, I_l , such that \bar{w}_j enforces I_i and $\{I_j, I_l\}$ is a witness set. If there is such trio, we query $\{I_j, I_l\}$, and only query I_i if the prediction that $\bar{w}_j \in I_i$ is correct. If the prediction is correct, then we have a set of size 3 with at least 2 mandatory intervals; otherwise we only query a witness pair, so we can enforce robustness, and if one interval in this pair is not in OPT then we can charge this error to the hop distance, since the prediction \bar{w}_j is incorrect. If there is no such trio, then we try to find a witness pair $\{I_i, I_j\}$ such that \bar{w}_j enforces I_i , but initially we only query I_i . If the prediction \bar{w}_j is correct, then we are

Algorithm 2: Algorithm for the minimum problem under uncertainty w.r.t. error measure k_M

Input: Intervals I_1, \dots, I_n , predicted value \bar{w}_i for each I_i , family of sets \mathcal{S} , and parameter γ

- 1 $P \leftarrow$ set of initial prediction mandatory intervals (characterized in Lemma 4.1);
- 2 **while** $\exists p \in P$ and an unqueried interval b where $\{p, b\}$ is a witness set **do**
- 3 **if** $|P| \geq \gamma - 1$ **then**
- 4 pick $P' \subseteq P$ with $p \in P'$ and $|P'| = \gamma - 1$;
- 5 query $P' \cup \{b\}$, $P \leftarrow P \setminus (P' \cup \{b\})$;
- 6 **while** there is a known mandatory interval I_i **do** query I_i , $P \leftarrow P \setminus \{I_i\}$;
- 7 **else** query P , $P \leftarrow \emptyset$;
- 8 **while** there is a known mandatory interval I_i **do** query I_i ;
- 9 query a minimum vertex cover Q for the current instance;
- 10 **while** there is a known mandatory interval I_i **do** query I_i ;

querying a mandatory interval. Otherwise, we show that I_j is either queried in Line 2 in the next iteration of the loop (so it is a mandatory interval), or is never queried by the algorithm; either way, the fact that this is a witness pair is enough to guarantee robustness, and if I_i is not in OPT then we can charge this error to the hop distance. Summing up, for each iteration of the loop we can identify a witness set of size at most γ , such that at least $\frac{\gamma+1}{\gamma}$ of its elements are prediction mandatory, and those that are not mandatory can be charged to the hop distance.

We may have an iteration of the loop that we do not query any intervals in Lines 9 and 11, but we show that this occurs at most once: If we cannot satisfy the conditions of Lines 8 and 11, then there are no more prediction mandatory intervals. After that, the algorithm will proceed to the second phase of the framework, querying a minimum vertex cover and intervals that become known mandatory. Here we combine the at most $\gamma - 2$ intervals queried in the iteration described with the intervals queried in the second phase, and it is not hard to prove γ -robustness and that every interval that is not in OPT can be charged to the hop distance.

Regarding mandatory-query distance Now we present our algorithm with performance guarantee depending on the mandatory-query distance k_M as the measure for the prediction accuracy.

Theorem 3.4. There is an algorithm for the minimum problem under uncertainty that, given an integer parameter $\gamma \geq 2$, achieves a competitive ratio of $\min\{(1 + \frac{1}{\gamma-1}) \cdot (1 + \frac{k_M}{\text{opt}}), \gamma\}$.

The upper bound is tight for $k_M = 0$ and large k_M due to Theorem 2.4. The theorem can be generalized for arbitrary real $\gamma \geq 2$ with a marginally increased competitive ratio; see Appendix C.2. The full proof of Theorem 3.4 appears also in Appendix C.2; here we describe the algorithm and some high-level arguments.

The algorithm (see pseudocode in Algorithm 2) firstly computes the set P of initial prediction mandatory intervals (Lemma 4.1). Then it tries to find an interval $p \in P$ that is part of a witness set $\{p, b\}$. If $|P| \geq \gamma - 1$, we query a set $P' \subseteq P$ of size $\gamma - 1$ that includes p , plus b (we allow $b \in P'$). This is clearly a witness set of size at most γ , at least $\frac{\gamma}{\gamma-1}$ of the intervals are in P , and every interval in $P \setminus \text{OPT}$ is in $\mathcal{I}_P \setminus \mathcal{I}_R$. We then repeatedly query known mandatory intervals, remove the queried intervals from P and repeat the process without recomputing P , until P is empty or no interval in P is part of a witness set.

We may have one last iteration of the loop where $|P| < \gamma - 1$. After that, the algorithm will proceed to the second phase of the framework, querying a minimum vertex cover and intervals that become known mandatory. Here we combine the at most $\gamma - 2$ intervals in P with the intervals queried in the second phase, and it is not hard to prove γ -robustness, and that the number of those queries can be bounded by the number of intervals in OPT for the current instance plus the number of intervals that are $\mathcal{I}_P \setminus \mathcal{I}_R$ or in $\mathcal{I}_R \setminus \mathcal{I}_P$.

We remark that, for a single set, the minimum problem admits an algorithm with competitive ratio $1 + \frac{1}{\text{opt}} \leq 2$ [39].

5 The minimum spanning tree problem

In this section, we present how the framework of Section 3 can be implemented to achieve an algorithm for the MST problem with the following performance guarantee. Full proofs are given in Appendix D.

Theorem 3.3. There is an algorithm for the MST problem under uncertainty with competitive ratio $\min\{1 + \frac{1}{\gamma} + \left(5 + \frac{1}{\gamma}\right) \cdot \frac{k_h}{\text{opt}}, \gamma + \frac{1}{\text{opt}}\}$, for any $\gamma \in \mathbb{Z}$ with $\gamma \geq 2$, where k_h is the hop distance of an instance.

Additionally, we introduce a second algorithm that achieves a better robustness at the cost of not providing an error-sensitive guarantee. Both algorithms use the same first phase for the framework described in Section 3 and differ only in the second one.

Theorem 5.1. There is a 1.5-consistent and 2-robust algorithm for MST under uncertainty.

Our algorithms build on structural insights presented in [49] which we summarize here. Let the *lower limit tree* $T_L \subseteq E$ be an MST for values w^L with $w_e^L = L_e + \epsilon$ for an infinitesimally small $\epsilon > 0$. Analogously, let the *upper limit tree* T_U be an MST for values w^U with $w_e^U = U_e - \epsilon$. It has been shown in [49] that any non-trivial edge in $T_L \setminus T_U$ is part of any feasible query set, i.e., it is mandatory. Thus, we may repeatedly query edges in (the adapting) $T_L \setminus T_U$ until $T_L = T_U$ and this will not worsen the robustness or consistency. By this preprocessing, we may assume $T_L = T_U$. Further, we can extend the preprocessing to achieve uniqueness for T_L and T_U .

Lemma 5.2. By querying only mandatory elements we can obtain an instance with $T_L = T_U$ such that T_L and T_U are the unique lower limit tree and upper limit tree, respectively.

Consider T_L and f_1, \dots, f_l in $E \setminus T_L$ ordered by increasing lower limits. For each $i \in \{1, \dots, l\}$, let C_i be the unique cycle in $T_L \cup \{f_i\}$. For each $e \in T_L$ let X_e be the set of edges in the cut of G defined by the two connected components of $T_L \setminus \{e\}$. We say an instance is *prediction mandatory free* if it contains no prediction mandatory elements. Otherwise, we say that the instance is *non-prediction mandatory free*. The following lemma further characterizes prediction mandatory free instances.

Lemma 5.3. An instance G is prediction mandatory free if and only if $\bar{w}_{f_i} \geq U_e$ and $\bar{w}_e \leq L_{f_i}$ holds for each $e \in C_i \setminus \{f_i\}$ and each cycle C_i with $i \in \{1, \dots, l\}$.

Figure 1(a) illustrates this definition. Our algorithms consist of two phases. The goal of the first phase is to query edges until each C_i is prediction mandatory free. The second phase handles prediction mandatory free instances.

5.1 Identifying witness sets

Before we describe our algorithms in the following sections, we introduce some preliminaries. As already stated, the first phase of our algorithm handles non-prediction mandatory free instances until they become prediction mandatory free. In order to achieve this goal while maintaining the desired performance guarantees, the algorithms rely on identifying witness sets on non-prediction mandatory free cycles C_i .

Existing algorithms for MST under uncertainty [24, 49] essentially follow the algorithms of Kruskal or Prim, and only identify witness sets in the cycle or cut that is currently under consideration. For a given instance this corresponds to the cycle C_1 or the cut X_{l_1} , where l_1 is the edge with the largest upper limit in T_L .

Algorithm 3: Phase 1 of the algorithms for MST under uncertainty

Input: Uncertainty graph $G = (V, E)$ and predictions \bar{w}_e for each $e \in E$

- 1 Sequentially query prediction mandatory elements while ensuring unique $T_L = T_U$ until either $\gamma - 2$ prediction mandatory elements are queried or the instance is prediction mandatory free;
 - 2 Let T_L be the lower limit tree and f_1, \dots, f_l be the edges in $E \setminus T_L$ ordered by lower limit non-decreasingly;
 - 3 **foreach** C_i with $i = 1$ to l **do**
 - 4 **if** C_i is not prediction mandatory free **then** Apply Lemma 5.7, 5.8 or 5.9 and restart ;
-

Regarding Phase 1 of our algorithm, it might hold for the first non-prediction mandatory free cycle C_i that $C_i \neq C_1$. Since additionally $C_i \cap X_{l_1} = \emptyset$ might hold, existing methods for identifying witness sets are not sufficient for our purpose. We show the following new structural insights that are crucial for our algorithms.

Lemma 5.4. Consider cycle C_i with $i \in \{1, \dots, l\}$. Let $l_i \in C_i \setminus \{f_i\}$ such that $I_{l_i} \cap I_{f_i} \neq \emptyset$ and l_i has the largest upper limit in $C_i \setminus \{f_i\}$, then $\{f_i, l_i\}$ is a witness set. Further, if $w_{f_i} \in I_{l_i}$, then $\{l_i\}$ is a witness set.

Lemma 5.5. Let $l_i \in C_i \setminus \{f_i\}$ with $I_{l_i} \cap I_{f_i} \neq \emptyset$ such that $l_i \notin C_j$ for all $j < i$, then $\{l_i, f_i\}$ is a witness set. Furthermore, if $w_{l_i} \in I_{f_i}$, then $\{f_i\}$ is a witness set.

5.2 Handling non-prediction mandatory free instances

Algorithm 3 implements Phase 1 of our algorithms. In each iteration the algorithm starts by querying elements that are prediction mandatory for the current instance. The set of prediction mandatory elements can be computed using the verification algorithm [21]. Our algorithm sequentially queries such elements until either $\gamma - 2$ prediction mandatory elements have been queried or no more exist. After each query, the algorithm ensures unique $T_L = T_U$ by using Lemma 5.2. Note that the set of prediction mandatory elements with respect to the current instance can change when elements are queried, and therefore we query the elements sequentially. We can prove that each of the at most $\gamma - 2$ elements is either mandatory or contributes one to the hop distance k_h .

Then, the algorithm iterates through $i \in \{1, \dots, l\}$ and stops if the current cycle C_i is non-prediction mandatory free. If it finds such a cycle, it queries edges on the cycle and possibly future cycles and restarts. The algorithm terminates when all C_i are prediction mandatory free that is at the latest when all edges in E have been queried. When the algorithm finds a non-prediction mandatory free cycle C_i , it carefully selects edges to query such that the following statements hold:

1. The algorithm only queries witness sets of size one or two, and sets of size three such that at least two elements are part of any feasible query set.
2. If the algorithm queries a witness set $W = \{e_1, e_2\}$ of size two, then either $W \subseteq Q$ for each feasible query set Q or the hop distances of e_1 and e_2 satisfy $h_{e_1} + h_{e_2} \geq 1$.
3. In an only exception, the algorithm queries single elements e that form witness sets $\{e, f(e)\}$ with distinct elements $f(e)$, and the algorithm *guarantees* that $f(e)$ remains unqueried during the complete execution. Since OPT must query at least one element of $\{e, f(e)\}$ and we guarantee that the algorithm queries exactly one element, querying such elements does not hurt the robustness or consistency as long as each such queried e can be matched with a distinct $f(e)$.

Let E be the set of such queried edges. For the sake of our analysis, we assume without loss of generality that $E \subseteq \text{OPT}$ and treat each $e \in E$ as a witness set of size one. We can do this without loss of generality since if $e \notin \text{OPT}$ we know $f(e) \in \text{OPT}$ and can charge e against $f(e)$.

Ignoring the last iteration where the instance becomes prediction mandatory free and possibly less than $\gamma - 2$ prediction mandatory elements are queried, the algorithm queries in each iteration $\gamma - 2$ prediction mandatory elements and a set W that satisfies the three statements. This implies the following lemma.

Lemma 5.6. After executing Algorithm 3 the instance is prediction mandatory free and, ignoring the last iteration of Line 1, $|\text{ALG}| \leq \min\{(1 + \frac{1}{\gamma}) \cdot (|\text{ALG} \cap \text{OPT}| + k_h), \gamma \cdot |\text{ALG} \cap \text{OPT}|\}$ holds for the set of edges ALG queried by Algorithm 3 and any optimal solution OPT .

In the last iteration, a (possibly empty) set P of at most $\gamma - 2$ prediction mandatory elements is queried. Lemma 5.3 implies that only the last execution of Line 1 might query less than $\gamma - 2$ prediction mandatory elements. As each $e \in P$ is either mandatory or contributes one to the hop distance, querying P does not violate the consistency. We ensure the $(\gamma + \frac{1}{\text{opt}})$ -robustness by charging P against the queries of the 3-robust Phase 2 of the algorithm. Charging the first $\gamma - 3$ elements of P against the 3-robust Phase 2 leads to γ -robustness while the final element of P leads to the additive term $\frac{1}{\text{opt}}$.

The following lemmas give algorithmic actions with a guarantee that the three statements are fulfilled. Each of them considers a cycle C_i such that all C_j with $j < i$ are prediction mandatory free, l_i is the edge with the highest upper limit in $C_i \setminus \{f_i\}$ and predictions are as indicated in Figure 1 b)-d). The proofs of the three lemmas rely on the structural insights of Subsection 5.1.

Lemma 5.7. If $\bar{w}_{f_i} \in I_{l_i}$ and $\bar{w}_{l_i} \in I_{f_i}$, then querying $\{f_i, l_i\}$ satisfies the three statements.

Lemma 5.8. Assume $\bar{w}_{f_i} \in I_{l_i}$ but $\bar{w}_{l_i} \notin I_{f_i}$. Let l'_i be the edge with the highest upper limit in $C_i \setminus \{f_i, l_i\}$ and $I_{l'_i} \cap I_{f_i} \neq \emptyset$. If no l'_i exists, then querying l_i , and querying f_i only if $w_{l_i} \in I_{f_i}$, satisfies the three statements. If l'_i exists, then querying $\{f_i, l_i\}$, and querying l'_i only if $w_{f_i} \in I_{l_i}$ and $w_{l_i} \notin I_{f_j}$ for each j with $l_i \in C_j$, satisfies the three statements.

Lemma 5.9. Assume $\bar{w}_{l'_i} \in I_{f_i}$ for some $l'_i \in C_i \setminus \{f_i\}$ but $\bar{w}_{f_i} \notin I_{l_i}$. Let f_j be the edge with the smallest lower limit in $X_{l'_i} \setminus \{l'_i, f_i\}$ and $I_{f_j} \cap I_{l'_i} \neq \emptyset$. If f_j does not exist, then querying f_i , and querying l'_i only if $w_{f_i} \in I_{l'_i}$, satisfies the three statements. If f_j exists, querying $\{f_i, l'_i\}$, and also querying f_j only if $w_{l'_i} \in I_{f_i}$ and $w_{f_i} \notin I_e$ for each $e \in C_i$, satisfies the three statements.

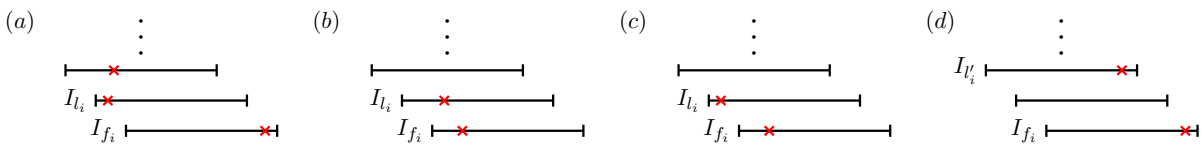


Figure 1: Intervals with predictions indicated as red crosses. a) Prediction mandatory free cycle. Illustration of the situations in the Lemmas 5.7 (b), 5.8 (c) and 5.9 (d).

5.3 Handling prediction mandatory free instances

This section describes Phase 2 of our algorithms. For prediction mandatory free instances, we present Algorithm 4 with *recovery strategies* A and B (in Line 6) that lead to the guarantees of Lemma 5.10. Our full algorithms execute Phase 1 followed by Phase 2 and differ only in the recovery strategy. Using the introduced ideas and lemmas, we can prove the Theorems 5.1 (recovery A) and 3.3 (recovery B).

Lemma 5.10. If Algorithm 4 is executed on a prediction mandatory free instance, then in each iteration the instance remains prediction mandatory free. Furthermore, recovery strategy A guarantees 1-consistency and 2-robustness and recovery strategy B guarantees $|\text{ALG}| \leq \min\{\text{opt} + 5 \cdot k_h, 3 \cdot \text{opt}\}$.

Algorithm 4: Phase 2 of the algorithms for MST under uncertainty

- Input:** Prediction mandatory free graph $G = (V, E)$ and predictions \bar{w}_e for each $e \in E$
- 1 Compute maximum matching h and minimum vertex cover VC for \bar{G} and initialize $W = \emptyset$;
 - 2 Let f'_1, \dots, f'_g and l'_1, \dots, l'_k be as described in Lemma 5.11;
 - 3 **for** e chosen sequentially from the ordered list $f'_1, \dots, f'_g, l'_1, \dots, l'_k$ **do**
 - 4 Query e , and ensure unique $T_L = T_U$;
 - 5 If $\{e, h(e)\} \cap W = \emptyset$, add $h(e)$ to W . Otherwise query $h(e)$ and ensure unique $T_L = T_U$;
 - 6 If the vertex cover instance changed, execute a recovery strategy and restart at Line 2;
-

Assume again unique $T_L = T_U$. In a prediction mandatory free instance $G = (V, E)$, each $f_i \in E \setminus T_L$ is predicted to be maximal on cycle C_i , and each $l \in T_L$ is predicted to be minimal in X_l . If these predictions are correct, then the optimal query set is a minimum vertex cover in a bipartite graph $\bar{G} = (\bar{V}, \bar{E})$ with $\bar{V} = E$ (excluding trivial edges) and $\bar{E} = \{\{f_i, e\} \mid i \in \{1, \dots, l\}, e \in C_i \setminus \{f_i\} \text{ and } I_e \cap I_{f_i} \neq \emptyset\}$ [21, 49]. We refer to \bar{G} as the *vertex cover instance*. Note that if a query reveals that an f_i is not maximal on C_i or an l_i is not minimal in X_{l_i} , then the vertex cover instance changes. Because of this and in contrast to the minimum problem, non-adaptively querying VC can lead to a competitiveness worse than 2.

Let VC be a minimum vertex cover of \bar{G} . The idea of the algorithm is to sequentially query each $e \in VC$ and charge for querying e by a distinct non-queried element $h(e)$ such that $\{e, h(e)\}$ is a witness set. Querying exactly one element per distinct witness set implies optimality. To identify $h(e)$ for each element $e \in VC$, we use the fact that König's Theorem (e.g. [13]) and the duality between minimum vertex covers and maximum matchings in bipartite graphs imply that there is a matching h that maps each $e \in VC$ to a distinct $e' \notin VC$. While the sets $\{e, h(e)\}$ with $e \in VC$ in general are not witness sets, querying VC in a specific order until the vertex cover instance changes guarantees that $\{e, h(e)\}$ is a witness set for each already queried e .

Lemma 5.11. Let f'_1, \dots, f'_g be the edges in $VC \setminus T_L$ ordered by lower limit non-decreasingly and let l'_1, \dots, l'_k be the edges in $VC \cap T_L$ ordered by upper limit non-increasingly. Let b be such that each f'_i with $i < b$ is maximal in cycle $C_{f'_i}$, then $\{f'_i, h(f'_i)\}$ is a witness set for each $i \leq b$. Let d be such that each l'_i with $i < d$ is minimal in cut $X_{l'_i}$, then $\{l'_i, h(l'_i)\}$ is a witness set for each $i \leq d$.

Algorithm 4 queries VC in the order $f'_1, \dots, f'_g, l'_1, \dots, l'_k$ (Lemma 5.11) until the vertex cover instance \bar{G} changes. If it does not change, $|VC|$ is a lower bound on opt and the algorithm only queries VC and a set M of mandatory elements that were queried as elements of $T_L \setminus T_U$. Since we can show that each $e \in M$ contributes one to k_h , it follows $|\text{ALG}| \leq \min\{\text{opt} + k_h, 2 \cdot \text{opt}\}$.

If the vertex cover instance changes, Line 6 executes a recovery strategy and restarts. The challenge here is that, after restarting, an element $h(e)$ that was used to charge for an already queried e might be used again to charge for a different element e' . This can happen if $h(e)$ is not queried and matched to element e' after the restart. To handle this problem the algorithm uses the set W to keep track of the elements $h(e)$ for already queried elements e and uses them in charging schemes when executing the recovery strategies.

Recovery strategy A The first recovery strategy can be used to achieve 2-robustness and show Theorem 5.1. It queries before a restart the set W of non-queried elements that were used to charge for already queried elements, ensures unique $T_L = T_U$ and restarts with re-computed VC and h . Since each $h(e) \in W$ forms a witness set with a distinct e (Lemma 5.11), this strategy ensures 2-robustness.

Recovery strategy B This strategy can be used to achieve the error-sensitive guarantee $|\text{ALG}| \leq \min\{\text{opt} + 5 \cdot k_h, 3 \cdot \text{opt}\}$. In this case querying W to ensure 2-robustness might violate $|\text{ALG}| \leq \text{opt} + 5 \cdot k_h$. Instead of preventing that an element $h(e)$ is used to charge for a second element, the algorithm prevents $h(e)$ from

being used to charge for three elements. To achieve this, Line 5 queries $h(e)$ when it is used to charge for a second element e' . As $U = \{h(e), e, e'\}$ is a witness set, this ensures 3-robustness. Furthermore, two elements of U might not be part of an optimal solution. However, executing the Otherwise-part of Line 5 at most $2 \cdot k_h$ times ensures $|\text{ALG}| \leq \text{opt} + 5 \cdot k_h$. Note that the factor is 5 instead of 4 because the errors used to bound the number of executions of the Otherwise-part of Line 5 and the errors used to charge for the queried elements of $T_L \setminus T_U$ in Lines 4 and 5 are not necessarily disjoint. This uses Lemma 5.12 and a restart with a specific matching h' that does not contain too many non-queried elements of W .

Lemma 5.12. Let $\bar{G}' = (\bar{V}', \bar{E}')$ be the changed vertex cover instance of Line 6, then $\bar{h} = \{\{e, e'\} \in h \mid \{e, e'\} \in \bar{E}'\}$ defines a partial matching for \bar{G}' . Let h' be the maximum matching for \bar{G}' computed by completing \bar{h} using a standard augmenting path algorithm [2] and let VC' be the vertex cover defined by h' . Then restarting Algorithm 4 with $VC = VC'$ and $h = h'$ implies that Line 5 queries at most $2 \cdot k_h$ times.

