ON HERMITE-HADAMARD-TYPE INEQUALITIES FOR STRONGLY-LOG CONVEX STOCHATIC PROCESSES

MUHARREM TOMAR, ERHAN SET, AND NURGÜL OKUR BEKAR

Abstract. In the present the work we introduce strongly logarithmic convex stochastic processes. Also, we obtain Hermite-Hadamard type integral inequalities for these processes.

1. Introduction

In recent years, inequalities are playing a very significant role in all fields of mathematics, and present a very active and attractive field of research. One of the significant inequalities is well known the Hermite-Hadamard integral inequality.

A function $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, is said to be a convex function on $I$ if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$. If the reversed inequality in (1.1) holds, then $f$ is concave. For some recent results related to this classic result, see the books [3, 4, 5, 6] and the papers [14, 15, 16, 17, 18, 19, 20, 21] where further references are given.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval $I$ of real numbers and $a < b$. The following double inequality

$$(1.2) \quad f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_{a}^{b} f(x)dx \leq \frac{f(a) + f(b)}{2}$$

is well known in the literature as Hadamard’s inequality. Both inequalities hold in the reversed direction if $f$ is concave.

Recently, log-convex functions have gained much interest in mathematics and its sub-areas such as optimization theory. A function $f : I \to [0, \infty)$ is said to be log-convex (or multiplicatively convex) if $\log(f)$ is convex or namely the following inequality

$$(1.3) \quad f(tx + (1-t)y) \leq [f(x)]^t[f(y)]^{1-t}$$

holds for all $x, y \in I$ and $t \in [0,1]$. Moreover, any log-convex function is a convex function since the inequality

$$(1.4) \quad [f(x)]^t[f(y)]^{1-t} \leq tf(x) + (1-t)f(y)$$
holds for all \( x, y \in I \) and \( t \in [0, 1] \). [2, p.7]

Recall that a function \( f : I \to \mathbb{R} \) is called strongly convex with modulus \( c > 0 \), if

\[
f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) - ct(1 - t)(x - y)^2
\]

for all \( x, y \in I \) and \( t \in (0, 1) \).[1]  
Recall also that a function \( f : I \to (0, \infty) \) is called strongly log-convex function with modulus \( c > 0 \) if

\[
f(tx + (1 - t)y) \leq [f(x)]^t[f(y)]^{(1-t)} - ct(1 - t)(x - y)^2
\]

for all \( x, y \in I \) and \( t \in (0, 1) \).[28]

2. Preliminaries

Let \((\Omega, \rightarrow ciFourier, P)\) be an arbitrary probability space. A function \( X : \Omega \to \mathbb{R} \) is called a random variable if it is \( ciFourier\)-measurable. Let \((\Omega, \rightarrow ciFourier, P)\) be an arbitrary probability space and let \( T \subset \mathbb{R} \) be time. A collection of random variables \( X(t, w), \ t \in T \) with values in \( \mathbb{R} \) is called a stochastic process. If \( X(t, w) \) takes values in \( S = \mathbb{R}^d \), it is called a vector-valued stochastic process. If the time \( T \) can be a discrete subset of \( \mathbb{R} \), then \( X(t, w) \) is called a discrete time stochastic process. If time is an interval, \( \mathbb{R}^+ \) or \( \mathbb{R} \), it is called a stochastic process with continuous time.

For any fixed \( \omega \in \Omega \), one can regard \( X(t, w) \) as a function of \( t \). It is called a sample function of the stochastic process. In the case of a vector-valued process, it is a sample path, a curve in \( \mathbb{R}^d \). Throughout the paper, we restrict our attention to stochastic processes with continuous time, i.e., index set \( T = [0, \infty) \).

Now, let’s start some essential definitions.

**Definition 1.** A real-valued stochastic process \( \{X(t) | t \in I\} \) is said to be, [?]

(i) continuous in probability in \( I \) if

\[
P - \lim_{t \to t_0} X(t, \cdot) = X(t_0, \cdot)
\]

(where \( P-\lim \) denotes limit in probability) or equivalently

\[
limit_{t \to t_0} P\{|X(t, \cdot) - X(t_0, \cdot)| > \varepsilon\} = 0
\]

for any arbitrary small enough \( \varepsilon > 0 \) and all \( t_0 \in I \).

