Semi-extending Modules

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Abstract

Throughout this paper R represents commutative ring with identity, and M is a unitary left R-module. The purpose of this paper is to study a new concept, (up to our knowledge), named a semi-extending modules, as generalization of extending modules, where an R-module M is called semi-extending if every sub module of M is a semi-essential in a direct summand of M. Various properties of semi-extending module are considered. Moreover, we investigate the relationships between semi-extending modules and other related concepts, such as CLS-modules and FI- extending modules.

Key words: Essential submodules, Semi-essential submodules, Closed submodules, St-closed submodules, Extending modules, Semi-extending modules, Fully essential modules.

INTRODUCTION

Let R be a commutative ring with identity, and let M be a unitary left R-module. Assume that all R-modules under study contain prime submodules. A proper submodule P of M is called prime, if whenever rm $\in$ P for r $\in$R and m $\in$M, then either m $\in$ P or r $\in$ $(P:_R M)$ [13]. It is well known that a nonzero submodule N of M is called essential (briefly $N \leq_e M$), if $N \cap L \neq (0)$ for each nonzero submodule L of M [9], and a nonzero submodule N of M is called semi-essential (briefly $N \leq_{sem} M$), if $N \cap P \neq (0)$ for each nonzero prime submodule P of M [10]. Equivalently, a submodule N of an R-module M is called semi-essential if whenever $N \cap P = (0)$, implies that P = (0) for every prime submodule P of M [15], A submodule N of M is called closed (briefly $N \leq_c M$), if N has no proper essential extensions in M, i.e if $N \leq_e K \leq M$ then N = K [8]. And a submodule N of M is called St-closed in M (briefly $N \leq_{Stc} M$), if N has no proper semi-essential extensions in M, i.e if there exists a submodule K of M such that $N \leq_{sem} K \leq M$ then N = K [16].

An R-module M is called extending, (briefly CS-module), if every submodule of M is an essential in a direct summand of M. Equivalently; if every closed submodule in M is a direct summand of M, see [6, P.55].

In our work we introduce a new class of modules (up to our knowledge), named semi-extending modules, which is a generalization of extending modules, where an R-module M is called **semi-extending**, if every submodule of M is a semi-essential in a direct summand of M. This paper consists of two sections; in section one, we introduce semi-extending modules, so we give the main properties of this class of modules. Also we give an example showing that the semi-extending module is a generalization of extending module.

Section two is devoted to discuss other results on semi-extending modules. So we study the conditions under which semi-extending modules can be extending modules. Moreover, we discuss the relationships between semi-extending modules and some other types of modules, such as CLS-modules and FI-extending modules

§1. Semi-extending modules

The main goal of this section is to introduce a generalization of extending modules named semi-extending modules. We give some examples and remarks about this class of modules. Also, we give a characterization of semi-extending modules, and we study the hereditary property of semi-extending modules between R-module M and R itself. Beside that we give an example showing that the semi-extending module is a generalization of the extending module.

Definition (1.1): An R-module M is called **semi-extending**, if every submodule of M is a semi-essential in a direct summand of M. A ring R is called semi-extending, if R is a semi-extending R-module.

Recall that an R-module M is **semisimple**, if every submodule of M is a direct summand of M [9, Ex (5.1.2) (2), P.107]. And a nonzero R-module M is called **semi-uniform**, if every nonzero submodule of M is a semi-essential in M [10, Def 1, P.55].

Examples and Remarks (1.2):

1. The Z-module Z_6 is a semi-extending module, since every submodule of Z_6 is a semi-essential in a direct summand of Z_6.
2. Every extending module is a semi-extending module.

Proof (2): It follows directly from [10, Ex (2), P.49]

3. Every semisimple module is a semi-extending module, for example all of Z_2, Z_3, Z_6, Z_{10}, Z_{30} are semi-extending Z-module.

Proof (3): Since every semisimple module is an extending module [6]. Then the result follows from (2).