6 The sorting problem

There is a simple reduction from the sorting problem under uncertainty to the minimum problem. For each intersecting pair of intervals in the same set in the sorting problem, we construct a corresponding set in the minimum problem, consisting of only this pair. If we have a solution for the sorting problem, then we clearly can determine the minimum interval in each pair. Conversely, if we know the minimum interval in each pair, then we can sort the intervals accordingly. Thus the sorting problem is a particular case of the minimum problem with sets of size 2, and all our algorithmic results for the minimum problem transfer to the sorting problem. However, the sorting problem has a simpler characterization for mandatory intervals: any interval that contains the true value of another interval in the same set is mandatory [37]. Still, we cannot hope for better guarantees for sorting due to the lower bounds in Theorems 2.1 and 2.4. Moreover, the NP-hardness of the verification problem of the minimum holds for sets of size 2 and, thus, it holds for sorting overlapping sets.

Nevertheless, if we are sorting a single set or disjoint sets, then there is a better algorithm than the ones presented for the minimum problem. The algorithm performs at most $\min\{\text{opt} + k, 2 \cdot \text{opt}\}$ queries, for any $k \in \{k_\#, k_M, k_h\}$, and thus is optimal due to Theorem 2.5. The main fact that makes the sorting problem easier for a single set is that the intersection graph is an interval graph [43]. Moreover, witness sets are not influenced by other intervals, because any two intersecting intervals constitute a witness set [37]. We give a full proof of Theorem 3.5 in Appendix E; in the remainder of the section we present the algorithm and high level arguments on why it obtains the desired result.

Theorem 3.5. For sorting under uncertainty for a single set, there is a polynomial-time algorithm with competitive ratio $\min\{1 + k/\text{opt}, 2\}$, for any error measure $k \in \{k_\#, k_M, k_h\}$.

To obtain a guarantee of $\text{opt} + k$ for any measure k , the algorithm must trust the predictions as much as possible. In particular, our algorithm queries all prediction mandatory intervals. We can do this and still guarantee 2-robustness because intervals that contain the same predicted value form a clique in the initial interval graph, and in any clique at most one query can be avoided [37]. However, in the second phase of the framework, when there are no more prediction mandatory intervals and we query a minimum vertex cover and intervals that become known mandatory, we may have different minimum vertex covers, so in order to maintain 2-robustness we must be more careful. We show that each component of the intersection graph at this point must be a path. The more intricate case is when we have an even path P , because we have two minimum vertex covers; we devise a charging scheme to decide which of them we query. This scheme is based on a forest of arborescences that is built according to the relation between prediction mandatory intervals and predicted values, which we define more precisely in the next paragraph. This forest is then used to partition the prediction mandatory intervals into sets that correspond to cliques in the initial interval graph, in such a way that only roots or children of the roots can be isolated in this clique partition. (When a root is

Algorithm 5: A nicely degrading algorithm for sorting with predictions.

Input: Ground set of intervals $\mathcal{I} = \{I_1, \dots, I_n\}$, and predictions $\bar{w}_1, \dots, \bar{w}_n$

- 1 $\mathcal{E} \leftarrow \emptyset$; $C_1, C_2, \dots, C_n \leftarrow \emptyset$;
- 2 **let** \mathcal{I}_P be the set of prediction mandatory intervals;
- 3 **foreach** $I_i \in \mathcal{I}_P$ **do** $\pi(i) \leftarrow j$ for some $j \neq i$ with $\bar{w}_j \in I_i$;
- 4 **while** $\exists i \neq j$ with $I_j \subseteq I_i$, or I_j was queried and $w_j \in I_i$ **do** query I_i ;
- 5 **let** \mathcal{S} be the set of intervals in \mathcal{I}_P that were not queried yet;
- 6 **foreach** $I_i \in \mathcal{S}$ **do**
- 7 query I_i ;
- 8 **if** $(I_{\pi(i)}, I_i)$ does not create a cycle in $(\mathcal{I}, \mathcal{E})$ **then** $\mathcal{E} \leftarrow \mathcal{E} \cup \{(I_{\pi(i)}, I_i)\}$;
- 9 **while** $\exists i \neq j$ where I_j was queried and $w_j \in I_i$ **do** query I_i ;
- 10 **while** $\mathcal{S} \neq \emptyset$ **do**
- 11 **let** I_i be a deepest vertex in the forest of arborescences $(\mathcal{I}, \mathcal{E})$ among those in \mathcal{S} ;
- 12 **if** $(I_{\pi(i)}, I_i) \in \mathcal{E}$ **then**
- 13 $C_{\pi(i)} \leftarrow \{I_{i'} \in \mathcal{S} : (I_{\pi(i)}, I_{i'}) \in \mathcal{E}\}$;
- 14 **if** $I_{\pi(i)} \in \mathcal{S}$ **then** $C_{\pi(i)} \leftarrow C_{\pi(i)} \cup \{I_{\pi(i)}\}$;
- 15 $\mathcal{S} \leftarrow \mathcal{S} \setminus C_{\pi(i)}$;
- 16 **else** $C_i \leftarrow \{I_i\}$; $\mathcal{S} \leftarrow \mathcal{S} \setminus C_i$;
- 17 **while** the problem is unsolved **do**
- 18 **let** $P = x_1 x_2 \dots x_p$ be a component of the current intersection graph which is a path with $p \geq 2$;
- 19 **if** p is odd **then** query $I_{x_2}, I_{x_4}, \dots, I_{x_{p-1}}$;
- 20 **else**
- 21 **if** $|C_{x_1}| = 1$ **then** query $I_{x_1}, I_{x_3}, \dots, I_{x_{p-1}}$;
- 22 **else** query $I_{x_2}, I_{x_4}, \dots, I_{x_p}$;
- 23 **while** $\exists i \neq j$ where I_j was queried and $w_j \in I_i$ **do** query I_i ;

isolated, we show that we can build a different clique partition for that component without isolated intervals.) We then show that only the endpoints of P can be part of this forest of arborescences. Thus P along with the isolated vertices in the clique partition that are children of the endpoints of P constitute an induced subgraph of size at most $|P| + 2$. If the size is $|P| + 1$, then we have to ensure that we query the vertex cover that contains the endpoint that is a parent of an isolated vertex in the clique partition; otherwise we can choose an arbitrary vertex cover. On the other hand, if we have an odd path, then it has a single minimum vertex cover, and we show that this is always a good choice.

A pseudocode is given in Algorithm 5. The first phase of the algorithm consists of Lines 2–9, in which we query known mandatory and prediction mandatory intervals. We fix the set \mathcal{I}_P of initial prediction mandatory intervals, and for each interval $I_i \in \mathcal{I}_P$ we assign a *parent* $\pi(i)$, meaning that $\bar{w}_{\pi(i)} \in I_i$. Next we query known mandatory intervals to ensure that the intersection graph becomes a proper interval graph, and let \mathcal{S} be the remaining intervals in \mathcal{I}_P . We then query every $I_i \in \mathcal{S}$ and include in set \mathcal{E} a directed edge $(I_{\pi(i)}, I_i)$ if that does not create a cycle in the graph $(\mathcal{I}, \mathcal{E})$; this graph $(\mathcal{I}, \mathcal{E})$ is the forest of arborescences previously mentioned. In Lines 10–16, we partition \mathcal{S} into cliques of the initial interval graph. We traverse each component of the forest from the deepest leaves towards the root, so we guarantee that only roots or children of the roots can be isolated in the final partition. The second phase of the framework consists of Lines 17–23: We use the size of the sets in the clique partition to decide between different minimum vertex covers, and then we query intervals that become known mandatory.

It is clear that this algorithm can be implemented in polynomial time.

7 Experimental results

We tested the practical performance of our algorithms in simulations and highlight here the results for the minimum problem. Further results on the MST as well as details on the generation of instances and predictions are provided in the appendix. Our instances were generated by randomly drawing interval sets from interval graphs, obtained from re-interpreted SAT instances from the rich SATLIB library [38]. Our instances have between 48 and 287 intervals and a variable number of overlapping sets. For each instance we generated 125 different predictions while ensuring that the predictions cover a wide range of relative errors k_M/opt .

Figure 2 shows the results of over 230,000 simulations (instance and predictions pairs). The figure compares the results of our prediction-based Algorithms 1 and 2 for different choices of the parameter γ with the standard *witness set algorithm*. The latter sequentially queries witness sets of size two and achieves the best possible competitive ratio of 2 without predictions [39]. The Algorithms 1 and 2, for every selected choice of γ , outperform the witness set algorithm up to a relative error of approximately 2.8 and 1.5, respectively. For small values of γ , Algorithm 1 outperforms the witness set algorithm even for *every* relative error. Further, the parameter γ reflects well the robustness-performance tradeoff for both algorithms: a high value γ is beneficial for accurate predictions while a low value for γ gives robustness against very inaccurate predictions. In the extreme case, $\gamma = |\mathcal{I}|$, Algorithm 2 directly follows the predictions; while it is superior for small errors, it gets outperformed by algorithms with smaller γ when the relative error is growing. The results indicate that Algorithm 1 performs better than Algorithm 2. For both algorithms the performance gap between the different values for γ appears less significant for small relative errors, which suggests that selecting γ not too close to the maximum value $|\mathcal{I}|$ might be beneficial. For Algorithm 1 the results even suggest that selecting $\gamma = 2$ might be the most beneficial choice.

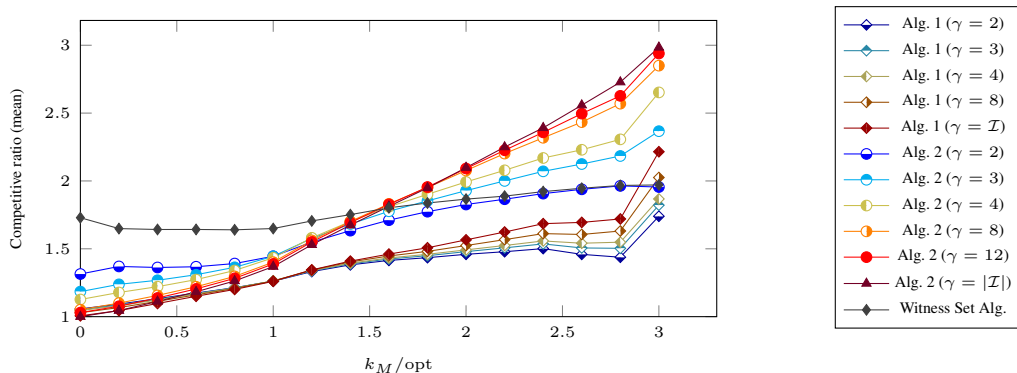


Figure 2: Experimental results for the minimum problem under uncertainty. Instances and predictions were grouped into equal size bins (0.2) according to their relative error k_M/opt .

Conclusion

In this paper we propose to use (possibly machine-learned) predictions for improving query-based algorithms when coping with explorable uncertainty. Our methods prove that untrusted predictions allow for rigorous worst-case guarantees that overcome known lower bounds. Thus, we contribute to the challenge of building trustable AI systems and the applicability also for safety-critical applications where such guarantees are obligatory. It would be interesting to study the power of predictions for other (theoretically interesting and practically relevant) problems and for other predictor models. Further, we hope to expedite research on explorable uncertainty with untrusted predictions for other natural optimization problems.

Acknowledgement. We thank Alexander Lindermayr for his great support in conducting the experiments.

A Appendix for Distance Measures and Lower Bounds (Section 2)

A.1 Lower bound on the consistency-robustness tradeoff

Theorem 2.1. Let $\beta \geq 2$ be a fixed integer. For the minimum (even in a single set), sorting and MST problems under uncertainty, there is no deterministic β -robust algorithm that is α -consistent for $\alpha < 1 + \frac{1}{\beta}$. And vice versa, no deterministic α -consistent algorithm, with $\alpha > 1$, is β -robust for $\beta < \max\{\frac{1}{\alpha-1}, 2\}$.

Proof. We state the proof for the minimum problem first. Assume, for the sake of contradiction, that there is a deterministic β -robust algorithm that is α -consistent with $\alpha = 1 + \frac{1}{\beta} - \varepsilon$, for some $\varepsilon > 0$. Consider the instance in Figure 3(a) with $\beta + 1$ intervals and a single set. The algorithm must query the intervals $\{I_1, \dots, I_\beta\}$ first as otherwise, it would query $\beta + 1$ intervals in case all predictions are correct, while there is an optimal query set of size β . Suppose w.l.o.g. that the algorithm queries the intervals $\{I_1, \dots, I_\beta\}$ in order of increasing indices. Consider the adversarial choice $w_i = \bar{w}_i$, for $i = 1, \dots, \beta - 1$, and then $w_\beta \in I_0$ and $w_0 \notin I_1 \cup \dots \cup I_\beta$. This forces the algorithm to query also I_0 , while an optimal solution only queries I_0 . Thus any such algorithm has robustness at least $\beta + 1$, a contradiction.

The second part of the theorem directly follows from the first part and the known general lower bound of 2 on the competitive ratio [24, 39]. Assume there is an α -consistent deterministic algorithm with some $\alpha = 1 + \frac{1}{\beta'}$, for some $\beta' \in [1, \infty)$. Consider the instance above with $\beta = \beta' - 1$. Then the algorithm has to query intervals $\{I_1, \dots, I_\beta\}$ first to ensure α -consistency as otherwise it would have a competitive ratio of $\frac{\beta+1}{\beta} > 1 + \frac{1}{\beta'} = \alpha$ in case that all predictions are correct. By the argumentation above, the robustness factor of the algorithm is at least $\beta + 1 = \beta' = \frac{1}{\alpha-1}$.

The same arguments can be used for the sorting problem, the only difference is that we take the input sets $\{I_0, I_i\}$ for $1 \leq i \leq \beta$.

For the MST problem, we first translate the above construction for the minimum problem to the maximum problem (defined in the obvious way) and then consider the MST instance consisting of a single cycle whose edges are associated with the weight intervals I_0, I_1, \dots, I_β . \square

A.2 Number of inaccurate predictions

Let $k_\#$ denote the *number of inaccurate predictions*, i.e., the number of intervals $I_i \in \mathcal{I}$ with $w_i \neq \bar{w}_i$.

This is a very natural but impractical prediction measure as the following theorem shows. It rules out using predictions to improve the competitive ratio on the known lower bound of 2.

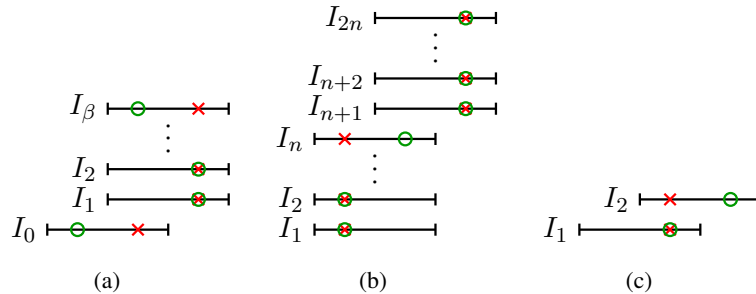


Figure 3: Instances for lower bounds. Red crosses indicate predicted values, and green circles show correct values. (a) Lower bound on robustness-consistency tradeoff. (b) Lower bound based on the number of inaccurate predictions. (c) Lower bound based on error measures.

Theorem A.1. If $k_{\#} \geq 1$, then any deterministic algorithm for the minimum, sorting or MST problem under uncertainty has competitive ratio $\rho \geq 2$.

Proof. First, we discuss the proof for the minimum problem. Consider $2n$ intervals as depicted in Figure 3(b) and sets $S_i = \{I_i, I_{n+1}, I_{n+2}, \dots, I_{2n}\}$, for $i = 1, \dots, n$. Assume w.l.o.g. that the algorithm queries the left-side intervals in the order I_1, I_2, \dots, I_n and the right side in the order $I_{n+1}, I_{n+2}, \dots, I_{2n}$. Before the algorithm queries I_n or I_{2n} , the adversary sets all predictions as correct, so the algorithm will eventually query I_n or I_{2n} . If the algorithm queries I_n before I_{2n} , then the adversary chooses a value for I_n that forces a query in I_{n+1}, \dots, I_{2n} , and the predicted values for the remaining right-side intervals as correct, so the optimum solution only queries I_{n+1}, \dots, I_{2n} . A symmetric argument holds if the algorithm queries I_{2n} before I_n .

For the sorting problem, use the same intervals but take the n^2 sets $\{I_i, I_j\}$ for $1 \leq i \leq n, n+1 \leq j \leq 2n$.

For the MST problem, first translate the above construction to the maximum problem. Call the resulting intervals I'_1, \dots, I'_{2n} , and note that both the true and the predicted values of I'_i for $1 \leq i \leq n$ are larger than both the real and the predicted values of I'_j for $n+1 \leq j \leq 2n$. Consider a graph that consists of a path with n edges with weight intervals I'_{n+1}, \dots, I'_{2n} , and let s and t denote the two end vertices of that path. Then add n parallel edges between s and t with weight intervals I'_1, \dots, I'_n . The adversary then proceeds as in the construction for the minimum problem. \square

A.3 Mandatory query distance

Theorem 2.3. For any instance of the minimum, sorting and MST problems under uncertainty, the hop distance is at least as large as the mandatory query distance, $k_M \leq k_h$.

Proof. We first discuss the minimum problem. Consider an instance with uncertainty intervals \mathcal{I} , true values w and predicted values \bar{w} . Let \mathcal{I}_P and \mathcal{I}_R be defined as above. Observe that k_M counts the intervals that are in $\mathcal{I}_P \setminus \mathcal{I}_R$ and those that are in $\mathcal{I}_R \setminus \mathcal{I}_P$. We will show the claim that, for every interval I_i in those sets, there is an interval I_j such that the value of I_j passes over L_i or U_i (or both) when going from \bar{w}_j to w_j . This means that each interval $I_i \in \mathcal{I}_P \Delta \mathcal{I}_R$ is mapped to a unique pair (j, i) such that the value of I_j passes over at least one endpoint of I_i , and hence each such pair contributes at least one to k_h . This implies $k_M \leq k_h$.

It remains to prove the claim. Consider an $I_i \in \mathcal{I}_P \setminus \mathcal{I}_R$. (The argumentation for intervals in $\mathcal{I}_R \setminus \mathcal{I}_P$ is symmetric, with the roles of w and \bar{w} exchanged.) As I_i is not in \mathcal{I}_R , replacing all intervals in $\mathcal{I} \setminus \{I_i\}$ by their true values yields an instance that is solved. This means that in every set $S \in \mathcal{S}$ that contains I_i , one of the following cases holds:

- (a) I_i is known not to be the minimum of S w.r.t. true values w . It follows that there is an interval I_j in S with $w_j \leq L_i$.
- (b) I_i is known to be the minimum of S w.r.t. true values w . It follows that all intervals $I_j \in S \setminus \{I_i\}$ satisfy $w_j \geq U_i$.

As I_i is in \mathcal{I}_P , replacing all intervals in $\mathcal{I} \setminus \{I_i\}$ by their predicted values yields an instance that is not solved. This means that there exists at least one set $S' \in \mathcal{S}$ that contains I_i and satisfies the following:

- (c) All intervals I_j in $S' \setminus \{I_i\}$ satisfy $\bar{w}_j > L_i$, and there is at least one such I_j with $L_i < \bar{w}_j < U_i$.

If S' falls into case (a) above, then by (a) there is an interval I_j in S' with $w_j \leq L_i$, and by (c) we have $\bar{w}_j > L_i$. This means that the value of I_j passes over L_i . If S' falls into case (b) above, then by (c) there exists an interval I_j in S' with $\bar{w}_j < U_i$, and by (b) we have $w_j \geq U_i$. Thus, the value of I_j passes over U_i . This establishes the claim, and hence we have shown that $k_M \leq k_h$ for the minimum problem.

We show in Appendix E that every instance of the sorting problem can be transformed into an equivalent instance of the minimum problem on the same uncertainty intervals. The transformation ensures that, for any two intervals I_i and I_j , there exists a set in the sorting instance that contains both I_i and I_j if and only if there exists a set in the minimum instance that contains both I_i and I_j . Therefore, the error measures for both instances are the same, and the above result for the minimum problem implies that $k_M \leq k_h$ also holds for the sorting problem.

Finally, we consider the MST problem. Consider an instance $G = (V, E)$ with uncertainty interval $I_e = (L_e, U_e)$, true value w_e and predicted value \bar{w}_e for $e \in E$. Let E_P and E_R be the mandatory queries with respect to the predicted and true values, respectively. Again, k_M counts the edges in $E_P \Delta E_R$. We claim that, for every interval I_e of an edge e in this set, there is an interval I_g of an edge g such that the value of I_g passes over L_e or U_e (or both) when going from \bar{w}_e to w_e and g lies on a cycle with e . This means that each edge $e \in E_P \Delta E_R$ is mapped to a unique pair (e, g) such that g lies in the same biconnected component as e and the value of I_g passes over at least one endpoint of I_e , and hence each such pair contributes at least one to k_h . This implies $k_M \leq k_h$.

It remains to prove the claim. Consider an edge $e \in E_P \setminus E_R$. (The argumentation for edges in $E_R \setminus E_P$ is symmetric, with the roles of w and \bar{w} exchanged.) As e is not in E_R , replacing all intervals I_g for $g \in E \setminus \{e\}$ by their true values yields an instance that is solved. This means that for edge e one of the following cases applies:

- (a) e is known to be in the MST. Then there is a cut X_e containing edge e (namely, the cut between the two vertex sets obtained from the MST by removing the edge e) such that e is known to be a minimum weight edge in the cut, i.e., every other edge g in the cut satisfies $w_g \geq U_e$.
- (b) e is known not to be in the MST. Then there is a cycle C_e in G (namely, the cycle that is closed when e is added to the MST) such that e is a maximum weight edge in C_e , i.e., every other edge g in the cycle satisfies $w_g \leq L_e$.

As e is in E_P , replacing all intervals I_g for $g \in E \setminus \{e\}$ by their predicted values yields an instance Π that is not solved. Let T' be the minimum spanning tree of $G' = (V, E \setminus \{e\})$ for Π . Let C' be the cycle closed in T' by adding e , and let f be an edge with the largest predicted value in $C' \setminus \{e\}$. Then there are only two possibilities for the minimum spanning tree of G for Π : Either T' is also a minimum spanning tree of G (if $\bar{w}_e \geq \bar{w}_f$), or the minimum spanning tree is $T' \cup \{e\} \setminus \{f\}$. As knowing whether e is in the minimum spanning tree would allow us to determine which of the two cases applies, it must be the case that we cannot determine whether e is in the minimum spanning tree or not without querying e . If e satisfied case (a) with cut X_e above, then there must be an edge g in $X_e \setminus \{e\}$ with $\bar{w}_g < U_e$, because otherwise e would also have to be in the MST of G for Π , a contradiction. Thus, the value of I_g passes over U_e . If e satisfied case (b) with cycle C_e above, then there must be an edge g in $C_e \setminus \{e\}$ with $\bar{w}_g > L_e$, because otherwise e would also be excluded from the MST of G for Π , a contradiction. Thus, the value of I_g passes over L_e . This establishes the claim, and hence we have shown that $k_M \leq k_h$ for the MST problem. \square

Theorem 2.4. Let $\gamma \geq 2$ be a fixed rational value. If a deterministic algorithm for the minimum, sorting or MST problem is γ -robust, then it cannot have competitive ratio better than $1 + \frac{1}{\gamma-1}$ for $k_M = 0$. Furthermore, if an algorithm has competitive ratio $1 + \frac{1}{\gamma-1}$ for $k_M = 0$, then it cannot be better than γ -robust.