4. Every uniform module is a semi-extending module and indecomposable module, such as all of the Z-modules Z, Z_4, Z_8, Z_{16}.

Proof (4): Since every uniform module is an extending and indecomposable module [11, Prop (2.5), P. 20], so the result follows from (2).

5. The semi-extending module is not necessary uniform module, for example: The Z-module Z_{36} is a semi-extending module but not uniform module, since there exists a submodule $(\overline{18}) \leq Z_{36}$ such that $(\overline{18})$ is not essential submodule of Z_{36}. In fact $(\overline{12})$ is a submodule of Z_{36} and $(\overline{18}) \cap (\overline{12}) = (\overline{0})$.

6. Every semi-uniform module is a semi-extending module.

Proof (6): Let N be a submodule of M. If N = (0), then clearly N is a semi-essential in (0) which is a direct summand of M. If N ≠ (0), since M is semi-uniform module, then $N \leq_{sem} M$. But M is a direct summand of itself, so we are done.

7. The converse of (6) is not true in general, for example the Z-module Z_{24} is a semi-extending module, but not semi-uniform module. In fact $(\overline{8})$ is not semi-essential submodule in Z_{24}.

8. A semi-uniform module may not be an indecomposable module, for example the Z-module Z_{36} is a semi-uniform module but not indecomposable module. In fact $Z_{36} = (\overline{4}) \oplus (\overline{9})$.

9. Every π-injective module is a semi-extending module.

10. Every module over a semisimple ring is a semi-extending module.

Proof (10): Let R be a semisimple ring, and let M be an R-module. By [8, Th (1.18), P.29], M is an injective module, hence M is a CS-module [6], and the result follows from (2).

11. A semi-essential submodule of a semi-extending module need not be semi-extending module, for example: if M be an R-module such that M is not semi-extending module.

Issue 5 volume 6, September-October 2015
ISSN 2249-9954

The injective hull of M is an injective module, hence it is a semi-extending module. On the other hand, M $\leq_{sem}$ E(M), so we find a semi-essential submodule of a semi-extending module, but not semi-extending module.

12. If both of M_1 and M_2 are isomorphic modules and M_1 is a semi-extending module, then M_2 is a semi-extending module.

Proposition (1.3): If a nonzero R-module M is a semi-extending and an indecomposable, then M is a semi-uniform module.

Proof: Let (0) ≠ N ≤ M, since M is a semi-extending module, then N $\leq_{sem}$ H, where H is direct summand of M. But M is an indecomposable module, then either H = (0) or H = M. But N ≠ (0), therefore H = M and we are done.

Recall that a nonzero R-module M is called **fully essential**, if every nonzero semi-essential submodule of M is an essential submodule of M [2], and an R-module M is called **fully prime**, if every proper submodule of M is a prime submodule [5], and we show in [15] that every fully prime module is a fully essential module.

Corollary (1.4): Let M be a semi-extending and indecomposable module. If M is a fully essential module, then M is a uniform module.

The following theorem gives a characterization for semi-extending modules.

Theorem (1.5): An R-module M is a semi-extending module, if and only if every St-closed submodule in M is a direct summand of M.

Proof: **⇒)** Let N be an St-closed submodule in M, since M is a semi-extending module, then N $\leq_{sem}$ H, where H is a direct summand of M. By the definition of St-closed submodule, we have N = H, that is N is a direct summand of M.

⇐) Let N be a submodule of M, if N = (0), then clearly N is a semi-essential submodule in a direct summand of M. If N ≠ (0), then by [16, Prop (1.4)], there exists an St-closed submodule H in M such that N $\leq_{sem}$ H. By hypotheses, H is a direct summand of M, so N $\leq_{sem}$ H, therefore M is a semi-extending module.

Proposition (1.6): Let A and B be submodules of a semi-extending R-module M. If A ∩ B is an St-closed in M, then A ∩ B is a direct summand of A and B.

Proof: By Th (1.5), $A \cap B$ is a direct summand of M. But $A \cap B \leq A$, thus $A \cap B$ is a direct summand of A [12, Lemma (2.4.3)].

In a similar way, we can prove that $A \cap B$ is a direct summand of B.

Proposition (1.7): Let M be a nonzero semi-extending module. Assume that $M = A \oplus B$, where A and B are submodules of M. If M is a fully essential module, then both of A and B are semi-extending modules, provided that ann A + ann B = R.

Proof: We want to prove that A is a semi-extending module, If one of the submodules A and B is equal to zero, then we can easily show that A is a semi-extending module. Otherwise; let K be an St-closed submodule in A, if K = (0), then K is a direct submodule of A. If $K \neq (0)$, we have to show that $K \oplus B$ is an St-closed submodule in M. Since B is an St-closed submodule in B and K is an St-closed submodule in A, then by [16. Prop (1.18)], $K \oplus B$ is an St-closed submodule in M. But M is a semi-extending module, therefore $K \oplus B$ is a direct summand of M. Thus, $M = (K \oplus B) \oplus D = K \oplus (B \oplus D)$, where D is a submodule of M, that is K is a direct summand of M. Now since $K \leq A$, then by [12, Lemma (2.4.3)], K is a direct summand of A.

We can generalize Prop (1.7) for any finite number of submodules.

Proposition (1.8): Let $M = \bigoplus_{i=1}^{n} A_i$ be a nonzero semi-extending R-module, where A_i is a submodule of M for each i; i = 1, 2, …, n. If M is a fully essential module then A_i is a semi-extending module, provided that ann A_1 + ann A_2 + …… + ann A_n = R.

In the following theorem we give under useful condition another characterization for semi-extending modules.

Theorem (1.9): Let M be an R-module such that $M \cap A \leq_{Stc} M \ \forall A \leq^{\oplus} E(M)$, then the following statement are equivalent:

1. M is a semi-extending module.
2. If A is a direct summand of injective hull E(M) of M, then $A \cap M$ is a direct summand of M.

Proof: **⇒)** Let A be a direct summand of E(M) by hypotheses $A \cap M \leq_{Stc} M$ since M is a semi-extending module, then $A \cap M$ is a direct summand of M.

⇐) Let A be a submodule of M and let B be a relative complement for A in M, then by [8, Prop (1.3), P.17], $A \oplus B \leq_e M$. Since M is an essential submodule in E(M), so by [8,Prop (1.1), P.16], $A \oplus B \leq_e E(M)$, thus $E(M) = E(A \oplus B) = E(A) \oplus E(B)$. Since E(A) is a direct summand of E(M), then by assumption $E(A) \cap M$ is a direct summand of M. Now A is an essential submodule of E(A) and M is an essential submodule of M, thus by [8,Prop (1.1), P.16], $A = A \cap M \leq_e E(A) \cap M$, so by [10] $A \leq_{sem} E(A) \cap M$ which is a direct summand of M.

From Th (1.5) and Th (1.9) we get the following theorem.

Theorem (1.10): Let M be an R-module such that $M \cap A \leq_{Stc} M \ \forall A \leq^{\oplus} E(M)$, then the following statements are equivalent:

1. M is a semi-extending module.
2. Every St-closed submodule in M is a direct summand of M.
3. If A is a direct summand of injective hull E(M) of M, then $A \cap M$ is a direct summand of M.

Proof: **(1) ⇒ (2)** See Th (1.5).

(2) ⇒ (3) Let A be a direct summand of the injective hull E(M) of M. By hypothesis $A \cap M$ is an St-closed submodule in M. And by using (2) we get the result.

(3) ⇒ (1) See Th (1.9).

The following proposition deals with the hereditary of the semi-extending property between R-module M and R itself.

Proposition (1.11): Let M be a finitely generated, faithful and multiplication R-module. If R is a semi-extending module, then M is a semi-extending module.