Proof. We first establish the following auxiliary claim, which is slightly weaker than the statement of the theorem:

Claim A.2. Let $\gamma' \geq 2$ be a fixed rational number. Every deterministic algorithm for the minimum, sorting or MST problem has competitive ratio at least γ' for $k_M = 0$ or has competitive ratio at least $1 + \frac{1}{\gamma'-1}$ for arbitrary k_M .

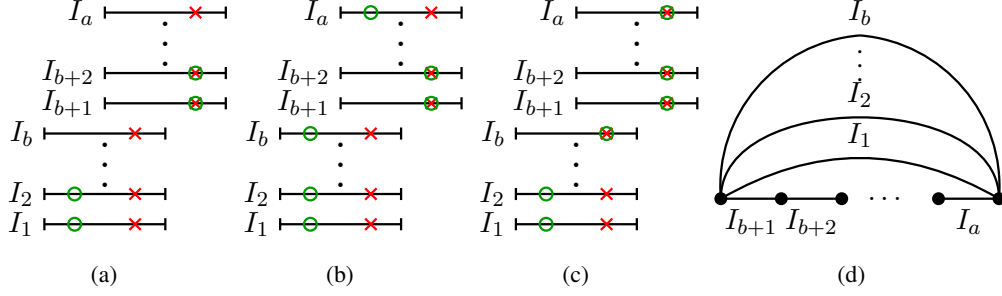


Figure 4: Instance for lower bound based on the mandatory query distance and MST instance.

Let $\gamma' = \frac{a}{b}$, with integers $a \geq 2b > 0$. We first give the proof of the claim for the minimum problem. Consider an instance with a intervals as depicted in Figure 4(a), with sets $S_i = \{I_i, I_{b+1}, I_{b+2}, \dots, I_a\}$ for $i = 1, \dots, b$. Suppose, w.l.o.g., that the algorithm queries the left-side intervals in the order I_1, I_2, \dots, I_b , and the right side in the order $I_{b+1}, I_{b+2}, \dots, I_a$. Let the predictions be correct for I_{b+1}, \dots, I_{a-1} , and $w_1, \dots, w_{b-1} \notin I_1 \cup \dots \cup I_a$.

If the algorithm queries I_a before I_b , then the adversary sets $w_a \in I_b$ and $w_b \notin I_a$. (See Figure 4(b).) This forces a query in all left-side intervals, so the algorithm queries all a intervals, while the optimum solution queries only the b left-side intervals. Thus the competitive ratio is at least $\frac{a}{b} = \gamma$ for arbitrary k_M .

If the algorithm queries I_b before I_a , then the adversary sets $w_a = \bar{w}_a$ and $w_b \in I_a$; see Figure 4(c). This forces the algorithm to query all remaining right-side intervals, i.e., a queries in total, while the optimum queries only the $a - b$ right-side intervals. Note, however, that $k_M = 0$, since the right-side intervals are mandatory for both predicted and correct values, while I_1, \dots, I_b are not mandatory in either of the solutions. Thus, the competitive ratio is at least $\frac{a}{a-b} = 1 + \frac{1}{\gamma-1}$ for $k_M = 0$.

For the sorting problem, the proof is the same except that we take the $b(a - b)$ input sets $\{I_i, I_j\}$ for $1 \leq i \leq b, b + 1 \leq j \leq a$.

For the MST problem, we again translate the above construction from the minimum problem to the maximum problem and then consider an MST instance consisting of a path with edge weight intervals I_j for $b + 1 \leq j \leq a$, and parallel edges with weight intervals I_i for $1 \leq i \leq b$ (where I_i and I_j , with slight abuse of notation, refer to the intervals of the maximum instance) between the endpoints of the path, see Figure 4(d). This completes the proof of Claim A.2.

Now we are ready to prove the theorem. The argument is the same for all three problems. Let $\gamma \geq 2$ be a fixed rational. Assume that there is a deterministic algorithm A that is γ -robust and has competitive ratio strictly smaller than $1 + \frac{1}{\gamma-1}$, say $1 + \frac{1}{\gamma+\varepsilon-1}$ with $\varepsilon > 0$, for $k_M = 0$. Let γ' be a rational number with $\gamma < \gamma' < \gamma + \varepsilon$. Then A has competitive ratio strictly smaller than γ' for arbitrary k_M and competitive ratio strictly smaller than $1 + \frac{1}{\gamma'-1}$ for $k_M = 0$, a contradiction to Claim A.2. This shows the first statement of the theorem.

Let $\gamma \geq 2$ again be a fixed rational. Assume that there is a deterministic algorithm A that has competitive ratio $1 + \frac{1}{\gamma-1}$ for $k_M = 0$ and is $(\gamma - \varepsilon)$ -robust, where $\varepsilon > 0$. As there is a lower bound of 2 on the robustness of any deterministic algorithm for all three problems, no such algorithm can exist for $\gamma = 2$. So we only need to consider the case $\gamma > 2$ and $\gamma - \varepsilon \geq 2$. Let γ' be a rational number with $\gamma - \varepsilon < \gamma' < \gamma$. Then A has competitive ratio strictly smaller than $1 + \frac{1}{\gamma'-1}$ for $k_M = 0$ and competitive ratio strictly smaller than γ' for arbitrary k_M , a contradiction to Claim A.2. This shows the second statement of the theorem. \square

A.4 Lower bound for all error measures

Theorem 2.5. Any deterministic algorithm for minimum, sorting or MST under uncertainty has a competitive ratio $\rho \geq \min\{1 + \frac{k}{\text{opt}}, 2\}$, for any error measure $k \in \{k_{\#}, k_M, k_h\}$, even for disjoint sets.

Proof. Consider the input instance of the minimum or sorting problem consisting of a single set $\{I_1, I_2\}$ as shown in Figure 3(c). If the algorithm starts querying I_1 , then the adversary sets $w_1 = \bar{w}_1$ and the algorithm is forced to query I_2 . Then $w_2 \in I_2 \setminus I_1$, so the optimum queries only I_2 . It is easy to see that $k_{\#} = k_M = k_h = 1$. A symmetric argument holds if the algorithm starts querying I_2 . In that case, $w_2 = \bar{w}_2$ which enforces to query I_1 with $w_1 \in I_1 \setminus I_2$. Taking multiple copies of this instance gives the result for any $k \leq \text{opt}$. For the MST problem, we can place copies of the instance (translated to the maximum problem as usual) in disjoint cycles that are connected by a tree structure. \square

B NP-hardness of the Verification Problem for Minimum and Sorting

Theorem B.1. It is NP-hard to solve the verification problem for finding the minimum in overlapping sets with uncertainty intervals and given values.

Proof. The proof uses a reduction from the vertex cover problem for 2-subdivision graphs, which is NP-hard [53]. A 2-subdivision is a graph H which can be obtained from an arbitrary graph G by replacing each edge by a path of length four (with three edges and two new vertices). The graph in Figure 5(b) is a 2-subdivision of the graph in Figure 5(a).

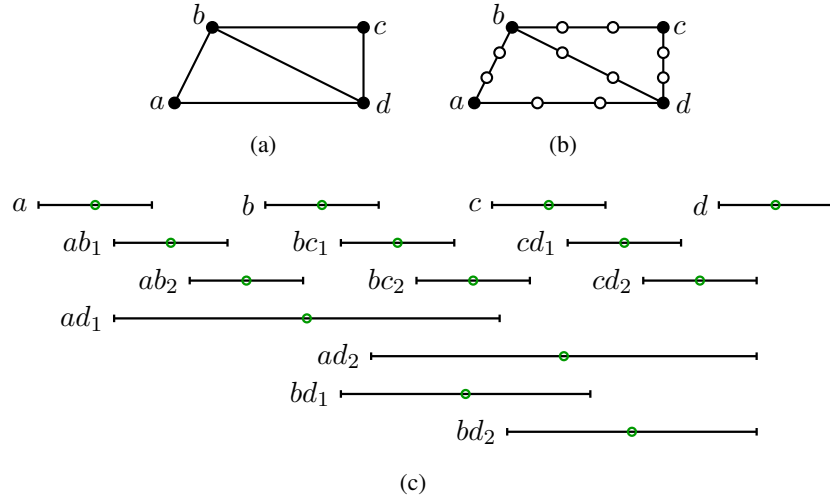


Figure 5: NP-hardness reduction for the minimum problem, from the vertex cover problem on 2-subdivision graphs. (a) A graph and (b) its 2-subdivision. (c) The corresponding instance for the minimum problem.

Given a graph H which is a 2-subdivision of a graph G , we build an instance of the minimum problem in the following way. If n is the number of vertices in G , then we start by creating n intervals that do not intersect each other. For each edge uv of G , we create two new intervals uv_1 and uv_2 , in such a way that we have intersection pairs $u(uv_1)$, $(uv_1)(uv_2)$, $(uv_2)v$. We then create a set of size 2 for each edge in H , consisting of the corresponding intervals. Finally, we can clearly assign true values such that, for any set xy , neither $w_x \in I_y$ nor $w_y \in I_x$. Therefore, to solve each set, it is enough to query one of the intervals, so clearly any solution to the problem corresponds to a vertex cover of H , and vice-versa. See Figure 5(c) for an example. \square

This reduction constructs an instance of the minimum problem where all sets have size 2. For instances with sets of size 2, the minimum problem and the sorting problem are equivalent. Hence, Theorem B.1 implies the following.

Corollary B.2. It is NP-hard to solve the verification problem for sorting overlapping sets with uncertainty intervals and given values.

C Appendix for the Minimum Problem (Section 4)

Lemma 4.1. An interval I_i is *mandatory* for the minimum problem if and only if (a) I_i is a true minimum of a set S and contains w_j of another interval $I_j \in S \setminus \{I_i\}$ (in particular, if $I_j \subseteq I_i$), or (b) I_i is not a true minimum of a set S but contains the value of the true minimum of S . *Prediction mandatory* intervals are characterized equivalently, replacing true values by predicted values.

Proof. If I_i is the true minimum of S and contains w_j of another set $I_j \in S$, then S cannot be solved even if we query all intervals in $S \setminus \{I_i\}$. If I_i is not a true minimum and contains the true minimum value w^* , then S cannot be solved even if we query all intervals in $S \setminus \{I_i\}$, as we cannot prove that $w^* \leq w_i$.

If I_i is the true minimum of a set S , but $w_j \notin I_i$ for every $I_j \in S \setminus \{I_i\}$, then $S \setminus \{I_i\}$ is a feasible solution for S . If I_i is not a true minimum of a set S and does not contain the true minimum value of S , then again $S \setminus \{I_i\}$ is a feasible solution for S . If every set S that contains I_i falls into one of these two cases, then querying all intervals except I_i is a feasible query set for the whole instance. \square

The following corollary is used to identify known mandatory intervals.

Corollary C.1. If the leftmost interval I_l in a set S contains the true value of another interval in S , then I_l is mandatory.

Proof. If I_l is the true minimum of S , then the claim follows directly from Lemma 4.1. Otherwise, I_l must contain the true minimum value of S because it is leftmost, so the claim follows from Lemma 4.1. \square

Lemma C.2. Let I_l be the leftmost interval in a set S , and assume that I_l does not contain another interval in S . If I_l is queried, then S can be solved by querying only intervals that become known mandatory.

Proof. Since I_l does not contain another interval in S , either w_l is found to be the minimum in S , or w_l is contained in the next leftmost interval in S , which becomes a known mandatory interval due to Corollary C.1, and then the claim follows by induction. \square

The following lemma guarantees that Algorithms 1 and 2 indeed solve the problem. In Algorithm 1, the instance in Line 14 has no known mandatory intervals because of Line 13. In Algorithm 2, the instance in Line 9 has no known mandatory intervals because of Line 8.

Lemma C.3. Suppose an instance without known mandatory intervals. After querying a vertex cover, the minimum problem can be solved by querying only intervals that become known mandatory.

Proof. The vertex cover is computed in the dependency graph, a graph with a vertex for each interval and, for each unsolved set S , edges between the leftmost interval I_l in S and all other intervals in S that intersect I_l . Consider an unsolved set S with leftmost interval I_l . The vertex cover must either (a) contain I_l (and maybe other intervals in S), or (b) contain all intervals in $S \setminus \{I_l\}$ that intersect I_l . If case (b) holds, then either I_l becomes known to be the minimum interval because no true value is in I_l , or some other true value is in I_l and the claim follows from Corollary C.1. If case (a) holds, then the claim follows from Lemma C.2. \square

C.1 Analysis of Algorithm 1 (Theorem 3.2)

Recall the following lemma for identifying witness sets.

Lemma C.4 ([39]). A set $\{I_i, I_j\} \subseteq S$ with $I_i \cap I_j \neq \emptyset$, and I_i or I_j leftmost in S , is always a witness set.

An important lemma for proving upper bounds for this algorithm is the following.

Lemma C.5. If \bar{w}_j enforces I_i , then $\{I_i, I_j\}$ is a witness set. Also, if $w_j \in I_i$, then I_i is mandatory.

Proof. Since \bar{w}_j enforces I_i , there must be a set S with $I_i, I_j \in S$ such that $\bar{w}_j \in I_i$ and either I_i is leftmost in S or I_j is leftmost in S and I_i is leftmost in $S \setminus \{I_j\}$. The first claim follows from Lemma C.4. If I_i is a true minimum of S or is leftmost in S , then the second claim follows from Lemma 4.1 and Corollary C.1. Otherwise, the fact that $w_j \in I_i$ and that I_i is leftmost in $S \setminus \{I_j\}$ implies that I_i contains the minimum true value, so the claim follows from Lemma 4.1. \square

Lemma C.6. Consider a point of execution of the algorithm in which \bar{w}_j enforces I_i . It holds that I_i is prediction mandatory for the current instance.

Proof. Consider the set S as in the definition of \bar{w}_j enforcing I_i . If I_i is leftmost in S , then the claim holds from a similar argument as in the proof of Corollary C.1. If I_j is leftmost in S and I_i is leftmost in $S \setminus \{I_j\}$, then we have three cases, in all of which the claim follows from Lemma 4.1: (a) \bar{w}_j is the predicted minimum value in S , and $\bar{w}_j \in I_i$ holds by definition; (b) \bar{w}_i is the predicted minimum value in S , and $\bar{w}_j \in I_i$ holds by definition; (c) some \bar{w}_l with $l \neq i, j$ is the predicted minimum value in S , and in this case we claim that $\bar{w}_l \in I_i$. To see that the claim for the last case holds, note that $L_i \leq L_l$ because I_j is leftmost in S and I_i is leftmost in $S \setminus \{I_j\}$, and $\bar{w}_l < U_i$ because \bar{w}_l is the predicted minimum. \square

Lemma C.7. There is at most one execution of the loop consisting of Lines 1–12 in which Line 5 is executed but no query is performed in Lines 9–11, and that is the last execution of this loop in which any query is performed. After this point, the instance has no prediction mandatory intervals.

Proof. Suppose that there is an iteration in which Line 5 is executed but no query is performed in Lines 9–11. We claim that, before the test in Line 8 is executed, the current instance has no prediction mandatory intervals. Suppose, for the sake of contradiction, that a set S satisfies some condition in Lemma 4.1 to contain a prediction mandatory interval. Let I_i be leftmost in S , and I_j be leftmost in $S \setminus \{I_i\}$. We claim that \bar{w}_i enforces I_j or some \bar{w}_l enforces I_i with $I_l \in S$, which contradicts the fact that no query is performed in Lines 9–11. Let \bar{w}^* be the predicted minimum value in S . If $\bar{w}_i \neq \bar{w}^*$, then $\bar{w}^* \in I_i$ because I_i is leftmost, so \bar{w}^* enforces I_i . Otherwise $\bar{w}_i = \bar{w}^*$ and we have two cases: (a) if $\bar{w}_i > L_j$, then $\bar{w}_i \in I_j$ because Line 6 prevents that $I_j \subseteq I_i$, so \bar{w}_i enforces I_j ; (b) if $\bar{w}_i \leq L_j$, then some interval $I_l \in S$ must have $\bar{w}_l \in I_i$, otherwise there would be no prediction mandatory intervals in S , so \bar{w}_l enforces I_i .

This claim implies that no query is performed in the next iteration because:

1. No query can be performed in Line 2 because known mandatory intervals are queried in the last execution of Line 6.
2. From the previous item, the instance does not change and remains without prediction mandatory intervals, so no query is performed in Lines 5 and 6.
3. No query can be made in Lines 9–11 because, if \bar{w}_j enforces I_i , then Lemma C.6 implies that I_i is prediction mandatory, which contradicts the previous item.

Therefore the instance after this point has no prediction mandatory intervals and the lemma holds. \square

Lemma C.8. Let I_i, I_j be a pair that satisfies the condition in Line 11 leading to a query of I_i . Every set S containing I_j will be solved after querying I_i , or will be solved using only known mandatory queries in Line 2 in the next consecutive iterations of the loop, or I_j is certainly not the minimum in S .

Proof. Consider the instance before I_i is queried. Due to the test in Line 8, for every set S containing I_j , the following facts hold:

- (1) If I_j is leftmost in S , then S is already solved, or $I_i \in S$ and I_i is the only interval in S that intersects I_j .
- (2) If I_j is not leftmost in S but intersects the leftmost interval $I_{i'}$ in S , then $I_{i'} = I_i$.
- (3) If I_j is not leftmost in S and does not intersect the leftmost interval in S , then I_j is certainly not the minimum in S .

If condition (1) holds and S is not solved then, after querying I_i , either $w_i \notin I_j$ and S becomes solved, or $w_i \in I_j$ and I_j will be queried in Line 2 due to Corollary C.1, and then S becomes solved. If condition (2) holds, then the result follows from Lemma C.2 because I_i is leftmost in S . The result follows trivially if condition (3) holds. \square

This lemma clearly implies the following corollary.

Corollary C.9. Let I_i, I_j as in Lemma C.8. After I_i is queried, I_j will no longer be a prediction mandatory interval identified in Line 3 or 7, or be part of a triple satisfying the condition in Line 8, or a pair satisfying the condition in Line 11, or be part of the vertex cover Q queried in Line 14, or be a known mandatory interval queried in Line 6, 13 or 15.

Theorem 3.2. There is an algorithm for the minimum problem under uncertainty that, given an integer parameter $\gamma \geq 2$, achieves a competitive ratio of $\min\{(1 + \frac{1}{\gamma})(1 + \frac{k_h}{\text{opt}}), \gamma\}$. Furthermore, if $\gamma = 2$, then the competitive ratio is $\min\{1.5 + k_h/\text{opt}, 2\}$.

Proof. **γ -robustness.** Intervals queried in Line 10 are in any feasible solution.

Fix an optimum solution OPT. Let \mathcal{I}' be the set of unqueried intervals in Line 3 at the iteration of the loop consisting of Lines 1–12 in which Line 5 is executed but no query is performed in Lines 9–11 (or before Line 13 if no such iteration exists). Recall that Lemma C.7 states that there is at most one such iteration, and it has to be the last iteration in which some interval is queried. If the problem is undecided at this point, then $|\text{OPT} \cap \mathcal{I}'| \geq 1$, and $|Q| \leq \gamma - 2$ implies $|Q| \leq (\gamma - 2) \cdot |\text{OPT} \cap \mathcal{I}'|$. Also, since Q' is a minimum vertex cover, then $|Q'| \leq |\text{OPT} \cap \mathcal{I}'|$. Let M be the set of intervals in \mathcal{I}' that are queried in Lines 6, 13 and 15; clearly $M \subseteq \text{OPT} \cap \mathcal{I}'$. Thus $|Q| + |Q'| + |M| \leq \gamma \cdot |\text{OPT} \cap \mathcal{I}'|$.

Now consider an iteration of the loop in which some query is performed in Lines 9–11. Let P' be the set of intervals queried in Lines 5, 9 and 11. If Line 9 is executed, then note that $\{I_j, I_l\}$ is a witness set. If a query is performed in Line 11, then note that $\{I_i, I_j\}$ is a witness set. Due to Lemma C.8, if I_j is queried, then it is in Line 2 at the next iteration; if that happens, then we include I_j in P' . Due to Corollary C.9, I_j is not considered more than once in this case, and is not considered in any of the previous cases. Either way, it holds that P' is a witness set of size at most γ .

The remaining intervals queried in Lines 2 and 6 are in any feasible solution.

Bound of $(1 + \frac{1}{\gamma})(1 + \frac{k_h}{\text{opt}})$. Fix an optimum solution OPT. Let $h'(I_j)$ be the number of intervals I_i such that $I_i, I_j \in S$ for some $S \in \mathcal{S}$, and the value of I_i passes over an endpoint of I_j . From the arguments in the proof of Theorem 2.3, it can be seen that, for each interval I_j that is prediction mandatory at some point and is not in OPT, we have that $h'(I_j) \geq 1$. For a subset $\mathcal{J} \subseteq \mathcal{I}$, let $h'(\mathcal{J}) = \sum_{I_j \in \mathcal{J}} h'(I_j)$. Note that $k_h = h'(\mathcal{I})$ holds by reordering summations.

In the following, we will show for various disjoint subsets $\mathcal{J} \subseteq \mathcal{I}$ that $|\mathcal{J} \cap \text{ALG}| \leq (1 + \frac{1}{\gamma}) \cdot (|\text{OPT} \cap \mathcal{J}| + h'(\mathcal{J}))$. The subsets \mathcal{J} will form a partition of \mathcal{I} , so it is clear that the bound of $(1 + \frac{1}{\gamma}) \cdot (1 + \frac{k_h}{\text{opt}})$ on the competitive ratio of the algorithm follows. Furthermore, if $\gamma = 2$, then we will show for every \mathcal{J} that $|\mathcal{J} \cap \text{ALG}| \leq 1.5 \cdot |\text{OPT} \cap \mathcal{J}| + h'(\mathcal{J})$, so we have a bound of $1.5 + k_h/\text{opt}$ on the competitive ratio.

Intervals queried in Lines 6 and 13 are in any feasible solution, so the set P_0 of these intervals satisfies $|P_0| \leq |\text{OPT} \cap P_0|$.

If there is an execution of the loop consisting of Lines 1–12 that does not perform queries in Lines 9–11, then let P_1 be the set of intervals queried in Line 5. Every interval $I_j \in P_1$ is prediction mandatory, so if $I_j \notin \text{OPT}$ then $h'(I_j) \geq 1$. Thus we have that $|P_1| \leq |P_1 \cap \text{OPT}| + h'(P_1)$.

Let \mathcal{I}' be the set of unqueried intervals before Line 14 is executed. Since Q' is a minimum vertex cover, we have that $|Q'| \leq |\text{OPT} \cap \mathcal{I}'|$. Let M be the set of intervals queried in Line 15. Due to Lemma 4.1, each interval $I_i \in M$ is known mandatory because it contains the value w_j of an interval $I_j \in Q$. But Lemma C.7 implies that $\bar{w}_j \notin I_i$ when Line 14 was executed, so $h'(I_i) \geq 1$. Thus we have that $|\mathcal{I}' \cap \text{ALG}| = |Q' \cup M| \leq |\mathcal{I}' \cap \text{OPT}| + h'(M) \leq |\mathcal{I}' \cap \text{OPT}| + h'(\mathcal{I}')$.