Proof: Let N be an St-closed submodule in M, then $N = (N :_R M)M$ [3]. Since N is an St-closed submodule in M, then by [16, Prop (2.5)], $(N :_R M)$ is an St-closed ideal in R. But R is a semi-extending ring, then $(N :_R M)$ is a direct summand of R. Thus $R = (N :_R M) \oplus J$, where J is an ideal of R, hence $M = RM = ((N :_R M) \oplus J)M = (N :_R M)M + JM$. Since M is a faithful and multiplication module, so by [3], $(N :_R M)M \cap JM = ((N :_R M) \cap J)M$, and by the definition of the direct sum, we have $(N :_R M) \cap J = (0)$. Thus $(N :_R M)M \cap JM = (0)M = (0)$, therefore $M = (N :_R M)M \oplus JM$, that is $(N :_R M)M = N$ is a direct summand of M.

In the following theorem, we see that the converse of Prop (1.11) is true when the condition (*) holds, where:

Condition (*): For any two ideals P and J of R, if P is a prime ideal of J, then PM is a prime submodule of JM.

Theorem (1.12): Let M be a finitely generated, faithful and multiplication R-module, then M is a semi-extending module if and only if R is a semi-extending ring, provided that M satisfies condition (*).

Proof: **⇒)** Let I be an St-closed ideal in R. Since M is a faithful, multiplication, and satisfies the condition (*), then by [16, Prop (2.4)], IM is an St-closed submodule in M. But M is a semi-extending module, so IM is a direct summand of M. Thus $M = IM \oplus N$, where N is a submodule of M. By [3], $N = (N :_R M)M$, so $M = IM \oplus (N :_R M)M = (I + (N :_R M))M$. Now by [3], $IM \cap (N :_R M)M = (0) = (I \cap (N :_R M))M$. This implies that $I \cap (N :_R M) \subseteq \text{ann } M$. But M is a faithful module, so $I \cap (N :_R M) = (0)$. Thus, $M = RM = (I + (N :_R M))M$, since M is a finitely generated, faithful and multiplication module, then $R = I \oplus (N :_R M)$ [3, Th (3.1)], therefore I is a direct summand of R.

⇐) It is Prop (1.11).

Note that, the condition (*) which mentioned in Prop (1.12) is not hold in general, as shown in the following example.

Example (1.13): The Z_4-module Z_4 is not satisfying the condition (*), since there exists a prime ideal $I = \{\bar{0}, \bar{2}\}$ of the ring Z_4, with IZ_4 is not prime submodule of Z_4. In fact $IM = \{\sum_{i,j} a_i m_j \mid a_i \in I, m_j \in Z_4\} = (\bar{0})$ is clearly not prime submodule of Z_4.

We end this section by the following example, which shows that the semi-extending module is a generalization of extending module.

Example (1.14): Consider the Z-module $M = Z_8 \oplus Z_2$. The number of submodules of M are eleven which are $<(\bar{0},\bar{0})>$, $<(\bar{1},\bar{0})>$, $<(\bar{0},\bar{1})>$, $<(\bar{1},\bar{1})>$, $<(\bar{2},\bar{0})>$, $<(\bar{2},\bar{1})>$, $<(\bar{4},\bar{0})>$, $<(\bar{4},\bar{1})>$, $<(\bar{0},\bar{1}),(\bar{4},\bar{0})>$, $<(\bar{2},\bar{0}),(\bar{4},\bar{1})>$, and M. This module is not extending module, since $<(\bar{2},\bar{1})>$ is a closed submodule in M, but not direct summand submodule in M [6]. On the other hand, M is a semi-extending module, since all St-closed submodules in M are $<(\bar{0},\bar{1})>$, <

$(\bar{4},\bar{1})$ >, and M, see [16, Ex and Rem (1.2) (1)], and both of them are direct summand of M, in fact M = < $(\bar{1},\bar{1})$ > ⊕ < $(\bar{0},\bar{1})$ > and M = < $(\bar{1},\bar{0})$ > ⊕ < $(\bar{4},\bar{1})$ >.

§2. Other results

This section is devoted to discuss other results on semi-extending modules. So we study the conditions under which semi-extending module can be extending module. Moreover, we discuss the relationships between semi-extending modules and some other types of modules, such as CLS-modules and FI-extending modules.