Consider an execution of the loop in which some query is performed in Lines 9–11. Let Q be the set of intervals queried in Line 5, and let W be the set of intervals queried in Lines 9–11. If a query is performed in Line 11 and I_j is queried in Line 2 at the next iteration, then include I_j in W as well. Note that $|Q| = \gamma - 2$. If \bar{w}_j enforces I_i in Line 8 or 11, then I_i is prediction mandatory due to Lemma C.6. Also, note that $h'(Q) \geq |Q \setminus \text{OPT}|$, since every interval in Q is prediction mandatory at some point. We divide the proof in three cases. For a pair $\{I_i, I_j\}$ as in Line 11, note that, due to Corollary C.9, I_j is not considered more than once, and is not considered in any of the previous cases.

- (a) If $|W| = 1$, then some interval I_i was queried in Line 11 because \bar{w}_j enforces I_i , and I_j is not queried by the algorithm due to Lemma C.8. Then it suffices to note that $\{I_i, I_j\}$ is a witness set to see that $|Q \cup W| \leq |\text{OPT} \cap (Q \cup \{I_i, I_j\})| + h'(Q)$.
- (b) Consider $|W| = 2$. If W is a pair of the form $\{I_j, I_l\}$ queried in Line 9, then $h'(I_i) \geq 1$ because \bar{w}_j enforces I_i but $w_j \notin I_i$. We can conceptually move this contribution in the hop distance to I_j , making $h'(I_i) := h'(I_i) - 1$ and $h'(I_j) := h'(I_j) + 1$. (If I_i is considered another time in Line 8 or in another point of the analysis because it is enforced by some predicted value, then it has to be the predicted value of an interval $I_{j'} \neq I_j$, so we are not counting the contribution to the hop distance more than once.) If W is a pair of the form $\{I_i, I_j\}$ queried in Line 11 and in Line 2 at the next iteration, then either $W \subseteq \text{OPT}$ or $h'(I_i) = 1$: It holds that I_j is mandatory, so if I_i is not in OPT then it suffices to see that \bar{w}_j enforces I_i . Either way, the fact that W is a witness set is enough to see that $|Q \cup W| \leq |\text{OPT} \cap (Q \cup W)| + h'(Q) + h'(W)$.
- (c) If $|W| = 3$, then $W = \{I_i, I_j, I_l\}$ as in Line 8, and $|Q \cup W| = \gamma + 1$. Also, it holds that I_i and at least one of $\{I_j, I_l\}$ are in any feasible solution. This implies that at least $\frac{\gamma}{\gamma+1} \cdot |Q \cup W| - h'(Q)$ of the intervals in $Q \cup W$ are in OPT , so $|Q \cup W| \leq (1 + \frac{1}{\gamma})(|\text{OPT} \cap (Q \cup W)| + h'(Q))$. If $\gamma = 2$, then $Q = \emptyset$ and we have that $|W| \leq 1.5 \cdot |\text{OPT} \cap W|$.

The remaining intervals queried in Line 2 are in any feasible solution. □

C.2 Analysis of Algorithm 2 (Theorem 3.4)

Lemma C.10. Every interval queried in Line 10 of Algorithm 2 is in $\mathcal{I}_R \setminus \mathcal{I}_P$.

Proof. Clearly every such interval is in \mathcal{I}_R because it is known to be mandatory, so it remains to prove that it is not in \mathcal{I}_P . Consider a set S . If an interval $I_j \in S \cap \mathcal{I}_P$ is unqueried in Line 9, then the condition for

identifying a witness set in Line 2 implies that, before Line 9 is executed, I_j is not leftmost in S and does not intersect the leftmost interval in S . Thus, it is not necessary to query I_j to solve S , and we can conclude that I_j is not queried in Line 10. \square

Theorem 3.4. There is an algorithm for the minimum problem under uncertainty that, given an integer parameter $\gamma \geq 2$, achieves a competitive ratio of $\min\{(1 + \frac{1}{\gamma-1}) \cdot (1 + \frac{k_M}{\text{opt}}), \gamma\}$.

Proof. γ -robustness. Given $P' \cup \{b\}$ queried in Line 5, at least one interval is in any feasible solution since $\{b, p\}$ is a witness set, thus $P' \cup \{b\}$ is a witness set of size γ .

Line 7 is executed at most once, since the size of P never increases. Fix an optimum solution OPT, and let \mathcal{I}' be the set of unqueried intervals before Line 7 is executed (or before Line 8 if Line 7 is never executed). Let P be the set of intervals queried in Line 7. If the problem is undecided at this point, then $|\text{OPT} \cap \mathcal{I}'| \geq 1$, so $|P| \leq \gamma - 2$ implies $|P| \leq (\gamma - 2) \cdot |\text{OPT} \cap \mathcal{I}'|$. Also, since Q is a minimum vertex cover, then $|Q| \leq |\text{OPT} \cap \mathcal{I}'|$. Let M be the set of intervals in \mathcal{I}' that are queried in Lines 8 or 10; clearly $M \subseteq \text{OPT} \cap \mathcal{I}'$. Thus $|P| + |Q| + |M| \leq \gamma \cdot |\text{OPT} \cap \mathcal{I}'|$.

The intervals queried in Line 6 are in any feasible solution, and the claim follows.

Bound of $(1 + \frac{1}{\gamma-1}) \cdot (1 + \frac{k_M}{\text{opt}})$. Fix an optimum solution OPT. In the following, we will show for various disjoint subsets $\mathcal{J} \subseteq \mathcal{I}$ that $|\mathcal{J} \cap \text{ALG}| \leq (1 + \frac{1}{\gamma-1}) \cdot (|\text{OPT} \cap \mathcal{J}| + k_{\mathcal{J}})$, where $k_{\mathcal{J}} \leq |\mathcal{J} \cap (\mathcal{I}_P \Delta \mathcal{I}_R)|$. The subsets \mathcal{J} will form a partition of \mathcal{I} , so it is clear that the bound of $(1 + \frac{1}{\gamma-1}) \cdot (1 + \frac{k_M}{\text{opt}})$ on the competitive ratio of the algorithm follows.

Intervals queried in Lines 6 and 8 are part of any feasible solution, hence the set P_0 of these intervals satisfies $|P_0| \leq |\text{OPT} \cap P_0|$.

Given $P' \cup \{b\}$ queried in Line 5, at least $\frac{\gamma-1}{\gamma}$ of the intervals in $P_{\gamma} \cup \{b\}$ are prediction mandatory for the initial instance. Among those, let $k' \leq k_M$ be the number of intervals in $\mathcal{I}_P \setminus \mathcal{I}_R$. Thus $|\text{OPT} \cap (P' \cup \{b\})| \geq \frac{\gamma-1}{\gamma} \cdot |P' \cup \{b\}| - k'$, which gives the desired bound, i.e., $|P' \cup \{b\}| \leq (1 + \frac{1}{\gamma-1}) \cdot (|\text{OPT} \cap (P' \cup \{b\})| + k')$.

Every interval queried in Line 7 that is not in OPT is in $\mathcal{I}_P \setminus \mathcal{I}_R$. Hence, if there are k'' such intervals, then the set P of intervals queried in Line 7 satisfies $|P| \leq |\text{OPT} \cap P| + k'' < (1 + \frac{1}{\gamma}) \cdot (|\text{OPT} \cap P| + k'')$.

Let \mathcal{I}' be the set of unqueried intervals before Line 9 is executed. Then $|Q| \leq |\text{OPT} \cap \mathcal{I}'|$ because Q is a minimum vertex cover. Let M be the set of intervals that are queried in Line 10. It holds that $|Q \cup M| \leq |\text{OPT} \cap \mathcal{I}'| + |M|$, so the claimed bound follows from Lemma C.10. \square

The parameter γ in Theorem 3.4 is restricted to integral values since it determines sizes of query sets. Nevertheless, a generalization to arbitrary $\gamma \in \mathbb{R}_+$ is possible at a small loss in the guarantee. We give the following rigorous upper bound on the achievable tradeoff of robustness and error-dependent competitive ratio.

Theorem C.11. For any real number $\gamma \geq 2$, there is a randomized algorithm for the minimum and sorting problem under uncertainty that achieves a competitive ratio of $\min\{(1 + \frac{1}{\gamma-1} + \xi) \cdot (1 + \frac{k_M}{\text{opt}}), \gamma\}$, for $\xi \leq \frac{\gamma - \lfloor \gamma \rfloor}{(\gamma-1)^2} \leq 1$.

Proof. For $\gamma \in \mathbb{Z}$, we run Algorithm 2 and achieve the performance guarantee from Theorem 3.4. Assume $\gamma \notin \mathbb{Z}$, and let $\{\gamma\} := \gamma - \lfloor \gamma \rfloor = \gamma - \lceil \gamma \rceil + 1$ denote its fractional part. We run the following randomized variant of Algorithm 2. We randomly chose γ' as $\lceil \gamma \rceil$ with probability $\{\gamma\}$ and as $\lfloor \gamma \rfloor$ with probability $1 - \{\gamma\}$, and then we run the algorithm with γ' instead of γ . We show that the guarantee from Theorem 3.4 holds in expectation with an additive term less than $\{\gamma\}$, more precisely, we show the competitive ratio

$$\min \left\{ \left(1 + \frac{1}{\gamma-1} + \xi \right) \cdot \left(1 + \frac{k_M}{\text{opt}} \right), \gamma \right\}, \text{ for } \xi = \frac{\{\gamma\}(1 - \{\gamma\})}{(\gamma-1)\lfloor \gamma \rfloor(\lceil \gamma \rceil - 1)} \leq \frac{\{\gamma\}}{(\gamma-1)^2}.$$

Following the arguments in the proof of Theorem 3.4 on the robustness, the ratio of the algorithm's number of queries $|\text{ALG}|$ and $|\text{OPT}|$ is bounded by γ' . In expectation the robustness is

$$\begin{aligned}\mathbb{E}[\gamma'] &= (1 - \{\gamma\}) \cdot \lfloor \gamma \rfloor + \{\gamma\} \cdot \lceil \gamma \rceil \\ &= (1 - \{\gamma\}) \cdot (\gamma - \{\gamma\}) + \{\gamma\} \cdot (\gamma - \{\gamma\} + 1) \\ &= \gamma.\end{aligned}$$

The error-depending bound on the competitive ratio is in expectation (with opt and k_M being independent of γ)

$$\mathbb{E} \left[\left(1 + \frac{1}{\gamma' - 1}\right) \cdot \left(1 + \frac{k_M}{\text{opt}}\right) \right] = \left(1 + \mathbb{E} \left[\frac{1}{\gamma' - 1} \right]\right) \cdot \left(1 + \frac{k_M}{\text{opt}}\right).$$

Applying simple algebraic transformations, we obtain

$$\begin{aligned}\mathbb{E} \left[\frac{1}{\gamma' - 1} \right] &= \frac{1 - \{\gamma\}}{\lfloor \gamma \rfloor - 1} + \frac{\{\gamma\}}{\lceil \gamma \rceil - 1} = \frac{1 - \{\gamma\}}{\gamma - \{\gamma\} - 1} + \frac{\{\gamma\}}{\gamma - \{\gamma\}} \\ &= \frac{(1 - \{\gamma\})(\gamma - \{\gamma\}) + \{\gamma\}(\gamma - \{\gamma\} - 1)}{(\gamma - \{\gamma\} - 1)(\gamma - \{\gamma\})} \\ &= \frac{\gamma - 2\{\gamma\}}{(\gamma - \{\gamma\} - 1)(\gamma - \{\gamma\})} = \frac{1}{\gamma - 1} - \frac{1}{\gamma - 1} + \frac{\gamma - 2\{\gamma\}}{(\gamma - \{\gamma\} - 1)(\gamma - \{\gamma\})} \\ &= \frac{1}{\gamma - 1} + \frac{\{\gamma\}(1 - \{\gamma\})}{(\gamma - 1)(\gamma - \{\gamma\} - 1)(\gamma - \{\gamma\})} = \frac{1}{\gamma - 1} + \frac{\{\gamma\}(1 - \{\gamma\})}{(\gamma - 1)\lfloor \gamma \rfloor(\lfloor \gamma \rfloor - 1)}.\end{aligned}$$

Hence, the competitive ratio is in expectation

$$\left(1 + \frac{1}{\gamma - 1} + \xi\right) \cdot \left(1 + \frac{k_M}{\text{opt}}\right) \text{ with } \xi = \frac{\{\gamma\}(1 - \{\gamma\})}{(\gamma - 1)\lfloor \gamma \rfloor(\lfloor \gamma \rfloor - 1)} \leq \frac{\{\gamma\}}{(\gamma - 1)^2},$$

which concludes the proof. \square

D Appendix for the MST Problem (Section 5)

In this section we prove the main theorems of Section 5. We first show Theorems 5.1 and 3.3 assuming that the lemmas of Section 5 hold, and then we prove that these lemmas are indeed true.

Theorem 5.1. There is a 1.5-consistent and 2-robust algorithm for MST under uncertainty.

Consider the algorithm that first executes Algorithm 3 with $\gamma = 2$ and then Algorithm 4 using recovery strategy A.

Proof. Let $\text{ALG} = \text{ALG}_1 \cup \text{ALG}_2$ be the query set queried by the algorithm where, ALG_1 and ALG_2 are the queries of Algorithm 3 and Algorithm 4, respectively. Let $\text{OPT} = \text{OPT}_1 \cup \text{OPT}_2$ be an optimal query set with $\text{OPT}_1 = \text{OPT} \cap \text{ALG}_1$ and $\text{OPT}_2 = \text{OPT} \setminus \text{ALG}_1$. Since $\gamma = 2$, Line 1 of Algorithm 3 does not query any elements. Therefore Lemma 5.6 implies $|\text{ALG}_1| \leq \min\{(1 + \frac{1}{2}) \cdot (|\text{OPT}_1| + k_h), 2 \cdot |\text{OPT}_1|\}$.

We continue by analyzing Algorithm 4 using recovery strategy A. According to Lemma 5.6, the input instance of Algorithm 4 is prediction mandatory free and we can apply Lemma 5.10. The lemma implies $|\text{ALG}_2| \leq |\text{OPT}_2|$ if all predictions are correct and $|\text{ALG}_2| \leq 2 \cdot |\text{OPT}_2|$ otherwise.

Summing up, this implies 1.5-consistency and 2-robustness. \square

Theorem 3.3. There is an algorithm for the MST problem under uncertainty with competitive ratio $\min\{1 + \frac{1}{\gamma} + (5 + \frac{1}{\gamma}) \cdot \frac{k_h}{\text{opt}}, \gamma + \frac{1}{\text{opt}}\}$, for any $\gamma \in \mathbb{Z}$ with $\gamma \geq 2$, where k_h is the hop distance of an instance.

Consider the algorithm that first executes Algorithm 3 with some $\gamma \in \mathbb{Z}, \gamma \geq 2$ and then Algorithm 4 using recovery strategy B.

Proof. Let $\text{ALG} = \text{ALG}_1 \cup P \cup \text{ALG}_2$ be the query set queried by the algorithm, where ALG_1 is the set of elements queried by Algorithm 3 without the elements queried in the last iteration of Line 1, P is the set of elements queried in the last iteration of Line 1 and ALG_2 is the set of elements queried by Algorithm 4. Let $\text{OPT} = \text{OPT}_1 \cup \text{OPT}_2$ be an optimal query set with $\text{OPT}_1 = \text{OPT} \cap \text{ALG}_1$ and $\text{OPT}_2 = \text{OPT} \setminus \text{ALG}_1$. Lemma 5.6 implies $|\text{ALG}_1| \leq \min\{(1 + \frac{1}{\gamma}) \cdot (|\text{OPT}_1| + k_h), \gamma \cdot |\text{OPT}_1|\}$.

We continue by analyzing ALG_2 and P and first show $|\text{ALG}_2 \cup P| \leq \gamma \cdot |\text{OPT}_2| + 1$. According to Lemma 5.6, the input instance of Algorithm 4 is prediction mandatory free and we can apply Lemma 5.10. The lemma implies $|\text{ALG}_2| \leq 3 \cdot |\text{OPT}_2 \setminus P|$. If $P \neq \emptyset$, then $|\text{OPT}_2| \geq 1$. This is because if the optimal query set was empty, the instance is solved at this point and no prediction mandatory elements can exist after querying ALG_1 , which contradicts $P \neq \emptyset$. This implies $|P| \leq \gamma - 2 \leq (\gamma - 3) \cdot |\text{OPT}_2| + 1$ and $|\text{ALG}_2 \cup P| \leq \gamma \cdot |\text{OPT}_2| + 1$. Using $|\text{ALG}_1| \leq \gamma \cdot |\text{OPT}_1|$ we can conclude $|\text{ALG}| = |\text{ALG}_1 \cup P \cup \text{ALG}_2| \leq \gamma \cdot |\text{OPT}| + 1$.

We continue by showing the consistency. Since each element of P is prediction mandatory, the proof of Theorem 2.3 implies that each such element is either mandatory or contributes at least one to the hop distance. It follows that querying P only improves the consistency and at most k_h elements of P are not part of OPT_2 .

Lemma 5.10 implies $|\text{ALG}_2| \leq |\text{OPT}_2 \setminus P| + 5 \cdot k_h$. Summing up the guarantee for P with the guarantee for ALG_2 directly gives us $|\text{ALG}_2 \cup P| \leq |\text{OPT}_2| + 6 \cdot k_h$. By combining this guarantee with $|\text{ALG}_1| \leq (1 + \frac{1}{\gamma}) \cdot (|\text{OPT}_1| + k_h)$, we directly obtain $|\text{ALG}| = |\text{ALG}_1 \cup P \cup \text{ALG}_2| \leq (1 + \frac{1}{\gamma}) \cdot \text{opt} + (7 + \frac{1}{\gamma}) \cdot k_h$. However, in Corollary D.6 we observe that we can exploit disjointness between the errors that we charge against to achieve the guarantees for ALG_1 , ALG_2 and P , to improve the guarantee for ALG to $|\text{ALG}| = |\text{ALG}_1 \cup P \cup \text{ALG}_2| \leq (1 + \frac{1}{\gamma}) \cdot \text{opt} + (5 + \frac{1}{\gamma}) \cdot k_h$. \square

The proofs of Theorems 5.1 and 3.3 show that the introduced lemmas imply the theorem. The remainder of this section proves that those lemmas indeed hold.

D.1 Preliminaries

Before the lemmas of Subsections 5.2 and 5.3 are shown, we introduce some preliminaries that are necessary for the proofs. We start by showing that we can assume uniqueness of T_L and T_U as well as $T_L = T_U$.

Lemma 5.2. By querying only mandatory elements we can obtain an instance with $T_L = T_U$ such that T_L and T_U are the unique lower limit tree and upper limit tree, respectively.

Proof. Let T_L be a lower limit tree for a given instance G and let T_U be an upper limit tree. According to [49], all elements of $T_L \setminus T_U$ are mandatory and we can repeatedly query them for (the adapting) T_L and T_U until $T_L = T_U$. We refer to this process as the *first preprocessing step*.

Consider an $f \in E \setminus T_U$ and the cycle C in $T_U \cup \{f\}$. If f is trivial, then the true value w_f is maximal in C and we may delete f without loss of generality. Assume otherwise. If the upper limit of f is uniquely maximal in C , then f is not part of any upper limit tree. If there is an $l \in C$ with $U_f = U_l$, then $T'_U = T_U \setminus \{l\} \cup \{f\}$ is also an upper limit tree. Since $T_L \setminus T'_U = \{l\}$, we may execute the first preprocessing step for T_L and T'_U . We repeatedly do this until each $f \in E \setminus T_U$ is uniquely maximal in the cycle C in $T_U \cup \{f\}$. Then, T_U is unique.

To achieve uniqueness for T_L , consider some $l \in T_L$ and the cut X of G between the two connected components of $T_L \setminus \{l\}$. If l is trivial, then the true value w_l is minimal in X and we may contract l without loss of generality. Assume otherwise. If l is uniquely minimal in X , then l is part of every lower limit tree.

If there is an $f \in X$ with $L_l = L_f$, then $T'_L = T_L \setminus \{l\} \cup \{f\}$ is also a lower limit tree. Since $T'_L \setminus T_U = \{f\}$, we may execute the first preprocessing step for T'_L and T_U . We repeatedly do this until each $l \in T_L$ is uniquely minimal in the cut X of G between the two components of $T_L \setminus \{l\}$. Then, T_L is unique. \square

Consider T_L and the edges f_1, \dots, f_l in $E \setminus T_L$ ordered by lower limit non-decreasingly. For each $i \in \{1, \dots, l\}$, define C_i to be the unique cycle in $T_L \cup \{f_i\}$ and $G_i = (V, E_i)$ to be the sub graph with $E_i = T_L \cup \{f_1, \dots, f_i\}$. Additionally, we define $G_0 = (V, T_L)$. During the course of this section, we will make use of the following two lemmas that were shown in [49]. According to Lemma 5.2, we can assume $T_L = T_U$ and that T_L and T_U are unique.

Lemma D.1 ([49, Lemma 5]). Let $i \in \{1, \dots, l\}$. Given a feasible query set Q for the uncertainty graph $G = (V, E)$, then the set $Q_i := Q \cap E_i$ is a feasible query set for $G_i = (V, E_i)$.

Lemma D.2 ([49, Lemma 6]). For some realization of edge weights, let T_i be a verified MST for graph G_i and let C be the cycle closed by adding f_{i+1} to T_i . Furthermore let h be some edge with the largest upper limit in C and $g \in C \setminus \{h\}$ be an edge with $U_g > L_h$. Then any feasible query set for G_{i+1} contains h or g . Moreover, if I_g is contained in I_h , any feasible query set contains edge h .

While Lemma D.2 shows how to identify a witness set on the cycle closed by f_{i+1} after an MST for graph G_i is already verified, our algorithms rely on identifying witness sets involving edges f_{i+1} without first verifying an MST for G_i . The remainder of the section derives properties that allow us to identify such witness sets.

Observation D.3. Let Q be a feasible query set that verifies an MST T^* . Consider any path $P \subseteq T_L$ between two endpoints a and b , and let $e \in P$ be the edge with the highest upper limit in P . If $e \notin Q$, then the path $\hat{P} \subseteq T^*$ from a to b is such that $e \in \hat{P}$ and e has the highest upper limit in \hat{P} after Q has been queried.

Proof. For each $i \in \{0, \dots, l\}$ let T_i^* be the MST for G_i as verified by Q_i and let \hat{C}_i be the unique cycle in $T_{i-1}^* \cup \{f_i\}$. Then $T_i^* = T_{i-1}^* \cup \{f_i\} \setminus \{h_i\}$ holds where h_i is the maximal edge on \hat{C}_i . Assume $e \notin Q$. We claim that there cannot be any \hat{C}_i with $e \in \hat{C}_i$ such that Q_i verifies that an edge $e' \in \hat{C}_i$ with $U_{e'} \leq U_e$ is maximal in \hat{C}_i . Assume otherwise. If $e' \neq e$, Q_i would need to verify that $w_e \leq w_{e'}$ holds. Since $U_{e'} \leq U_e$, this can only be done by querying e , which is a contradiction to $e \notin Q$. If $e' = e$, then \hat{C}_i still contains edge f_i . Since $T_L = T_U$, f_i has a higher lower limit than e . To verify that e is maximal in \hat{C}_i , Q_i needs to prove $w_e \geq w_{f_i} > L_{f_i}$. This can only be done by querying e , which is a contradiction to $e \notin Q$.

We show via induction on $i \in \{0, \dots, l\}$ that each T_i^* contains a path P_i^* from a to b with $e \in P_i^*$ such that e has the highest upper limit in P_i^* after Q_i has been queried. For this proof via induction we define $Q_0 = \emptyset$. *Base case* $i = 0$: Since $G_0 = (V, T_L)$ is a spanning tree, $T_0^* = T_L$ follows. Therefore $P_0^* = P$ is part of T_L and by assumption $e \in P$ has the highest upper limit in P_0^* .

Inductive step: By induction hypothesis, there is a path P_i^* from a to b in T_i^* with $e \in P_i^*$ such that e has the highest upper limit in P_i^* after querying Q_i . Consider cycle \hat{C}_{i+1} . If an edge $e' \in \hat{C}_{i+1} \setminus P_i^*$ is maximal in \hat{C}_{i+1} , then $T_{i+1}^* = T_i^* \cup \{f_{i+1}\} \setminus \{e'\}$ contains path P_i^* . Since e by assumption is not queried, e still has the highest upper limit on $P_i^* = P_{i+1}^*$ after querying Q_{i+1} and the statement follows.

Assume some $e' \in P_i^* \cap \hat{C}_{i+1}$ is maximal in \hat{C}_{i+1} , then $U_{e'} \leq U_e$ follows by induction hypothesis since e has the highest upper limit in P_i^* . We already observed that \hat{C}_{i+1} then cannot contain e . Consider $P' = \hat{C}_{i+1} \setminus P_i^*$. Since $e' \in P_i^*$ is maximal in \hat{C}_{i+1} , we can observe that $P' \subseteq T_{i+1}^*$ holds. It follows that path $P_{i+1}^* = P' \cup (P_i^* \setminus \hat{C}_{i+1})$ with $e \in P_{i+1}^*$ is part of T_{i+1}^* . Since e is not queried, it still has a higher upper limit than all edges in P_i^* . Additionally, we can observe that after querying Q_{i+1} no $u \in P'$ can have an upper limit $U_u \geq U_{e'}$. If such an u would exist, querying Q_{i+1} would not verify that e' is maximal on \hat{C}_{i+1} , which contradicts the assumption. Using $U_e \geq U_{e'}$, we can conclude that e has the highest upper limit on P_{i+1}^* and the statement follows. \square

Using this observation, we derive two lemmas that allow us to identify witness sets of size two. These lemmas are valuable for our algorithms since they allow us to identify witness sets on a cycle C_i independent of what MST T_{i-1} is verified for graph G_{i-1} . While most algorithms for MST under uncertainty iteratively resolve the cycles closed by adding the edges f_1, \dots, f_l (or execute the analogous cut-based algorithm), the following lemmas allow us to query edges using a less local strategy.

Lemma 5.4. Consider cycle C_i with $i \in \{1, \dots, l\}$. Let $l_i \in C_i \setminus \{f_i\}$ such that $I_{l_i} \cap I_{f_i} \neq \emptyset$ and l_i has the largest upper limit in $C_i \setminus \{f_i\}$, then $\{f_i, l_i\}$ is a witness set. Further, if $w_{f_i} \in I_{l_i}$, then $\{l_i\}$ is a witness set.

Proof. To prove the lemma, we have to show that each feasible query set contains at least one element of $\{f_i, l_i\}$. Let Q be an arbitrary feasible query set. By Lemma D.1, $Q_{i-1} := Q \cap E_{i-1}$ is a feasible query set for G_{i-1} and verifies some MST T_{i-1} for G_{i-1} . We show that $l_i \notin Q_{i-1}$ implies either $l_i \in Q$ or $f_i \in Q$.

Assume $l_i \notin Q_{i-1}$ and let C be the unique cycle in $T_{i-1} \cup \{f_i\}$. Since $T_L = T_U$, edge f_i has the highest upper limit in C after querying Q_{i-1} . While we only assume $T_L = T_U$ for the initially given instance, [49] show that this still implies that f_i has the highest upper limit in C . If we show that $l_i \notin Q_{i-1}$ implies $l_i \in C$, we can apply Lemma D.2 to derive that $\{f_i, l_i\}$ is a witness set for graph G_i , and thus either $f_i \in Q_i \subseteq Q$ or $l_i \in Q_i \subseteq Q$. For the remainder of the proof we show that $l_i \notin Q_{i-1}$ implies $l_i \in C$. Let a and b be the endpoints of f_i , then the path $P = C_i \setminus \{f_i\}$ from a to b is part of T_L and l_i has the highest upper limit in P . Using $l_i \notin Q_{i-1}$ we can apply Observation D.3 to conclude that there must be a path \hat{P} from a to b in T_{i-1} such that l_i has the highest upper limit on \hat{P} after querying Q_{i-1} . Therefore, $C = \hat{P} \cup \{f_i\}$ and it follows $l_i \in C$.

If $w_{f_i} \in I_{l_i}$ and $l_i \notin Q_{i-1}$, then l_i must be queried to identify the maximal edge on C , thus it follows that $\{l_i\}$ is a witness set. \square

Lemma 5.5. Let $l_i \in C_i \setminus \{f_i\}$ with $I_{l_i} \cap I_{f_i} \neq \emptyset$ such that $l_i \notin C_j$ for all $j < i$, then $\{l_i, f_i\}$ is a witness set. Furthermore, if $w_{l_i} \in I_{f_i}$, then $\{f_i\}$ is a witness set.

Proof. Consider the set of edges X_i in the cut of G defined by the two connected components of $T_L \setminus \{l_i\}$. By assumption, $l_i, f_i \in X_i$. However, $f_j \notin X_i$ for all $j < i$, otherwise $l_i \in C_j$ for an $f_j \in X_i$ with $j < i$ would follow and contradict the assumption. We can observe $X_i \cap E_i = \{f_i, l_i\}$.

Let Q be any feasible query set. According to Lemma D.1, $Q_{i-1} = E_{i-1} \cap Q$ verifies an MST T_{i-1} for G_{i-1} . Consider the unique cycle C in $T_{i-1} \cup \{f_i\}$. As observed in [49], f_i has the highest upper limit on C after querying Q_{i-1} . Since $f_i \in X_i$, $f_i \in C$ and C is a cycle, it follows that another edge in $X_i \setminus \{f_i\}$ must be part of C . We already observed, $X_i \cap E_i = \{l_i, f_i\}$, and therefore $l_i \in C$. Lemma D.2 implies that $\{f_i, l_i\}$ is a witness set. If $w_{l_i} \in I_{f_i}$, then f_i must be queried to identify the maximal edge in C , so it follows that $\{f_i\}$ is a witness set. \square

D.2 Proofs of Subsection 5.2 (Phase 1)

In this section, we proof the lemmas of Subsection 5.2. Recall that C_i is the non-prediction mandatory free cycle with the smallest index such that all C_j with $j < i$ are prediction mandatory free and that l_i is the edge with the highest upper limit in $C_i \setminus \{f_i\}$.

Lemma 5.7. If $\bar{w}_{f_i} \in I_{l_i}$ and $\bar{w}_{l_i} \in I_{f_i}$, then querying $\{f_i, l_i\}$ satisfies the three statements.

Proof. To show that querying $\{f_i, l_i\}$ satisfies the two statements of Section 5.2, we show that either $\{f_i, l_i\} \subseteq Q$ for each feasible query set Q or $h_{f_i} + h_{l_i} \geq 1$ for the hop distance $h_{f_i} + h_{l_i}$ of f_i and l_i .

By assumption, all C_j with $j < i$ are prediction mandatory free. We claim that this implies $l_i \notin C_j$ for all $j < i$. Assume, for the sake of contradiction, that there is a C_j with $j < i$ and $l_i \in C_j$. Then, $T_L = T_U$ and $i < j$ imply that f_i and f_j have larger upper and lower limits than l_i and, since $L_{f_i} \geq L_{f_j}$, it follows

$I_{l_i} \cap I_{f_i} \subseteq I_{l_i} \cap I_{f_j}$. Thus, $\bar{w}_{l_i} \in I_{f_i}$ implies $\bar{w}_{l_i} \in I_{f_j}$, which contradicts C_j being prediction mandatory free. According to Lemma 5.5, $\{f_i, l_i\}$ is a witness set.

Consider any feasible query set Q , then Q_{i-1} verifies the MST T_{i-1} for graph G_{i-1} and Q needs to identify the maximal edge on the unique cycle C in $T_{i-1} \cup \{f_i\}$. Following the argumentation of Lemma 5.5, we can observe $l_i, f_i \in C$. Since we assume $T_L = T_U$, we can also observe that f_i has the highest upper limit in C . By Observation D.3 l_i has the highest upper limit in $C \setminus \{f_i\}$ after querying $Q_{i-1} \setminus \{l_i\}$.

If $w_{l_i} \in I_{f_i}$, then f_i is part of any feasible query set according to Lemma 5.5. Otherwise, $w_{l_i} \leq L_{f_i} < \bar{w}_{l_i}$ and $h_{l_i} \geq 1$. If $w_{f_i} \in I_{l_i}$, then l_i is part of any feasible query set according to Lemma 5.4. Otherwise, $w_{f_i} \geq U_{l_i} > \bar{w}_{f_i}$ and $h_{f_i} \geq 1$. In conclusion, either $\{f_i, l_i\} \subseteq Q$ for any feasible query set Q or $h_{f_i} + h_{l_i} \geq 1$ \square

Lemma 5.8. Assume $\bar{w}_{f_i} \in I_{l_i}$ but $\bar{w}_{l_i} \notin I_{f_i}$. Let l'_i be the edge with the highest upper limit in $C_i \setminus \{f_i, l_i\}$ and $I_{l'_i} \cap I_{f_i} \neq \emptyset$. If no l'_i exists, then querying l_i , and querying f_i only if $w_{l_i} \in I_{f_i}$, satisfies the three statements. If l'_i exists, then querying $\{f_i, l_i\}$, and querying l'_i only if $w_{f_i} \in I_{l_i}$ and $w_{l_i} \notin I_{f_j}$ for each j with $l_i \in C_j$, satisfies the three statements.

Proof. By assumption, all C_j with $j < i$ are prediction mandatory free. According to Lemma 5.4, $\{f_i, l_i\}$ is a witness set. Assume that the edge l'_i exists. To show that the query strategy for this case satisfies the statements of Subsection 5.2, we show that either $|\{f_i, l_i, l'_i\} \cap Q| \geq 2$ for any feasible query set Q (in case $w_{f_i} \in I_{l_i}$ and $w_{l_i} \notin I_{f_j}$ for each j with $l_i \in C_j$) or $h_{f_i} + h_{l_i} \geq 1$.

Assume that either $w_{f_i} \notin I_{l_i}$ or $w_{l_i} \in I_{f_j}$ for some j with $l_i \in C_j$. If $w_{f_i} \notin I_{l_i}$, then $w_{f_i} \geq U_{l_i} > \bar{w}_{f_i}$ and $h_{f_i} \geq 1$ follows. If $w_{l_i} \in I_{f_j}$, then $\bar{w}_{l_i} \leq L_{f_j} < w_{l_i}$ and $h_{l_i} \geq 1$ follows. Therefore, if either $w_{f_i} \notin I_{l_i}$ or $w_{l_i} \in I_{f_j}$ for some j with $l_i \in C_j$, then $h_{l_i} + h_{f_i} \geq 1$ follows. Now assume that $w_{f_i} \in I_{l_i}$ and $w_{l_i} \notin I_{f_j}$ for each j with $l_i \in C_j$. According to Lemma 5.4, $w_{f_i} \in I_{l_i}$ implies that l_i is part of any feasible query set. Consider the relaxed instance where l_i is already queried, then $w_{l_i} \notin I_{f_j}$ for each j with $l_i \in C_j$ implies that l_i is minimal in X_{l_i} and that the lower limit tree does not change by querying l_i . It follows that l'_i is the edge with the highest upper limit in $C_i \setminus \{f_i\}$ in the relaxed instance and, by Lemma 5.4, $\{f_i, l'_i\}$ is a witness set. In conclusion, either $|\{f_i, l_i, l'_i\} \cap Q| \geq 2$ for any feasible query set Q or $h_{l_i} + h_{f_i} \geq 1$.

Finally, assume that l'_i does not exist. To show that the query strategy for this case satisfies the statements of Subsection 5.2, we show that we either can guarantee that the algorithm will not query f_i and querying $e = l_i$ satisfies the third statement for $f(e) = f_i$, or $h_{l_i} + h_{f_i} \geq 1$ and querying $\{l_i, f_i\}$ satisfies the second statement. If $w_{l_i} \in I_{f_i}$, then $\bar{w}_{l_i} \leq L_{f_i} < w_{l_i}$ and $h_{l_i} \geq 1$. Assume otherwise. The non-existence of l'_i implies that l_i is the only element of $C_i \setminus \{f_i\}$ with an interval that intersects I_{f_i} . Therefore $w_{l_i} \notin I_{f_i}$ implies that f_i is uniquely maximal on C_i and not part of any MST. It follows that f_i can, without loss of generality, be deleted. This guarantees that the algorithm will not query, or even consider, f_i afterwards. \square

Lemma 5.9. Assume $\bar{w}_{l'_i} \in I_{f_i}$ for some $l'_i \in C_i \setminus \{f_i\}$ but $\bar{w}_{f_i} \notin I_{l_i}$. Let f_j be the edge with the smallest lower limit in $X_{l'_i} \setminus \{l'_i, f_i\}$ and $I_{f_j} \cap I_{l'_i} \neq \emptyset$. If f_j does not exist, then querying f_i , and querying l'_i only if $w_{f_i} \in I_{l'_i}$, satisfies the three statements. If f_j exists, querying $\{f_i, l'_i\}$, and also querying f_j only if $w_{l'_i} \in I_{f_i}$ and $w_{f_i} \notin I_e$ for each $e \in C_i$, satisfies the three statements.

Proof. By assumption, all C_j with $j < i$ are prediction mandatory free. We claim that this implies $l'_i \notin C_j$ for all $j < i$. Assume, for the sake of contradiction, that there is a C_j with $j < i$ and $l'_i \in C_j$. Then, $T_L = T_U$ and $i < j$ imply that f_i and f_j have larger upper and lower limits than l'_i and, since $L_{f_i} \geq L_{f_j}$, it follows $I_{l'_i} \cap I_{f_i} \subseteq I_{l'_i} \cap I_{f_j}$. Thus, $\bar{w}_{l'_i} \in I_{f_i}$ implies $\bar{w}_{l'_i} \in I_{f_j}$, which contradicts C_j being prediction mandatory free. According to Lemma 5.5, $\{f_i, l'_i\}$ is a witness set.

Assume that the edge f_j exists. To show that the query strategy for this case satisfies the statements of Subsection 5.2, we show that either $|\{f_i, f_j, l'_i\} \cap Q| \geq 2$ for any feasible query set Q (in case $\bar{w}_{l'_i} \in I_{f_i}$ and $w_{f_i} \notin I_e$ for each $e \in C_i$) or $h_{f_i} + h_{l'_i} \geq 1$.

Assume that either $w_{l'_i} \notin I_{f_i}$ or $w_{f_i} \in I_e$ for some $e \in C_i$. If $w_{l'_i} \notin I_{f_i}$, then $w_{l'_i} \leq L_{f_i} < \bar{w}_{l'_i}$ and $h_{l'_i} \geq 1$ follows. If $w_{f_i} \in I_e$, then $\bar{w}_{f_i} \geq U_e > w_{f_i}$ and $h_{f_i} \geq 1$ follows. Therefore $w_{l'_i} \notin I_{f_i}$ or $w_{f_i} \in I_e$ for some $e \in C_i$ implies $h_{l'_i} + h_{f_i} \geq 1$.

Now assume that $w_{l'_i} \in I_{f_i}$ and $w_{f_i} \notin I_e$ for each $e \in C_i$. According to Lemma 5.5, $w_{l'_i} \in I_{f_i}$ implies that f_i is part of any feasible query set. Consider the relaxed instance where f_i is already queried, then $w_{f_i} \notin I_e$ for each $e \in C_i$ implies that f_i is maximal in C_i and that the lower limit tree does not change by querying f_i . It follows that f_j is the edge with the smallest index and $l'_i \in C_j$ in the relaxed instance and, by Lemma 5.5, $\{f_j, l'_i\}$ is a witness set. In conclusion, either $|\{f_i, f_j, l'_i\} \cap Q| \geq 2$ for any feasible query set Q or $h_{l'_i} + h_{f_i} \geq 1$.

Finally, assume that f_j does not exist. To show that the query strategy for this case satisfies the statements of Subsection 5.2, we show that we either can guarantee that the algorithm will not query l'_i and querying $e = f_i$ satisfies the third statement for $f(e) = l'_i$, or $h_{l'_i} + h_{f_i} \geq 1$ and querying $\{l'_i, f_i\}$ satisfies the second statement. If $w_{f_i} \in I_{l'_i}$, then $\bar{w}_{f_i} \geq U_{l'_i} > w_{f_i}$ and $h_{f_i} \geq 1$. Assume otherwise. The non-existence of f_j implies that f_i is the only element of $X_{l'_i} \setminus \{f_i\}$ with an interval that intersects $I_{l'_i}$. Therefore $w_{f_i} \notin I_{l'_i}$ implies that l'_i is uniquely minimal in $X_{l'_i}$ and part of every MST. It follows that l'_i can, without loss of generality, be contracted. This guarantees that the algorithm will not query, or even consider, l'_i afterwards. \square

Lemma 5.3. An instance G is prediction mandatory free if and only if $\bar{w}_{f_i} \geq U_e$ and $\bar{w}_e \leq L_{f_i}$ holds for each $e \in C_i \setminus \{f_i\}$ and each cycle C_i with $i \in \{1, \dots, l\}$.

Proof. Assume $\bar{w}_{f_i} \geq U_e$ and $\bar{w}_e \leq L_{f_i}$ holds for each $e \in C_i \setminus \{f_i\}$ and each cycle C_i with $i \in \{1, \dots, l\}$. Then each $f_i \in E \setminus T_L$ is predicted to be maximal on C_i and each $e \in T_L$ is predicted to be minimal in X_e . Assuming the predictions are correct, we can observe that each vertex cover of bipartite graph \bar{G} is a feasible query set [21]. Define $\bar{G} = (\bar{V}, \bar{E})$ with $\bar{V} = E$ (excluding trivial edges) and $\bar{E} = \{\{f_i, e\} \mid i \in \{1, \dots, l\}, e \in C_i \setminus \{f_i\} \text{ and } I_e \cap I_{f_i} \neq \emptyset\}$. Since both $Q_1 := T_L$ and $Q_2 := E \setminus T_L$ are vertex covers for \bar{G} , Q_1 and Q_2 are feasible query sets under the assumption that the predictions are correct. This implies that no element is part of every feasible solution because $Q_1 \cap Q_2 = \emptyset$. We can conclude that no element is prediction mandatory and the instance is prediction mandatory free.

For the other direction we show the contraposition. Assume there is a cycle C_i such that $\bar{w}_{f_i} \in I_e$ or $\bar{w}_e \in I_{f_i}$ for some $e \in C_i \setminus \{e\}$. Let C_i be such a cycle with the smallest index. If $\bar{w}_{f_i} \in I_e$ for some $e \in C_i \setminus \{e\}$, then also $\bar{w}_{f_i} \in I_{l_i}$ for the edge l_i with the highest upper limit in $C_i \setminus \{f_i\}$. (This is because we assume $T_L = T_U$.) Under the assumption that the predictions are true, Lemma 5.4 implies that l_i is mandatory and thus prediction mandatory. It follows that G is not prediction mandatory free.

Assume $\bar{w}_e \in I_{f_i}$. We can conclude $e \notin C_j$ for each $j < i$. This is because $\bar{w}_e \in I_{f_i}$ and $j < i$ would imply $\bar{w}_e \in I_{f_j}$. As we assumed that C_i is the first cycle with this property, $e \in C_j$ leads to a contradiction. Under the assumption that the predictions are true, Lemma 5.5 implies that f_i is mandatory and thus prediction mandatory. It follows that G is not prediction mandatory free. \square

Lemma 5.6. After executing Algorithm 3 the instance is prediction mandatory free and, ignoring the last iteration of Line 1, $|\text{ALG}| \leq \min\{(1 + \frac{1}{\gamma}) \cdot (|\text{ALG} \cap \text{OPT}| + k_h), \gamma \cdot |\text{ALG} \cap \text{OPT}|\}$ holds for the set of edges ALG queried by Algorithm 3 and any optimal solution OPT.

Proof. Since Algorithm 3 only terminates if Line 4 determines each C_i to be prediction mandatory free, the instance after executing the algorithm is prediction mandatory free by definition and Lemma 5.3. All elements queried in Line 1 to ensure unique $T_L = T_U$ are mandatory by Lemma 5.2 and never decrease the performance guarantee.

We now consider the remaining queries. Since the last iteration is ignored, each iteration i queries a set P_i of $\gamma - 2$ prediction mandatory elements in Line 1 and a set W_i in Line 4. According to Theorem 2.3 each element of P_i is either mandatory or contributes one to the hop distance. To be more precise, let $h'(e)$ with

$e \in E$ be the number of edges e' such that the value of e' passes over an endpoint of I_e . From the arguments in the proof of Theorem 2.3, it can be seen that, for each edge e that is prediction mandatory at some point but not mandatory, we have that $h'(e) \geq 1$. For a subset $U \subseteq E$, let $h'(U) = \sum_{e \in U} h'(e)$. Note that $k_h = h'(E)$ holds by reordering summations.

Lemmas 5.7, 5.8 and 5.9 imply that the three statements of Section 5.2 hold for set W_i . Further, the proofs of the three lemmas imply that the second statement still holds if we slightly rephrase it and state the error-dependent guarantee in terms of h' :

2. If the algorithm queries a witness set $W_i = \{e_1, e_2\}$ of size two, then either $W_i \subseteq Q$ for each feasible query set Q or the hop distances of e_1 and e_2 satisfy $h'(e_1) + h'(e_2) \geq 1$.

Consider first the set of elements E that are queried because they fulfill the third statement of Subsection 5.2. We assume without loss of generality that $E \subseteq \text{OPT}$ and treat each $e \in E$ as a witness set of size one. We can do this without loss of generality since if $e \notin \text{OPT}$, we know $f(e) \in \text{OPT}$ for a distinct $f(e)$ and can charge e against $f(e)$ instead. All other sets $Q_i := P_i \cup W_i$ are compared only against $\text{OPT} \cap Q_i$ and therefore we can guarantee that $f(e)$ is not used to charge for another element. Since the proofs of the Lemmas 5.8 and 5.9 show that each $f(e)$ can be either deleted (because it is uniquely maximal on a cycle) or contracted (because it is uniquely minimal in a cut), no $f(e)$ will be relevant for the second phase.

Define $Q_i := P_i \cup W_i$. We show that $|Q_i| \leq \min\{(1 + \frac{1}{\gamma}) \cdot (|Q_i \cap \text{OPT}| + h'(Q_i)), \gamma \cdot |Q_i \cap \text{OPT}|\}$ for each Q_i . As all Q_i 's are disjoint, this implies the lemma.

If the set W_i in Line 4 contains one element, then we can assume that that element is part of any feasible solution. It follows that the set Q_i is a witness set of size $\gamma - 1$ and therefore does not violate the robustness. Additionally, at least $|Q_i| - h'(P_i)$ elements of Q_i are part of any feasible query set Q . This implies $|Q_i| \leq \min\{|Q_i \cap \text{OPT}| + h'(Q_i), \gamma \cdot |Q_i \cap \text{OPT}|\}$.

If the set W_i of Line 4 contains two elements, then either $W_i = \{e_1, e_2\} \subseteq Q$ for any feasible query set Q or $h'(e_1) + h'(e_2) \geq 1$. It follows that at least $|Q_i| - h'(Q_i)$ elements of Q_i are part of any feasible query set Q . This implies $|Q_i| \leq (|Q_i \cap \text{OPT}| + h'(Q_i))$. Additionally, Q_i is a witness set of size at most γ , which implies $|Q_i| \leq \gamma \cdot |Q_i \cap \text{OPT}|$.