Theorem (2.1): Let M be an R-module, assume that any nonzero semi-essential extension of any submodule of M is a fully essential. Then M is an extending module if and only if M is a semi-extending module.

Proof: ⇒) It is clear

⇐) let N be a submodule of M, since M is a semi-extending module, then N is a semi-essential submodule in a direct summand of M say H. If H = (0), then N must be equal to zero, and clearly N is an essential submodule in a direct summand of M. Otherwise, by assumption H is a fully essential module, therefore N is an essential submodule of H, hence M is an extending module.

In similar way we can prove the following theorem.

Theorem (2.2): Let M be an R-module, if every nonzero direct summand of M is a fully essential module, then M is an extending module if and only if M is a semi-extending module.

Theorem (2.3): Let M be a fully prime R-module. Then M is a semi-extending module if and only if M is an extending module.

Proof: ⇒) Let N be a submodule of M, if N = (0), then clearly N is an essential submodule in a direct summand of M. If N ≠ (0), and since M is a semi-extending module, then N is a semi-essential submodule in a direct summand of M, say H. But M is a fully prime module, so by [15, Prop (2.1)], N is essential submodule of H. That is M is an extending module.

⇐) It is clear

Theorem (2.4): Let M be an R-module such that for every submodule X of M, there exists an St-closed submodule H of M with $X \leq_e H$. Then M is a semi-extending module if and only if M is an extending module.

Proof: **⇒)** Assume that M is a semi-extending module, and let $X \leq M$. By hypothesis, there exists an St-closed submodule H of M such that $X \leq_e H$. Since M is a semi-extending module, then H is a direct summand of M, and hence M is an extending module.

⇐) It is clear

Recall that an R-module M is called a **CLS-module**, if every y-closed submodule of M is a direct summand of M [14], where a submodule N of M is called **y-closed**, if if $\frac{M}{N}$ is a nonsingular module [8, P.42]

We don't know if there is a direct implication between semi-extending module and CLS-module. However, we satisfy that under certain condition as the following proposition shows. Before that we need the following definition.

Definition (2.5): [8, P.31]

Let M be an R-module. The singular submodule of M (denoted by Z(M)) is defined by: $Z(M) = \{x \in M: ann(x) \leq_e R\}$ and it is called **singular submodule** of M. If Z(M) = M, then M is called **singular module**, and if Z(M) = (0), then M is called a **nonsingular module**.

Proposition (2.6): Let M be a nonsingular R-module. If M is a CLS-module, then M is a semi-extending module.

Proof: Let M be a CLS-module, and let N be an St-closed submodule in M. Since M is a nonsingular module. Then N is a y-closed submodule in M [16, Prop (1.24)]. But M is a CLS-module, thus N is a direct summand of M.

Recall that a submodule N of an R-module M is called **fully invariant** if $f(N) \leq N$ for each R-endomorphism f of M [7], and an R-module M is called **duo**, if every submodule of M is a fully invariant. An R-module M is called an **FI-extending,** if every fully invariant submodule of M is an essential in a direct summand of M [1, P.21]. It is clear that every extending module is an FI-extending module, then we can give the following proposition.

Proposition (2.7): Let M be a duo R-module. If M is an FI-extending module, then M is a semi-extending module.

Proof: Let N be a submodule of M. Since M is a duo module, then N is a fully invariant submodule of M. But M is an FI-extending module, then N $\leq_e$ A, where A is a direct summand of M. That is M is a semi-extending module.

The converse of Prop (2.7) is true when we replace the condition "duo module" by the condition "every nonzero direct summand of M is fully essential module", as we see in the following Proposition.

Proposition (2.8): Let M be an R-module, such that every nonzero direct summand of M is a fully essential module. If M is a semi-extending module, then M is an FI-extending module.

Proof: Since M is a semi-extending module, then by Th (2.2), M is an extending module, and hence M is an FI-extending module.

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