If the set W_i in Line 4 contains three elements, then $|W_i \cap Q| \geq 2$ for any feasible query set Q . It follows that at least $\frac{\gamma}{\gamma+1} \cdot |Q_i| - h'(P_i)$ elements of Q_i are part of any feasible query set Q . Additionally, Q_i is a set of size $\gamma + 1$ and at least two elements of Q_i are part of any feasible solution. It follows $|Q_i| \leq \min\{(1 + \frac{1}{\gamma}) \cdot (|Q_i \cap \text{OPT}| + h'(Q_i)), \gamma \cdot |Q_i \cap \text{OPT}|\}$.

Since $|Q_i| \leq \min\{(1 + \frac{1}{\gamma}) \cdot (|Q_i \cap \text{OPT}| + h'(Q_i)), \gamma \cdot |Q_i \cap \text{OPT}|\}$ holds for each Q_i and all Q_i are disjoint, the lemma follows. \square

D.3 Proofs of Subsection 5.3 (Phase 2)

Recall that the vertex cover instance \bar{G} for an MST under uncertainty instance $G = (V, E)$ is defined as $\bar{G} = (\bar{V}, \bar{E})$ with $\bar{V} = E$ (excluding trivial edges) and $\bar{E} = \{\{f_i, e\} \mid i \in \{1, \dots, l\}, e \in C_i \setminus \{f_i\} \text{ and } I_e \cap I_{f_i} \neq \emptyset\}$ [21, 49]. Recall that VC is a minimum vertex cover of \bar{G} and that h is a maximum matching of \bar{G} that matches each $e \in VC$ to a distinct $h(e) \notin VC$. By definition, \bar{G} is a bipartite graph.

Lemma 5.11. Let f'_1, \dots, f'_g be the edges in $VC \setminus T_L$ ordered by lower limit non-decreasingly and let l'_1, \dots, l'_k be the edges in $VC \cap T_L$ ordered by upper limit non-increasingly. Let b be such that each f'_i with $i < b$ is maximal in cycle $C_{f'_i}$, then $\{f'_i, h(f'_i)\}$ is a witness set for each $i \leq b$. Let d be such that each l'_i with $i < d$ is minimal in cut $X_{l'_i}$, then $\{l'_i, h(l'_i)\}$ is a witness set for each $i \leq d$.

Proof. We start by showing the first part of the lemma, i.e., if b is such that each f'_i with $i < b$ is maximal in cycle $C_{f'_i}$, then $\{f'_i, h(f'_i)\}$ is a witness set for each $i \leq b$.

Consider an arbitrary f'_i and $h(f'_i)$ with $i \leq b$. By definition of h , the edge $h(f'_i)$ is part of the lower limit tree. Let X_i be the cut between the two components of $T_L \setminus \{h(f'_i)\}$, then we claim that X_i only

contains $h(f'_i)$ and edges in $\{f'_1, \dots, f'_g\}$ (and possibly irrelevant edges that do not intersect $I_{h(f'_i)}$). To see this, assume an $f_j \in \{f_1, \dots, f_l\}$ with $f_j \notin VC$ was part of X_i . Since $f_j \notin VC$, each edge in $C_j \setminus \{f_j\}$ must be part of VC as otherwise VC would not be a vertex cover. But if f_j is in cut X_i , then C_j must contain $h(f'_i)$. By definition, $h(f'_i) \notin VC$ holds which is a contradiction. We can conclude that X_i only contains $h(f'_i)$ and edges in $\{f'_1, \dots, f'_g\}$.

Now consider any feasible query set Q , then $Q_{i'-1}$ verifies an MST $T_{i'-1}$ for graph $G_{i'-1}$ where i' is the index of f'_i in the order f_1, \dots, f_l . Consider again the cut X_i . Since each $\{f'_1, \dots, f'_{i'-1}\}$ is maximal in a cycle by assumption, $h(f'_i)$ is the only edge in the cut that can be part of $T_{i'-1}$. Since one edge in the cut must be part of the MST, we can conclude that $h(f'_i) \in T_{i'-1}$. Finally, consider the cycle C in $T_{i'-1} \cup \{f'_i\}$. We can use that f'_i and $h(f'_i)$ are the only elements in $X_i \cap (T_{i'-1} \cup \{f'_i\})$ to conclude that $h(f'_i) \in C$ holds. According to Lemma D.2, $\{f'_i, h(f'_i)\}$ is a witness set.

We continue by showing the second part of the lemma, i.e., if d is such that each l'_i with $i < d$ is minimal in cut $X_{l'_i}$, then $\{l'_i, h(l'_i)\}$ is a witness set for each $i \leq d$.

Consider an arbitrary l'_i and $h(l'_i)$ with $i \leq d$. By definition of h , the edge $h(l'_i)$ is not part of the lower limit tree. Consider $C_{h(l'_i)}$, i.e., the cycle in $T_L \cup \{h(l'_i)\}$, then we claim that $C_{h(l'_i)}$ only contains $h(l'_i)$ and edges in $\{l'_1, \dots, l'_k\}$ (and possibly irrelevant edges that do not intersect $I_{h(l'_i)}$). To see this, recall that $l'_i \in VC$, by definition of h , implies $h(l'_i) \notin VC$. For VC to be a vertex cover, each $e \in C_{h(l'_i)} \setminus \{h(l'_i)\}$ must either be in VC or not intersect $h(l'_i)$. Consider the relaxed instance where the true values for each l'_j with $j < d$ and $j \neq i$ are already known. By assumption each such l'_j is minimal in its cut $X_{l'_j}$. Thus, we can w.l.o.g contract each such edge. It follows that in the relaxed instance l'_i has the highest upper limit in $C_{h(l'_i)} \setminus \{h(l'_i)\}$. According to Lemma 5.4, $\{l'_i, h(l'_i)\}$ is a witness set. \square

Recall that the vertex cover instance \bar{G} for an MST under uncertainty instance $G = (V, E)$ is defined as $\bar{G} = (\bar{V}, \bar{E})$ with $\bar{V} = E$ (excluding trivial edges) and $\bar{E} = \{\{f_i, e\} \mid i \in \{1, \dots, l\}, e \in C_i \setminus \{f_i\} \text{ and } I_e \cap I_{f_i} \neq \emptyset\}$ [21, 49]. Recall that VC is a minimum vertex cover of \bar{G} and that h is a maximum matching of \bar{G} that matches each $e \in VC$ to a distinct $h(e) \notin VC$. By definition, \bar{G} is a bipartite graph. Note that if a query reveals that an f_i is not maximal in C_i or an l_i is not minimal in X_{l_i} , then the vertex cover instance changes. The following lemma considers the situation where the vertex cover instance in Line 6 of Algorithm 4 has changed in comparison to the initial vertex cover instance of the iteration. Thus, the algorithm restarts. Let \bar{G} be the initial vertex cover instance and let \bar{G}' be the changed vertex cover instance that the algorithm considers after the restart.

Lemma 5.12. Let $\bar{G}' = (\bar{V}', \bar{E}')$ be the changed vertex cover instance of Line 6, then $\bar{h} = \{\{e, e'\} \in h \mid \{e, e'\} \in \bar{E}'\}$ defines a partial matching for \bar{G}' . Let h' be the maximum matching for \bar{G}' computed by completing \bar{h} using a standard augmenting path algorithm [2] and let VC' be the vertex cover defined by h' . Then restarting Algorithm 4 with $VC = VC'$ and $h = h'$ implies that Line 5 queries at most $2 \cdot k_h$ times.

Proof. The goal of this proof is to show that the number of times Algorithm 4 executes the Otherwise-part of Line 5 is limited by $2 \cdot k_h$ if the algorithm uses recovery strategy B and restarts with the vertex cover and matching as described in the lemma.

We can observe that the otherwise-part of Line 5 is only executed if the previous line queried an element e' such that $\{e', h(e')\} \cap W \neq \emptyset$. This can only happen if either $h(e')$ or e' was added to set W in a previous restart, i.e., $h(e') = h(e)$ or $e' = h(e)$ for an element e that was queried before e' . Thus, we can show that the number of times the Otherwise-part of Line 5 is executed is limited, by showing that the number of elements $h(e)$ that are re-matched after being matched to a first partner e is limited.

Now consider an $h(e)$ that is added to W in Line 5 after its first partner e gets queried in Line 4, then $h(e)$ can only get re-matched to another element e' if the algorithm restarts afterwards. If the algorithm does not restart, then it continues with the matching that already matched $h(e)$ to its original partner e and therefore $h(e)$ will not be matched to another element. Consider the first restart after e was queried and let \bar{G} be the

vertex cover instance before the restart, let h be the maximum matching for \bar{G} and let \bar{G}' be the vertex cover instance after the restart. We can observe that $\{e, h(e)\}$ is not an edge in vertex cover instance \bar{G}' because e became trivial and \bar{G}' by definition does not contain trivial elements. It follows that $h(e)$ is not matched by the partial matching $\bar{h} = \{\{e, e'\} \in h \mid \{e, e'\} \in \bar{E}'\}$. We can conclude that each $h(e) \in W$ that is matched a second time, was not matched by the partial matching \bar{h} in the restart after e was queried. The only way for $h(e)$ to be matched a second time is if it is added to the matching while completing \bar{h} using the standard augmenting path algorithm. Note that this does not need to happen in the restart directly after e was queried, but since $h(e)$ at some point left the matching it can only be matched a second time if it is re-introduced to the matching when executing the augmenting path algorithm in a restart, i.e., it was not matched before executing the augmenting path algorithm but is matched afterwards. Thus, if we show that the total number of elements that were not matched before an execution of the augmenting path algorithm but are matched after the execution is bounded by $2 \cdot k_h$, then the lemma follows.

Consider any restart i . Define H_i to be the set of elements that were queried since the last restart before i . We will show in the subsequent Lemma D.5 that the number of elements that are not matched by \bar{h} but become matched after executing the augmenting path algorithm in restart i is bounded by $2 \cdot \sum_{e \in H_i} h_e$ where h_e is the hop distance of element e .

We can observe that all sets H_i are pairwise disjoint because no element is queried multiple times. Let d be the total number of restarts. Since the sets H_i with $i \in \{1, \dots, d\}$ are disjoint, it follows

$$\sum_{i \in \{1, \dots, d\}} 2 \cdot \sum_{e \in H_i} h_e \leq 2 \cdot k_h.$$

Therefore Lemma D.5 implies that the total number of elements that were not matched before executing the augmenting path but were matched afterwards is, over all restarts, bounded by $2 \cdot k_h$. This implies the lemma. \square

In order to show Lemma D.5, we first show another auxiliary lemma.

Lemma D.4. Let $G = (V, E)$ be an instance with unique T_L and T_U such that $T_L = T_U$. Let $G' = (V', E')$ be an instance obtained from G by executing a query set Q with unique T'_L and T'_U such that $T'_L = T'_U$, where T'_L and T'_U are the lower and upper limit tree of G' . Then $e \in T_L \setminus T'_L$ implies $e \in Q$, and $e \in T'_L \setminus T_L$ implies $e \in Q$.

Proof. Let $e \in T_L \setminus T'_L$, then $e \in T_L$ and T_L being unique imply that e has the unique minimal lower limit in the cut X_e of G between the two connected components of $T_L \setminus \{e\}$. Thus, e is part of any lower limit tree for G . For e to be not part of T'_L , it cannot have the unique minimal lower limit in the cut X_e of G' . Since querying elements in $X_e \setminus \{e\}$ only increases their lower limits, this can only happen if $e \in Q$.

Let $e \in T'_L \setminus T_L$. Then $T_L = T_U$ and $T'_L = T'_U$ imply $e \in T'_U \setminus T_U$. Since $e \notin T_U$ and T_U is unique, it follows that e has the unique largest upper limit in the cycle C_e of $T_U \cup \{e\}$. Thus, e is not part of any upper limit tree for G . For e to be part of T'_U , it cannot have the unique largest upper limit in the cycle C_e of G' . Since querying elements in $C_e \setminus \{e\}$ only decreases their upper limits, this can only happen if $e \in Q$. \square

Lemma D.5. Let \bar{G} and \bar{G}' be the vertex cover instances before and after a restart, let H be the set of elements queried since the previous restart, and let h be the maximum matching for \bar{G} . Then the number of elements that are not matched by $\bar{h} = \{\{e, e'\} \in h \mid \{e, e'\} \in \bar{E}'\}$ but become matched after completing \bar{h} using the augmenting path algorithm is bounded by $2 \cdot \sum_{e \in H} h_e$.

Proof. Let h' be the maximum matching constructed from \bar{h} using the augmenting path algorithm. In each iteration, the augmenting path algorithm increases the size of the matching by one and matches at most two elements that were not matched before. If we can show that the number of iterations of the augmenting

path algorithm is bounded by $\sum_{e \in H} h_e$, it follows that at most $2 \cdot \sum_{e \in H} h_e$ elements that are not matched by \bar{h} are matched by h' , which implies the lemma.

According to König's Theorem, see, e.g., [13], the size of h' is upper bounded by the size of the minimum vertex cover for \bar{G}' . Since the augmenting path algorithm increases the size of the matching by one in each iteration, the number of iterations is bounded by $|VC'| - |\bar{h}|$, where $|VC'|$ is the size of a minimum vertex cover VC' for \bar{G}' and $|\bar{h}|$ is the size of the matching \bar{h} , i.e., the number of edges of \bar{G}' in \bar{h} . We show that the size of the minimum vertex cover is at most $|\bar{h}| + \sum_{e \in H} h_e$ and the statement follows.

In order to do so, we define $\overline{VC} = \{e \in VC \mid \exists e' \text{ s.t. } \{e, e'\} \in \bar{h}\}$ for \bar{G} where VC is the vertex cover for \bar{G} as defined by h . Note that $|\overline{VC}| = |\bar{h}|$. We show that we can construct a vertex cover for \bar{G}' by adding at most $\sum_{e \in H} h_e$ elements to \overline{VC} , which implies $|VC'| \leq |\bar{h}| + \sum_{e \in H} h_e$ for the minimum vertex cover VC' of \bar{G}' .

We argue how to extend \overline{VC} such that it covers each edge in \bar{G}' by adding at most $\sum_{e \in H} h_e$ elements. We sequentially consider edges $\{f, e\} \in \bar{E}'$ that are not covered by \overline{VC} and add an element of $\{f, e\}$ to \overline{VC} in order to cover $\{f, e\}$. In the process we argue that we can charge each added element to a distinct error. We will consider pairs of elements $\{e, e'\}$ such that the value of e' passes over a boundary of e . In that case we say that e contributes to the hop distance $h_{e'}$.

Consider any edge of \bar{G}' that is not covered by the current \overline{VC} . Either that edge is part of $\bar{E}' \setminus \bar{E}$ or is part of $\bar{E} \cap \bar{E}'$ but both endpoints of the edge are not part of \overline{VC} . Let $T'_L = T'_U$ be the unique upper and lower limit tree after the restart, i.e., the vertex cover instance \bar{G}' is based on T'_L .

Case 1: Consider an $\{f, e\} \in \bar{E}' \setminus \bar{E}$ that is not covered by the current \overline{VC} with $e \in T'_L$ and $f \notin T'_L$. By definition of the vertex cover instance, $\{f, e\} \in \bar{E}'$ and $f \notin T'_L$ imply $e \in C'_f$ such that $I_f \cap I_e \neq \emptyset$, where C'_f is the unique cycle in $T'_L \cup \{f\}$.

By definition, \bar{G}' only contains non-trivial elements and thus e and f are non-trivial. According to Lemma D.4, e being non-queried and $e \in T'_L$ imply $e \in T_L = T_U$. Similar, f being non-queried and $f \notin T'_L$ imply $f \notin T_L = T_U$.

Define X'_e to be the set of edges in the cut of G' between the two connected components of $T'_L \setminus \{e\}$. Remember that $f \in X'_e$ since $f \notin T'_L$ and $e \in C'_f$. Since $\{f, e\} \notin \bar{E}$ implies $e \notin C_f$, there must be an element $l \in T_L \cap (X'_e \setminus \{e, f\})$ such that $l \in C_f$, where C_f is the unique cycle in $T_L \cup \{f\}$. As the instance is prediction mandatory free, $l \in C_f$ implies $\bar{w}_l \notin I_f$.

Further, $X'_e \cap T'_L = \{e\}$ holds by definition and implies $l \notin T'_L$. According to Lemma D.4, $l \in T_L \setminus T'_L$ implies that l must have been queried after the previous restart. If $w_l \in I_e$, this implies that l has the unique smallest upper limit in cut X'_e and is therefore part of T'_U . This would imply $T'_L \neq T'_U$ which is a contradiction. If $w_l \notin I_e$, then also $w_l \geq U_e$ ($w_l \leq L_e$ cannot be the case because l would have the smallest lower limit in X'_e which would contradict $l \notin T'_L$). As $I_e \cap I_f \neq \emptyset$, we can conclude that $w_l \geq U_e$ implies $w_l > L_f$.

It follows $\bar{w}_l \leq L_f < w_l$ and l passes over L_f . Therefore f contributes to the hop distance h_l . We already argued that l was queried and thus $l \in H$ and we can afford to add f to the vertex cover in order to cover edge $\{f, e\}$. Afterwards, all edges incident to f are covered and we therefore do not need to consider any of those edges again. Thus, the error contributed by f to h_l will not be counted multiple times in Case 1. Further, $l \in T_L$ holds for the element l that passes over an interval boundary and $f \notin T_L$ holds for the non-trivial element f that is passed over. This combination of properties for the pair of elements $\{l, f\}$ that contribute an error is mutually exclusive to the properties of the element pairs considered in the following cases (cf. Table 1). Thus, the error contributed by f to h_l will not be counted in the other cases.

Case 2: Consider an edge $\{f, e\}$ with $f \notin T_L$ and $e \in T_L$ that is part of $\bar{E} \cap \bar{E}'$ but $f, e \notin \bar{VC}$, then $\{f, e\} \in \bar{E}$ implies that either a) $f \in VC$ or b) $e \in VC$.

Case 2a): Assume $f \in VC$, then h matches f to an element $h(f)$ and we can conclude $\{h(f), f\} \notin \bar{E}'$ since otherwise \bar{h} would also match f and $h(f)$ and therefore $f \in \bar{VC}$ would follow by definition of \bar{VC} .

From $\{h(f), f\} \notin \bar{E}'$ it follows that either $h(f) \notin C'_f$, or $h(f) \in C'_f$ and $h(f)$ is trivial after the restart. By definition, $h(f) \in C'_f$ and $h(f)$ being non-trivial would imply $\{f, h(f)\} \in \bar{E}'$, a contradiction.

Assume $h(f) \in C'_f$ and $h(f)$ is trivial. For $h(f)$ to be trivial it must have been queried after the last restart. We can observe that $h(f)$ was not queried in Line 5 of Algorithm 4 as a result of $\{f, h(f)\} \cap W = \emptyset$ because then f would have been queried in Line 4 and f being trivial would contradict $\{e, f\} \in \bar{E}'$.

Since $f \in VC$ implies $h(f) \notin VC$, we can conclude that $h(f)$ was also not queried as part of VC in Line 4 but to ensure unique $T_L = T_U$ using Lemma 5.2. Directly after the previous restart, $h(f)$ had the unique minimum upper and lower limit in the cut $X_{h(f)}$ of G between the two connected components of $T_L \setminus \{h(f)\}$ because $h(f) \in T_L = T_U$ and T_L, T_U are unique. Since $h(f)$ was queried to ensure $T_L = T_U$, it must, at some point before it was queried, have been part of the current lower limit tree but not of the upper limit tree. Thus, some element f' of $X_{h(f)} \setminus \{h(f)\}$ must have been queried and $w_{f'} \in I_{h(f)}$. (Otherwise $h(f)$ would still have the unique smallest upper limit in $X_{h(f)}$ and therefore be still part of the current upper limit tree.)

Since the input instance is prediction mandatory free, it follows $\bar{w}_{f'} \geq U_{h(f)} > w_{f'}$. Therefore f' passes over $U_{h(f)}$ and $h(f)$ contributes to the hop distance $h_{f'}$. Since f' was queried, $f' \in H$ and we can afford to add f to \bar{VC} in order to cover the edge $\{e, f\}$.

Afterwards, all edges incident to f are covered and we therefore do not need to consider any of those edges again in Case 2a). As $f \in VC$ is the unique partner of $h(f) \notin VC$ and f will not be considered again in Case 2a), the error contributed by $h(f)$ to $h_{f'}$ will not be counted again in Case 2a). For the element $h(f)$ that is passed over, it holds that $h(f) \in T_L$ and $h(f)$ is trivial and, for the element f' that passes over, it holds that $f' \notin T_L$ and f' is trivial. This combination of properties for the pair of elements $\{h(f), f'\}$ that contribute an error is mutually exclusive to the properties of the element pairs considered in the other cases (cf. Table 1). Thus, the error contributed by $h(f)$ to $h_{f'}$ will not be counted in the other cases.

Now assume that $h(f)$ is non-trivial and $h(f) \notin C'_f$. Note that $h(f) \in C_f \cap T_L$, since $f \notin T_L$ and $\{f, h(f)\} \in \bar{E}$. Thus $f \in X_{h(f)}$ and, since $h(f) \notin C'_f$, some edge $f' \in C'_f \cap T'_L$ must be in $X_{h(f)}$. From $f' \in X_{h(f)}$ follows $f' \notin T_L$, since T_L by definition only contains one element of cut $X_{h(f)}$ and that is $h(f)$. Additionally $f' \in X_{h(f)}$ implies $h(f) \in C_{f'}$, and since the instance is prediction mandatory free also $\bar{w}_{f'} \geq U_{h(f)}$. According to Lemma D.4, $f' \in T'_L \setminus T_L$ implies that f' must have been queried after the previous restart and therefore is trivial. If $w_{f'} \geq U_{h(f)}$, then $I_{h(f)} \cap I_{f'} \neq \emptyset$ implies $w_{f'} > L_f$. This contradicts $f' \in T'_L$ because f' would have the unique highest lower limit on C'_f . If $w_{f'} < U_{h(f)}$, then $w_{f'} < U_{h(f)} \leq \bar{w}_{f'}$. Therefore f' passes over $U_{h(f)}$ and $h(f)$ contributes to the hop distance $h_{f'}$. Since f' was queried, we have that $f' \in H$ and we can afford to add f to \bar{VC} in order to cover the edge $\{e, f\}$.

Afterwards, all edges incident to f are covered and we therefore do not need to consider any of those edges again in Case 2a). As $f \in VC$ is the unique partner of $h(f) \notin VC$ and f will not be considered again in Case 2a), the error contributed by $h(f)$ to $h_{f'}$ will not be counted again in Case 2a). For the element $h(f)$ that is passed over, it holds that $h(f) \in T_L \setminus VC$ and $h(f)$ is non-trivial and, for the element f' that passes over, it holds that $f' \notin T_L$ and f' is trivial. This combination of properties for the pair of elements $\{h(f), f'\}$ that contribute an error is mutually exclusive to the properties of the element pairs considered in the other cases (cf. Table 1). Thus, the error contributed by $h(f)$ to $h_{f'}$ will not be counted in the other cases.

Case 2b): Assume $e \in VC$, then h matches e to an element $h(e)$ and we can conclude that $\{h(e), e\} \notin \bar{E}'$, since otherwise \bar{h} would also match e and $h(e)$, and therefore $e \in \bar{VC}$ would follow. Since $\{h(e), e\} \in \bar{E}$, it follows $e \in C_{h(e)}$ and $I_e \cap I_{h(e)} \neq \emptyset$.

From $\{h(e), e\} \notin \bar{E}'$ follows that either $e \notin C'_{h(e)}$, or $e \in C'_{h(e)}$ and $h(e)$ is trivial after the restart. By

definition, $e \in C'_{h(e)}$ and $h(e)$ being non-trivial would imply $\{e, h(e)\} \in \bar{E}'$, a contradiction.

Assume $e \in C'_{h(e)}$ and $h(e)$ is trivial. For $h(e)$ to be trivial it must have been queried after the last restart. We can observe that $h(e)$ was not queried in Line 5 of Algorithm 4 as a result of $\{e, h(e)\} \cap W = \emptyset$ because then e would have been queried in Line 4, and e being trivial would contradict $\{e, f\} \in \bar{E}'$.

Since $e \in VC$ implies $h(e) \notin VC$, we can conclude that $h(e)$ was also not queried as part of VC in Line 4 but to ensure $T_L = T_U$ using Lemma 5.2. Directly after the previous restart, $h(e)$ had the unique maximum upper and lower limit in cycle $C_{h(e)}$ because $h(e) \notin T_L = T_U$ and T_L, T_U are unique. Since $h(e)$ was queried to ensure $T_L = T_U$, it must, at some point before it was queried, have been part of the current lower limit tree but not of the upper limit tree. Thus, some element e' of $C_{h(e)}$ must have been queried and $w_{e'} \in I_{h(e)}$. (Otherwise $h(e)$ would still have the unique largest lower limit in $C_{h(e)}$ and therefore not be part of the current lower limit tree.) Since the input instance is prediction mandatory free, $\bar{w}_{e'} \leq L_{h(e)} < w_{e'}$. Therefore e' passes over $L_{h(e)}$ and $h(e)$ contributes to the hop distance $h_{e'}$. Since e' was queried, $e' \in H$ and we can afford to add e to \bar{VC} in order to cover the edge $\{e, f\}$.

Afterwards, all edges incident to e are covered and we therefore do not need to consider any of those edges again in Case 2b). As $e \in VC$ is the unique partner of $h(e) \notin VC$ and e will not be considered again in Case 2b), the error contributed by $h(e)$ to $h_{e'}$ will not be counted again in Case 2a). For the element $h(e)$ that is passed over, it holds that $h(e) \notin T_L$ and $h(e)$ is trivial and, for the element e' that is passed over, it holds that $e' \in T_L$ and e' is trivial. This combination of properties for the pair of elements $\{h(e), e'\}$ that contribute an error by is mutually exclusive to the properties of the element pairs considered in the other cases (cf. Table 1). Thus, the error contributed by $h(e)$ to $h_{e'}$ will not be counted in the other cases.

Now assume that $e \notin C'_{h(e)}$. Note that $e \in C_{h(e)} \cap T_L$, since $h(e) \notin T_L$ and $\{e, h(e)\} \in \bar{E}$. Thus $h(e) \in X_e$ and, since $e \notin C'_{h(e)}$, some edge $e' \in C'_{h(e)} \cap T'_L$ must be in X_e .

From $e' \in X_e$ it follows that $e' \notin T_L$, since T_L by definition only contains one element of cut X_e and that is e . Additionally $e' \in X_e$ implies $e \in C_{e'}$, and since the instance is prediction mandatory free also $\bar{w}_{e'} \geq U_e$.

By Lemma D.4, $e' \in T'_L \setminus T_L$ implies that e' must have been queried after the previous restart and is trivial. If $w_{e'} \geq U_e$, then $I_{h(e)} \cap I_e \neq \emptyset$ implies $w_{e'} > L_{h(e)}$. This contradicts $e' \in T'_L$ because e' would have the unique highest lower limit on $C'_{h(e)}$. If $w_{e'} < U_e$, then $w_{e'} < U_e \leq \bar{w}_{e'}$. Therefore e' passes over U_e and e contributes to the hop distance $h_{e'}$. Since e' was queried, we have that $e' \in H$ and we can afford to add e to \bar{VC} in order to cover the edge $\{e, f\}$.

Afterwards, all edges incident to e are covered and we therefore do not need to consider any of those edges again in Case 2b). Therefore the error contributed by e to $h_{e'}$ will not be counted again in Case 2b). For the element e that is passed over, it holds that $e \in T_L \cap VC$ and e is non-trivial and, for the element e' that passes over, it holds that $e' \notin T_L$ and e' is trivial. This combination of properties for the pair of elements $\{e, e'\}$ that contribute an error is mutually exclusive to the properties of the element pairs considered in the other cases (cf. Table 1). Thus, the error contributed by e to e' will not be counted in the other cases.

In summary, we can exhaustively execute Cases 1 and 2 until all edges of \bar{G}' are covered while adding at most $\sum_{e \in H} h_e$ elements. As explained at the beginning of the proof, this implies the lemma. \square

Lemma 5.10. If Algorithm 4 is executed on a prediction mandatory free instance, then in each iteration the instance remains prediction mandatory free. Furthermore, recovery strategy A guarantees 1-consistency and 2-robustness and recovery strategy B guarantees $|\text{ALG}| \leq \min\{\text{opt} + 5 \cdot k_h, 3 \cdot \text{opt}\}$.

Proof. We first show the first statement of the lemma, that the instance remains prediction mandatory free. Let G be the prediction mandatory free instance at the beginning of an iteration and let T_L be the corresponding lower limit tree. Let G' be the instance after the next restart and assume G' is not prediction mandatory free. We show that G being prediction mandatory free implies that G' is prediction mandatory free via proof by contradiction.

	Case 1	Case 2a)	Case 2a)	Case 2b)	Case 2b)
Passed Element	$f \notin T_L$ non-trivial in G'	$h(f) \in T_L$ trivial in G'	$h(f) \in T_L \setminus VC$ non-trivial in G'	$h(e) \notin T_L$ trivial in G'	$e \in T_L \cap VC$ non-trivial in G'
Passing Element	$l \in T_L$ trivial in G'	$f' \notin T_L$ trivial in G'	$f' \notin T_L$ trivial in G'	$e' \in T_L$ trivial in G'	$e' \notin T_L$ trivial in G'

Table 1: Errors considered in the different cases of the proof of Lemma D.5. The first row shows properties of the element whose interval border is passed over, and the second row shows properties of the passing element. Note that Cases 2a) and 2b) are listed twice because both contain two sub-cases.

Let T'_L be the lower limit tree of G' , let f'_1, \dots, f'_l be the (non-trivial) edges in $E' \setminus T'_L$ ordered by lower limit non-decreasingly, and let C'_i be the unique cycle in $T'_L \cup \{f'_i\}$. By definition of the algorithm, $T'_L = T'_U$ holds and $T'_L = T'_U$ is unique. We can w.l.o.g. ignore trivial edges in $E' \setminus T'_L$ since those are maximal in a cycle and can be deleted. Since G' is not prediction mandatory free, there must be some C'_i such that either $\bar{w}_e \in I_{f'_i}$ or $\bar{w}_{f'_i} \in I_e$ for some non-trivial $e \in C'_i \setminus \{f'_i\}$.

Assume $e \notin T_L$. Since e is part of $T'_L = T'_U$, Lemma D.4 implies that e must have been queried and therefore is trivial, which is a contradiction. Assume $e \in T_L$ and $\bar{w}_e \in I_{f'_i}$. Since $e \in T_L$, cycle C'_i must contain some $f \in X_e \setminus \{e\}$, where X_e is the cut between the two components of $T_L \setminus \{e\}$ in G . Instance G being prediction mandatory free implies $\bar{w}_e \notin I_f$ where I_f denotes the uncertainty interval of f before querying it. If $\bar{w}_e \in I_{f'_i}$, this implies $L_{f'_i} < L_f$. It follows that f has the highest lower limit in C'_i , which contradicts f'_i having the highest lower limit in C'_i .

Assume $e \in T_L$ and $\bar{w}_{f'_i} \in I_e$. Remember that f'_i is non-trivial and $f'_i \notin T'_L = T'_U$. According to Lemma D.4, it follows $f'_i \notin T_L = T_U$. Let $C_{f'_i}$ be the cycle in $T_L \cup \{f'_i\}$. Since G is prediction mandatory free, $\bar{w}_{f'_i} \notin I_{e'}$ for each $e' \in C_{f'_i} \setminus \{f'_i\}$, which implies $U_e > U_{e'}$. It follows that the highest upper limit on the path between the two endpoints of f'_i in $T'_U = T'_L$ is strictly higher than the highest upper limit on the path between the two endpoints of f'_i in $T_U = T_L$. We argue that this cannot happen and we have a contradiction to $e \in T_L$ and $\bar{w}_{f'_i} \in I_e$.

Let P be the path between the endpoints a and b of f'_i in T_U and let P' be the path between a and b in T'_U . Define U_P to be the highest upper limit on P . Observe that the upper limit of each edge can only decrease from T_U to T'_U since querying edges only decreases their upper limits. Therefore each $e' \in P' \cap P$ cannot have a higher upper limit than U_P . It remains to argue that the upper limit of each $e' \in P' \setminus P$ cannot be larger than U_P . Consider the set \mathcal{S} of maximal subpaths $S \subseteq P'$ such that $P \cap S = \emptyset$. Each $e' \in P' \setminus P$ is part of such a subpath S . Let S be an arbitrary element of \mathcal{S} , then there is a cycle $C \subseteq S \cup P$ with $S \subseteq C$. Assume $e' \in S$ has a strictly larger upper limit than U_P , then an element of S has the unique highest upper limit on C . It follows that subpath S and path P' cannot be part of any upper limit tree in the instance G' , which contradicts the assumption of P' being a path in T'_U .

We conclude that the graph G' is prediction mandatory free. Note that this proof is independent of the individual recovery strategy that is used in Algorithm 4 and relies only on maintaining the property $T'_L = T'_U$ being unique.

We continue by showing the performance guarantees starting with **recovery A**. If all predictions are correct, the algorithm queries exactly the minimum vertex cover VC which, according to [21], is optimal for the input instance of Algorithm 4, i.e., $|\text{ALG}| \leq \text{opt}$. If not all predictions are correct, the algorithm additionally queries elements of $T_L \setminus T_U$ in Line 4. Since those elements are mandatory, querying them does not violate the 2-robustness. Additionally, the algorithm might execute recovery strategy A in Line 6, i.e., query all elements in W and afterwards ensure $T_L = T_U$. According to Lemma 5.11, each $h(e) \in W$ forms

a witness set of size two with a distinct already queried $e \in VC$. In summary, the algorithm only queries disjoint witness sets of size one and two which implies $|\text{ALG}| \leq 2 \cdot \text{opt}$.

We conclude the proof by showing the performance guarantee of **recovery B**. First, we show the robustness. All elements that were queried because they were in $T_L \setminus T_U$ are mandatory and querying them never decrease the robustness. According to Lemma 5.11, each element e queried in Line 4 forms a witness set with the element $h(e)$. If $h(e) \notin \text{ALG}$ or $h(e)$ is queried as an element of $T_L \setminus T_U$, then $\{e, h(e)\}$ is disjoint to all other $\{e', h(e')\}$ pairs that are considered at Line 4. If $h(e)$ is queried in Line 4 or 5 in a later iteration, then $h(e)$ must have been re-matched to an element e' after a restart. In this case, the algorithm queries all of $\{h(e), e, e'\}$ because of Line 5. It follows that $\{h(e), e, e'\}$ is a witness set and, since all elements are queried and will not be considered again, this is disjoint to all other considered witness sets. In summary, the algorithm queries only (subsets of) disjoint witness sets of at most size three. This implies that $|\text{ALG}| \leq 3 \cdot |\text{OPT}|$.

We continue by showing the error-dependent guarantee. Consider set ALG . Observe that $|W| \leq |\text{OPT}|$ holds since $|VC| \leq |\text{OPT}|$ and, for each element $h(e) \in W$, there is a distinct element $e \in VC$ such that $\{e, h(e)\}$ is a witness set. Each $h(e) \in W$ was added to W in Line 5 after the distinct $e \in VC$ was queried in Line 4. Let S be the set of those queried elements, then $|S| = |W| \leq |\text{OPT}|$.

Consider some $e \in \text{ALG} \setminus S$, then e was queried either (i) as an element of $T_L \setminus T_U$ in Line 4 or 5, or (ii) as part of an witness set $\{e, h(e)\}$ with $\{e, h(e)\} \cap W \neq \emptyset$ in Line 4 or 5.

(i) e was queried as an element of $T_L \setminus T_U$ in Line 4 or 5. We argue that e contributes at least one to the hop distance k_h . The element e is not prediction mandatory, as the instance is by Lemma 5.6 prediction mandatory free but, as $e \in T_L \setminus T_U$, it is mandatory. The proof of Theorem 2.3 shows that e contributes at least one to k_h . To be more precise, let $h'(e)$ with $e \in E$ be the number of edges e' such that the value of e' passes over an endpoint of I_e . From the arguments in the proof of Theorem 2.3, it can be seen that, for each edge e that is non-prediction mandatory at some point but is mandatory, we have that $h'(e) \geq 1$. For a subset $U \subseteq E$, let $h'(U) = \sum_{e \in U} h'(e)$. Note that $k_h = h'(E)$ holds by reordering summations. Thus at most $h'(E_i) \leq k_h$ elements that satisfy (i) are queried, where E_i denotes the set of all elements that satisfy (i).

(ii) e was queried as part of an witness set $\{e, h(e)\}$ with $\{e, h(e)\} \cap W \neq \emptyset$ in Line 4 or 5. Let $\{e, h(e)\}$ be a set that is queried in Line 4 and 5. Since both e and $h(e)$ are queried, there must be some $l \in \{e, h(e)\} \cap W$. This means that l must have been matched to a different element in an earlier restart. By Lemmas 5.6 and 5.10, the instance at the beginning of each restart is prediction mandatory free. We can apply Lemma 5.12 and conclude that the number of elements that are re-matched is at most $2 \cdot k_h$. It follows that at most $2 \cdot k_h$ sets $\{e, h(e)\}$ can be queried in Line 4 and 5 and, thus, in total at most $4 \cdot k_h$ elements.

We have that $\text{ALG} = S \cup E_{ii} \cup E_i$, where E_i denotes the set of elements that satisfy (i) and E_{ii} denotes the set of elements that satisfy (ii). Since $|S| \leq |\text{OPT}|$, $|E_i| \leq k_h$ and $|E_{ii}| \leq 4 \cdot k_h$, we obtain a performance guarantee of $|\text{ALG}| \leq |\text{OPT}| + 5 \cdot k_h$ \square

Corollary D.6. Let ALG be the set of queries made by the algorithm that first executes Algorithm 3, and then Algorithm 4 with recovery strategy B. Then, $|\text{ALG}| \leq (1 + \frac{1}{\gamma}) \cdot \text{opt} + (5 + \frac{1}{\gamma}) \cdot k_h$.

Proof. Let $\text{ALG} = \text{ALG}_1 \cup P \cup \text{ALG}_2$, where ALG_1 denotes the queries of Algorithm 3 without the last execution of Line 1, P denotes the queries in the last execution of Line 1 and ALG_2 denotes the queries of Algorithm 4. Furthermore, let $\text{OPT} = \text{OPT}_1 \cup \text{OPT}_2$ be an optimal query set with $\text{OPT}_1 = \text{OPT} \cap \text{ALG}_1$ and $\text{OPT}_2 = \text{OPT} \setminus \text{ALG}_1$.

Since each element of P is prediction mandatory, the proof of Theorem 2.3 implies that each such element is either mandatory or contributes at least one to the hop distance. From the arguments in the proof of Theorem 2.3, it can be seen that, for each edge e that is prediction mandatory at some point but not mandatory, we have that $h'(e) \geq 1$, where h' is defined as in the proof of Lemma 5.10. Thus, we have the following guarantees for the sets ALG_1 , P and ALG_2 :

- For ALG_1 , the proof of Lemma 5.6 implies $|\text{ALG}_1| \leq (1 + \frac{1}{\gamma}) \cdot (|\text{OPT}_1| + h'(\text{ALG}_1))$.
- For set P , we have $|P| \leq |P \cap \text{OPT}_2| + h'(P)$.
- For ALG_2 , we have that $|\text{ALG}_2| \leq |\text{OPT}_2| + |E_i| + |E_{ii}| \leq |\text{OPT}_2| + h'(E_i) + 4 \cdot k_h$, where E_i and E_{ii} denote the sets of elements that satisfy (i) and (ii), respectively, as defined in the proof of Lemma 5.10.

Summing up the guarantees, we get

$$|\text{ALG}| = |\text{ALG}_1 \cup P \cup \text{ALG}_2| \leq (1 + \frac{1}{\gamma}) \cdot |\text{OPT}| + (1 + \frac{1}{\gamma})(h'(\text{ALG}_1) + h'(P) + h'(E_i)) + 4 \cdot k_h.$$

By definition, it holds that ALG_1 , P and E_i are pairwise disjoint, which implies $h'(\text{ALG}_1) + h'(P) + h'(E_i) \leq k_h$. Thus, we can conclude

$$\text{ALG} \leq (1 + \frac{1}{\gamma}) \cdot \text{opt} + (5 + \frac{1}{\gamma}) \cdot k_h.$$

□

E Appendix for the Sorting Problem (Section 6)

We present an analysis of Algorithm 5, obtaining the following theorem. Remember that we assume that each interval is either trivial or open; however the algorithm can be modified to allow for closed and half-open intervals as in [37]. We say that an interval I_j *forces* a query in interval I_i if I_i is queried in Line 4, 9 or 23 because $I_j \subseteq I_i$ or $w_j \in I_i$.

Theorem 3.5. For sorting under uncertainty for a single set, there is a polynomial-time algorithm with competitive ratio $\min\{1 + k/\text{opt}, 2\}$, for any error measure $k \in \{k_{\#}, k_M, k_h\}$.

To prove that the algorithm indeed solves the problem, we must show that, at every execution of Line 18, every component of the intersection graph is a path. We must have a proper interval graph, because every interval that contains another interval is queried in Line 4, and every remaining interval that contains a queried value is queried in Line 4 or 9. We claim that, at every execution of Line 18, the graph contains no triangles; for proper interval graphs, this implies that each component is a path, because the 4-star $K_{1,3}$ is a forbidden induced subgraph [59]. Suppose by contradiction that there is a triangle abc , and assume that $L_a \leq L_b \leq L_c$; it holds that $U_a \leq U_b \leq U_c$ because no interval is contained in another. Since I_a and I_c intersect, we have that $U_a \geq L_c$, so $I_b \subseteq I_a \cup I_c$ and it must hold that $\bar{w}_b \in I_a$ or $\bar{w}_b \in I_c$, a contradiction since we query intervals that contain a predicted value in Line 7.

We also need to prove that $(\mathcal{I}, \mathcal{E})$ is a forest of arborescences. It cannot contain directed cycles because we avoid this in Line 8. Also, there cannot be two arcs with the same destination because we assign a single parent to each prediction mandatory interval.

Theorem E.1. Algorithm 5 spends at most $\text{opt} + k_M$ queries.

Proof. Fix an optimum solution OPT . Every interval queried in Lines 4 and 9 is in OPT . Every interval queried in Line 7 that is not in OPT is clearly in $\mathcal{I}_P \setminus \mathcal{I}_R$.

For each path P considered in Line 18, let P' be the intervals queried in Lines 19–22. It clearly holds that $|P'| \leq |P \cap \text{OPT}|$. Finally, every interval queried in Line 23 is in $\mathcal{I}_R \setminus \mathcal{I}_P$ because we query all prediction mandatory intervals at the latest in Line 7. □

The next result follows from Theorems 2.3 and E.1.

Theorem E.2. Algorithm 5 spends at most $\text{opt} + k_h$ queries.

For the remaining two theorems in this section, we need the following lemma.

Lemma E.3. Fix the state of \mathcal{S} as in Line 5. For each interval I_j , let $S_j = \{I_i \in \mathcal{S} : \pi(i) = j\}$. For any path P considered in Line 18 and any interval $I_j \in P$ that is not an endpoint of P , it holds that $S_j = \emptyset$.

Proof. Suppose by contradiction that some $I_j \in P$ that is not an endpoint of P has $S_j \neq \emptyset$. Let I_a and I_b be its neighbors in P , and let $I_i \in S_j$. We have that $\bar{w}_j \notin I_a \cup I_b$, otherwise I_a or I_b would have been queried in Line 7. Thus $(I_i \cap I_j) \setminus (I_a \cup I_b) \neq \emptyset$, because $\bar{w}_j \in I_i$. It is not the case that $I_i \subseteq I_j$ or $I_j \subseteq I_i$: If $I_i \subseteq I_j$, then I_j would have been queried in Line 4 before P is considered; if $I_j \subseteq I_i$, then I_i would have been queried in Line 4 and $I_i \notin \mathcal{S}$. Therefore it must be that $I_i \subseteq I_a \cup I_j \cup I_b$, otherwise $I_a \subseteq I_i$ or $I_b \subseteq I_i$ (again a contradiction for $I_i \in \mathcal{S}$), since $(I_i \cap I_j) \setminus (I_a \cup I_b) \neq \emptyset$. However, if $I_i \subseteq I_a \cup I_j \cup I_b$, then I_i would have forced a query in I_a , I_b or I_j in Line 9, a contradiction. \square

Theorem E.4. Algorithm 5 performs at most $\text{opt} + k_{\#}$ queries.

Proof. Fix an optimum solution OPT. We partition the intervals in \mathcal{I} into sets with the following properties. One of the sets \tilde{S} contains intervals that are not queried by the algorithm. We have a collection \mathcal{S}' in which each set has at most one interval not in OPT. Also, if it has one interval not in OPT, then we assign a prediction error to that set, in such a way that each error is assigned to at most one set. (The interval corresponding to the prediction error does not need to be in the same set.) Let \mathcal{T}' be the set of intervals with a prediction error assigned to some set in \mathcal{S}' . Finally, we have a collection \mathcal{W} such that for every $W \in \mathcal{W}$ it holds that $|\text{ALG} \cap W| \leq |W \cap \text{OPT}| + k_{\#}(W \setminus \mathcal{T}')$, where $k_{\#}(X)$ is the number of intervals in X with incorrect predictions. If we have such a partition, then it is clear that we spend at most $\text{opt} + k_{\#}$ queries.

We begin by adding a different set to \mathcal{S}' for each interval queried in Lines 4 and 9; all such intervals are clearly in OPT, and we do not need to assign a prediction error.

Fix the state of \mathcal{S} as in Line 5. To deal with the intervals queried in Line 7, we add to \mathcal{S}' the set $S_j = \{I_i \in \mathcal{S} : \pi(i) = j\}$ for $j = 1, \dots, n$. Note that each such set is a clique, because all intervals contain \bar{w}_j . Therefore, at most one interval in S_j is not in OPT, and if that occurs, then $\bar{w}_j \neq w_j$, and we assign this prediction error to S_j .

Let $P = x_1 x_2 \dots x_p$ with $p \geq 2$ be a path considered in Line 18, and let P' be the set of intervals in P that are queried in Lines 19, 21 or 22. It clearly holds that $|P'| = \lfloor |P|/2 \rfloor \leq |P \cap \text{OPT}|$. It also holds that at most $k_{\#}(P')$ intervals in P are queried in Line 23: Each interval $I_j \in P'$ can force a query in at most one interval I_i in Line 23, and in that case the predicted value of I_j is incorrect because $w_j \in I_i$ but $\bar{w}_j \notin I_i$, or I_i would have been queried in Line 7. We will create a set $W \in \mathcal{W}$ and possibly modify \mathcal{S}' , in such a way that $P \subseteq W$ and $P' \cap \mathcal{T}' = \emptyset$, so it is enough to show that

$$|\text{ALG} \cap W| \leq |W \cap \text{OPT}| + k_{\#}(P'). \quad (\text{E.1})$$

We initially take W as the intervals in P . By Lemma E.3, it holds that $S_j = \emptyset$ for any $j \neq x_1, x_p$. If $S_{x_1} \subseteq \text{OPT}$, then we do not need to assign a prediction error to S_{x_1} . Otherwise, let I_i be the only interval in $S_{x_1} \setminus \text{OPT}$. The predicted value of I_{x_1} is incorrect because $\bar{w}_{x_1} \in I_i$, and it must hold that $I_{x_1} \in \text{OPT}$, or OPT would not be able to decide the order between I_{x_1} and I_i . If $x_1 \notin P'$, then we will not use its error in the bound of $|\text{ALG} \cap W|$ if we prove Equation (E.1). Otherwise, we add I_i to W and remove it from S_{x_1} , and now we do not need to assign a prediction error to S_{x_1} . We do a similar procedure for x_p , and since at most one of x_1, x_p is in P' , we only have two cases to analyze: (1) $W = P$, or (2) $W = P \cup \{I_i\}$ with $\pi(i) \in \{x_1, x_p\}$.

- (1) $W = P$. Clearly $|\text{ALG} \cap W| \leq |P'| + k_{\#}(P') \leq |W \cap \text{OPT}| + k_{\#}(P')$.
- (2) $W = P \cup \{I_i\}$, with $\pi(i) \in \{x_1, x_p\}$. Suppose w.l.o.g. that $\pi(i) = x_1$. Remember that $x_1 \in P'$, that $I_{x_1} \in \text{OPT}$ and that its predicted value is incorrect. Since $x_1 \in P'$, it holds that $|P|$ is even and $x_2 \notin P'$. We have two cases.

(a) I_{x_2} is not queried in Line 23. Then I_{x_1} does not force a query in Line 23, so

$$\begin{aligned} |\text{ALG} \cap W| &\leq |P' \cup \{I_i\}| + k_{\#}(P' \setminus \{x_1\}) \\ &= |P'| + 1 + k_{\#}(P' \setminus \{x_1\}) \\ &\leq |P \cap \text{OPT}| + k_{\#}(P') \\ &\leq |W \cap \text{OPT}| + k_{\#}(P'). \end{aligned}$$

(b) I_{x_2} is queried in Line 23. Then $I_{x_1}, I_{x_2} \in \text{OPT}$, and $|\text{OPT} \cap (P \setminus \{I_{x_1}, I_{x_2}\})| \geq |P' \setminus \{I_{x_1}\}|$ because $|P|$ is even. Therefore,

$$\begin{aligned} |\text{ALG} \cap W| &\leq |P' \cup \{I_{x_2}, I_i\}| + k_{\#}(P' \setminus \{I_{x_1}\}) \\ &\leq |P \cap \text{OPT}| + 1 + k_{\#}(P' \setminus \{I_{x_1}\}) \\ &\leq |W \cap \text{OPT}| + k_{\#}(P'). \end{aligned}$$

To conclude, we add the remaining intervals that are not queried by the algorithm to \tilde{S} . \square

Now it remains to prove that the algorithm is 2-robust. Consider the forest of arborescences $(\mathcal{I}, \mathcal{E})$ that is constructed by Algorithm 5. For each of these arborescences, the prediction mandatory intervals contained in the arborescence are partitioned into cliques by Lines 10–16 of Algorithm 5. Each of these clique partitions may contain a single clique of size 1. As we would like to use the cliques in these clique partitions as witness sets, cliques of size 1 require special treatment. It turns out that the most difficult case is where the clique of size 1 is formed by a prediction mandatory interval I_i that is the root of an arborescence. This happens if the interval $I_{\pi(i)}$ that makes I_i prediction mandatory is a descendant of I_i in that arborescence. The following lemma shows that in this case we can revise the clique partition of that arborescence in such a way that all cliques in that clique partition have size at least 2.

Lemma E.5. Consider an out-tree (arborescence) T on a set of prediction mandatory intervals, where an edge (I_j, I_i) represents that $\bar{w}_j \in I_i$. Let the root be I_r . Let interval I_m with $\bar{w}_m \in I_r$ be a descendant of the root somewhere in T . Then the intervals in T can be partitioned into cliques (sets of pairwise overlapping intervals) in such a way that all cliques have size at least 2.

Proof. We refer to the clique partition method of Lines 10–16 in Algorithm 5 as algorithm CP. This method will partition the nodes of an arborescence into cliques, each consisting either of a subset of the children of a node, or of a subset of the children of a node plus the parent of those children. In the case considered in this lemma, all cliques will have size at least 2, except that the clique containing the root of the tree may have size 1.

We first modify T as follows: If there is a node I_i in T that is not a child of the root I_r but contains \bar{w}_r , then we make I_r the parent of I_i (i.e., we remove the subtree rooted at I_i and re-attach it below the root). After this transformation, all intervals that contain \bar{w}_r are children of I_r .

Apply CP to each subtree of T rooted at a child of I_r . For each of the resulting partitions, we call the clique containing the root of the subtree the *root clique* of that subtree. There are several possible outcomes that can be handled directly:

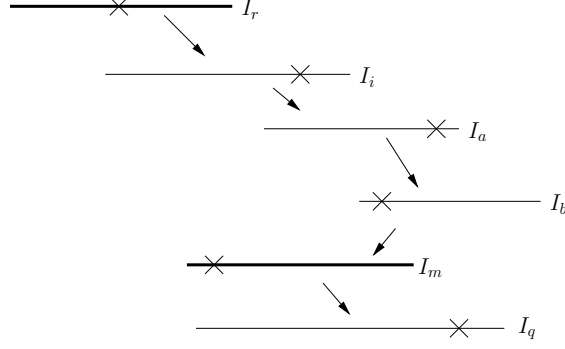


Figure 6: Illustration of path from I_r to I_m 's child I_q in T

- At least one of the clique partitions has a root clique of size 1. In that case we combine all these root cliques of size 1 with I_r to form a clique of size at least 2, and we are done: This new clique together with all remaining cliques from the clique partitions of the subtrees forms the desired clique partition.
- All of the clique partitions have root cliques of size at least 2, and at least one of them has a root clique of size at least 3. Let I_s be the root node of a subtree whose root clique has size at least 3. We remove I_s from its clique and form a new clique from I_s and I_r , and we are done.
- All of the clique partitions have root cliques of size exactly 2, and at least one of the children I_i of I_r has $\bar{w}_i \in I_r$. Then we add I_r to the root clique that contains I_i . We can do this because all intervals in that root clique contain \bar{w}_i .

Now assume that none of these cases applies, so we have the following situation: All of the clique partitions have root cliques of size exactly 2, and every child of I_r has its predicted value outside I_r , i.e., $\bar{w}_i \notin I_r$. In particular, I_m , the interval that makes I_r prediction mandatory, cannot be a child of I_r .

Let T' be the subtree of T that is rooted at a child of I_r and that contains I_m . Let the root of T' be I_i .

Observe that I_i is the only interval in T' that contains \bar{w}_r , because all such intervals are children of I_r in T . Assume w.l.o.g. that \bar{w}_i lies to the right of I_r . Then all intervals in T' , except for I_i , lie to the right of \bar{w}_r . See Figure 6 for an illustration of a possible configuration of the path from I_r to I_m (and a child I_q of I_m) in T .

Now re-attach the subtree T_m rooted at I_m as a child of I_r (ignoring the fact that \bar{w}_r is not inside I_m), and let $T_i = T' \setminus T_m$ denote the result of removing T_m from T' . Re-apply CP to the two separate subtrees T_m and T_i . The possible outcomes are:

- The root clique of at least one of the two subtrees has size 1. We can form a clique by combining I_r with those (one or two) root cliques of size 1. As both I_i and I_m intersect I_r from the right, the resulting set is indeed a clique. Together with all other cliques from the clique partitions of T_m and T_i , and those of the other subtrees of I_r in T , we obtain the desired clique partition.
- The root cliques of both subtrees have size at least 2. We add I_r to the root clique of T_m . That root clique contains only intervals that contain \bar{w}_m , and I_r also contains \bar{w}_m , so we do indeed get a clique if we add I_r to that root clique. This new clique, together with all other cliques from the clique partitions of T_m and T_i , and those of the other subtrees of I_r in T , forms the desired clique partition.

This concludes the proof of the lemma. □

Theorem E.6. Algorithm 5 is 2-robust.

Proof. Fix an optimum solution OPT. We partition the input into a set S of intervals that are not queried, plus a set S' of intervals in OPT, plus a collection \mathcal{C} of sets with size at least 2 that are cliques in the initial intersection graph, plus a collection \mathcal{W} such that, for each $W \in \mathcal{W}$, the algorithm queries at most $2 \cdot |W \cap \text{OPT}|$ intervals in W . If we have such a partition, then it is clear that we spend at most $2 \cdot \text{opt}$ queries.

For every arborescence that meets the condition of Lemma E.5, we take the revised clique partition whose existence is guaranteed by that lemma and add all its cliques to \mathcal{C} . These arborescences need no longer be considered in the rest of this proof.

We continue by adding all sets C_j computed by Algorithm 5 to \mathcal{C} , for all $j \in \{1, \dots, n\}$ where $C_j \neq \emptyset$ and where C_j has not been part of an arborescence that was already handled using Lemma E.5 in the previous paragraph. When building a set C_j for $j = \pi(i)$ in Algorithm 5, we always pick a vertex $I_i \in S$ that is furthest from the root; therefore, if I_j is queried in Line 7, then I_j will still be in S when we pick I_i and build C_j . Thus, if a set $C_j \in \mathcal{C}$ has size 1, then I_j is not queried in Line 7.

Now, if there is $C_j \in \mathcal{C}$ of size 1 and I_j was queried in Line 4 or 9, then we include I_j in C_j .

At this point, if there is a $C_j \in \mathcal{C}$ of size 1, then I_j must belong to some path P that is a component of the dependency graph just before Line 17 of Algorithm 5 is executed for the first time. If P is a path of length one, i.e., consist only of I_j , then we add I_j to C_j so that C_j becomes a clique of size 2. It remains to consider the case where P is a path of length at least two. Then P is considered in Line 18 and, by Lemma E.3, we have that I_j must be one of the endpoints of P . Let $P = x_1 x_2 \dots x_p$. We add all intervals in P to a set W . If $|C_{x_1}| = 1$, then we make $W := W \cup C_{x_1}$ and $\mathcal{C} := \mathcal{C} \setminus \{C_{x_1}\}$. We do a similar step if $|C_{x_p}| = 1$. Since P is a path, at least $\lfloor p/2 \rfloor$ of its intervals are in $P \cap \text{OPT}$. However, the graph induced by W may no longer be a path. (For example, an interval in C_{x_1} may intersect I_{x_2} and I_{x_3} , but not I_{x_4} .) Still, it is not hard to see that $|W \cap \text{OPT}| \geq \lfloor |W|/2 \rfloor$ as well, since any solution to the problem queries a vertex cover in the intersection graph. If $|W|$ is even, then we simply add W to \mathcal{W} and we are done; so assume that $|W|$ is odd. We divide the analysis in two cases:

- (1) p is odd. Then the algorithm queries $I_{x_2}, I_{x_4}, \dots, I_{x_{p-1}}$ in Line 19. Each of those intervals can force at most one query in Line 23, therefore we have at least one interval in W that is never queried by the algorithm. Since $|W \cap \text{OPT}| \geq \lfloor |W|/2 \rfloor$, clearly the algorithm queries at most $2 \cdot |W \cap \text{OPT}|$ intervals in W , and we include W in \mathcal{W} .
- (2) p is even. Then either $|C_{x_1}| = 1$ or $|C_{x_p}| = 1$ (but not both), otherwise $|W|$ would be even. Thus we have two subcases:
 - (2a) $W = P \cup C_{x_1}$. The algorithm queries $I_{x_1}, I_{x_3}, \dots, I_{x_{p-1}}$ in Line 21. We begin by adding $C_{x_1} \cup \{x_1\}$ to \mathcal{C} . If I_{x_1} forces a query in I_{x_2} in Line 23, then we add I_{x_2} to S' , and the remaining of P to a new set W' in \mathcal{W} ; this will be an even path because it was even to begin with and we remove I_{x_1} and I_{x_2} , so we will be fine. Otherwise, the remaining of P is an odd path for which less than half intervals are queried in Line 21, and we proceed similarly as in case (1).
 - (2b) $W = P \cup C_{x_p}$. Then the algorithm queries $I_{x_2}, I_{x_4}, \dots, I_{x_p}$ in Line 22, and the analysis is symmetric to the previous subcase.

At this point, it is clear that there are no more sets of size 1 in \mathcal{C} . The remaining intervals that are not queried are simply included in S , the remaining intervals queried in Lines 4 and 9 are included in S' , and we clearly obtain a partition of the intervals as desired. \square

F Appendix for the experimental results (Section 7)

In this section we describe in detail the instance and prediction generation as used in the simulations of Section 7. In addition, Section F.2 shows experimental results for the MST problem under uncertainty.

Our algorithms rely on finding minimum vertex covers. For the minimum problem, we solved the vertex cover problem using the *Coin-or branch and cut (CBC)*¹ mixed integer linear programming solver. Since our MST algorithms only solve the vertex cover problem in bipartite graphs, we determined minimum vertex covers using a standard augmenting path algorithm [2].

F.1 Experimental results for the minimum problem

We generated test instances by drawing interval sets from interval graphs. As source material we used instances of the *boolean satisfiability problem (SAT)* from the rich SATLIB [38] library. A *clause* of a SAT instance is a set of variables (with polarities) where each variable can be represented by its index, i.e., the variables are numbered. We interpret each clause c as an interval based on the indices of variables in c (ignoring the polarities). Each clause c can be interpreted as the interval (L_c, U_c) with $L_c = c_{\min} - \varepsilon$ and $U_c = c_{\max} + \varepsilon$ for a small $\varepsilon > 0$, where c_{\min} and c_{\max} denote the lowest and highest variable index in c . In non-trivial SAT instances, the complexity of the problem is created by clauses sharing variables. This in many instances leads to a high overlap between the interval representations of the clauses which makes the resulting interval graphs interesting source material for the minimum problem under uncertainty.

To generate instances for the minimum problem, we uniformly at random draw a sample of (not necessarily distinct) *root intervals* from the re-interpreted SAT instance. For each root interval, we add a *root set* S to the minimum problem instance which contains the root interval and up to r_w intersecting intervals where r_w is a parameter of the instance generation. Note that the size of the root sets also depends on whether the source SAT instance contains enough intervals that intersect the root interval. The number of intersecting intervals (between 1 and r_w) and the intersecting intervals themselves are again drawn uniformly at random. Note that we only generate preprocessed instances, i.e., instances where the leftmost interval I_i of a set S does not fully contain any interval $I_j \in S \setminus \{I_i\}$, and therefore might discard drawn intervals that would lead to the instance becoming non-preprocessed. We generate only preprocessed instances as these are the difficult instances: a non-preprocessed instance gives all algorithms access to “free information” in form of queries that are obviously part of any feasible solution which can be very useful in solving the instance.

To ensure that the generated instances have an interesting underlying interval graph structure, that is, an interesting vertex cover instance (cf. Section 4), each root set S is used as a starting point of paths with length up to r_d in the vertex cover instance where r_d is a parameter of the instance generation. For each non-leftmost interval $I_S \in S$ we draw an integer r'_d between 0 and $r_d - 1$ that denotes the length of the path starting at I_S . If $r'_d > 0$, we generate a set S' with I_S as leftmost interval and size of at most r_w by again drawing the number of intersecting intervals (between 1 and r_w) and the intersecting intervals themselves uniformly at random. The procedure is repeated recursively in set S' with parameter $r'_d < r_d$. Note that the generation of these paths also depends on whether generating them is possible using the source SAT instance. This part of the instance generation ensures a more complex underlying interval graph structure in the generated instances. In total, the family of sets \mathcal{S} of a generated instance consists of all root sets and all sets S' that are added in the recursive procedure. The set of intervals \mathcal{I} of a generated instance consists of all intervals that are added during the root set generation and all intervals added in the recursive procedure

The instances were generated by drawing between 75 and 150 root clauses using values $r_w = 10$ and $r_d = 2$. The resulting instances contain between 47 and 287 intervals, and between 15 and 126 sets. Since the number of variables and clauses in the source SAT instances influences the probability with which the

¹<https://github.com/coin-or/Cbc>, accessed November 4, 2020.

generated sets share intervals, we used SAT instances of different sizes, containing between 411 and 32316 clauses, and between 100 and 8704 variables.

Thus far, we only described how to generate the intervals and sets. This paragraph describes the generation of the true values. To generate the true values, we start with initial true values that are placed in a standard form such that no elements are mandatory. Note that it is not always possible to set the true values such that no elements are mandatory. In such cases, we start with true values such that only a small number of elements are mandatory. Afterwards, we uniformly at random draw the number of mandatory elements of the generated instance and then sequentially select random elements whose true values can be adjusted such that the number of mandatory elements increases until the determined number of mandatory elements is reached (if possible). To set a true value such that the number of mandatory elements increases, we exploit Lemma 4.1. We generate the true values in this way because an instance of the minimum problem is essentially characterized by its vertex cover instance and its set of mandatory elements. Since the set of mandatory elements has an important effect on the instance, it makes sense to generate them in a way such that a wide range of different mandatory element sets is covered and especially mandatory element sets of a wide range of sizes are considered.

Regarding the generation of predictions, we observed that just picking predicted values uniformly at random (or by using a Gaussian distribution) does not lead to predictions that cover a wide range of relative errors k_M/opt . Thus, we employed a more sophisticated prediction generation.

For each instance, we start with a target prediction error of $v = 0$. For this target prediction error, we generate predictions with an error of $k_M \approx v$. (An error of $k_M = v$ is not always possible.) We repeatedly generate such predictions while increasing v until we cannot find any predictions with $k_M \approx v$. To generate predictions with an error of $k_M \approx v$ for a given target value v , we start with tentative predicted values that equal the true values. Then, we determine the set of elements F whose predicted values can be placed such that the error k_M for the tentative predicted values increases. The procedure uniformly at random draws an $e \in F$ and places \bar{w}_e such that the error increases by at least one. We repeat this procedure until the targeted error (or, if not possible, an error close to the targeted error) is reached.

After we iteratively generate predictions with an increasing error, we equally divide the interval $[0, v_{\max}]$ into 25 bins of equal size, where v_{\max} is the maximum prediction error of the generated predictions, and, for each bin, select the 5 predictions with the highest error within the bin. The resulting 125 sets of predictions are then used for our experiments.

F.2 Experimental results for the MST problem

For the MST problem we generated instances based on the symmetric traveling salesman problem instances of the TSPLIB². We considered graphs of up to 90 vertices and 4000 edges. For TSP instances with more than 90 vertices, we instead selected and used connected sub-graphs with 90 vertices. The TSPLIB instances already contain the graph structure and the true edge weights.

We generate the interval boundaries as described in [27]. For each edge e with true value w_e , an interval boundary is set close to w_e with a probability of $\frac{1}{2}$. That is, with a probability of $\frac{1}{2}$ we set either $L_e = w_e - \varepsilon$ or $U_e = w_e + \varepsilon$ for a small $\varepsilon > 0$. This encourages the generation of instances where an intersecting interval contains the true value of e and increases the chance of generating instances with mandatory elements. The interval boundaries that are not set afterwards are drawn uniformly at random within a ratio of d around the true value w_e where d is a parameter of the instance generation.

Our test instances were generated using different values for d , between 0.05 and 1. While [27] observes that small choices for d lead to more difficult instances in terms of the competitive ratio and, in particular, utilizes the parameter $d = 0.065$ to generate difficult instances, a small choice for d also leads to smaller generated instances. This is because a small d increases the probability of small uncertainty intervals and

²<http://comopt.ifi.uni-heidelberg.de/software/TSPLIB95/tsp/>, accessed November 3, 2020.

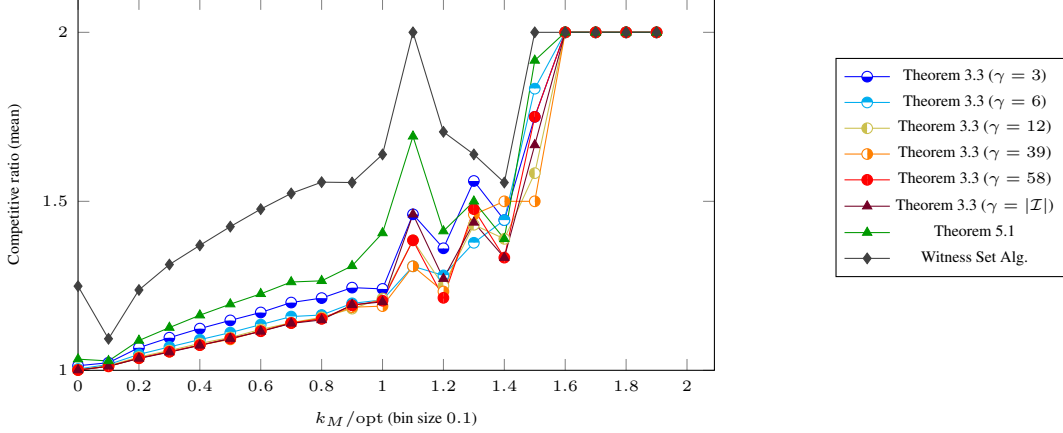


Figure 7: Experimental results for the MST problem under uncertainty. Instances and predictions were grouped into equal size bins according to their relative error.

therefore the probability that intervals on a cycle do not intersect. Thus, instances effectively become smaller. Since we want to observe the performance of our algorithms for different choices of γ , we rely on instances with a wider range of relative errors and, thus, on bigger instances.

For each of the 102 graphs we generated 5 instances of the MST problem under uncertainty without predictions and for each such generated instance we in turn generated 100 predictions. While the minimum problem required a more sophisticated prediction generation to cover a wide range of prediction errors, we were able to cover a large enough range for the MST problem by selecting the predicted values uniformly at random. By repeating the random prediction generation long enough, we ensure that a large enough relative error range is covered.

Figure 7 shows the results of the over 20,000 simulations (instance and prediction pairs). The figure compares the results of our prediction-based algorithms of Theorems 3.3 and 5.1 for different choices of the parameter γ with the standard *witness set algorithm*. The latter sequentially resolves cycles by querying witness sets of size two and achieves the best possible competitive ratio of 2 without predictions [24].

Our prediction-based algorithms outperform the witness set algorithm for every relative error up to 1.4. For higher relative errors, the prediction-based algorithms match the performance of the witness set algorithm.

Further, the parameter γ reflects the robustness-performance tradeoff in the sense that the curves for different choices of γ intersect and high values for γ perform better for smaller relative errors, while smaller values for γ perform better for high relative errors. The performance gap between the different values for γ appears less significant for small relative errors, which suggests that selecting γ not too close to the maximum value $|\mathcal{I}|$ might be beneficial.

In contrast to the results for the minimum problem (cf. Figure 2), the plots for the prediction-based algorithm are not monotonously increasing and instead contain jumps for higher relative errors. This is because the instances more strongly vary for the different error bins. In particular, most of the instances with predictions that lead to higher relative errors are small in the sense that opt is small. The variation in the instances between different error bins leads to jumps in the plots. For relative errors of at least 1.6 the instances are small enough such that the different choices for γ essentially behave the same.

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