(ii) mean-square continuous (or continuous in quadratic mean) in \( I \) if

\[
\lim_{t \to t_0} E[(X(t) - X(t_0))^2] = 0
\]
such that \( E[X(t)^2] < \infty \), for all \( t_0 \in I \).

(iii) mean-square differentiable in \( I \) if it is mean square continuous and there exists a process \( X'(t, \cdot) \) ("speed" of the process) such that

\[
\lim_{t \to t_0} E\left[\left(\frac{X(t) - X(t_0)}{t - t_0} - X'(t_0)\right)^2\right] = 0.
\]

Different types of continuity concepts can be defined for stochastic processes. Surely (everywhere) and almost surely (almost everywhere or sample path) convergences are rarely used in applications due to restrictive requirement, that is, as \( t \to t_0 \), \( X(t, \omega) \) has to approach \( X(t_0, \omega) \) for each outcome \( \omega \in S \subseteq \Omega \). For further
reading on stochastic calculus, reader may refer to [?].

**Definition 2.** Let \( X : I \times \Omega \to \mathbb{R} \) be a stochastic process with \( E[X(t)^2] < \infty \) for all \( t \in I \). Let \([a, b] \subset I, a = t_0 < t_1 < \ldots < t_n = b \) be a partition of \([a, b]\) and \( \Theta_k \in [t_{k-1}, t_k] \) arbitrarily for \( k = 1, \ldots, n \). A random variable \( Y : \Omega \to \mathbb{R} \) is called **mean-square integral of the process** \( X(t) \) on \([a, b]\) if the following identity holds:

\[
(2.1) \quad \lim_{n \to \infty} E\left[ \sum_{k=1}^{n} X(\Theta_k)(t_k - t_{k-1}) - Y \right] = 0.
\]

Then we can write

\[
\int_{a}^{b} X(t, \cdot) dt = Y(\cdot) \ (a.e.).
\]

Mean square integral operator is increasing, that is,

\[
\int_{a}^{b} X(t, \cdot) dt \leq \int_{a}^{b} Z(t, \cdot) dt \ (a.e.)
\]

where \( X(t, \cdot) \leq Z(t, \cdot) \) (a.e.) in \([a, b]\).

Let \((\Omega, A, P)\) be a probability space and \( T \subset \mathbb{R} \) be an interval. We say that a stochastic process \( X : T \times \Omega \to \mathbb{R} \) is **convex** if

\[
X(\lambda u + (1 - \lambda) v, \cdot) \leq \lambda X(u, \cdot) + (1 - \lambda) X(v, \cdot)
\]

for all \( u, v \in T \) and \( \lambda \in [0, 1] \). This class of stochastic process are denoted by \( C \).[13]

Let \((\Omega, A, P)\) be a probability space and \( T \subset \mathbb{R} \) be an interval. We say that a stochastic process \( X : T \times \Omega \to [0, \infty) \) is **log–convex** if

\[
X(\lambda u + (1 - \lambda) v, \cdot) \leq [X(u, \cdot)]^{\lambda} \cdot [X(v, \cdot)]^{1 - \lambda}
\]

for all \( u, v \in T \) and \( \lambda \in [0, 1] \). Log–convex stochastic processes have been introduced by Tomar et al. in [26] and they proved following theorem in this article. Let us denote by \( A(a, b) \) the arithmetic mean of the nonnegative real numbers, and by \( G(a, b) \) the geometric mean of the same numbers.

**Theorem 1.** Let \( X : T \times \Omega \to (0, \infty) \) be a log–convex stochastic process on \( T \times \Omega \) and \( u, v \in T \) with \( u < v \). Then, one has the inequalities:

\[
(2.3) \quad X\left(\frac{u + v}{2}, \cdot\right) \leq \exp\left[ \frac{1}{v - u} \int_{u}^{v} \ln[X(t, \cdot)] dt \right]
\]

\[
\leq \frac{1}{v - u} \int_{u}^{v} G(X(t, \cdot), X(u + v - t, \cdot)) dt
\]

\[
\leq \frac{1}{v - u} \int_{u}^{v} X(t, \cdot) dt
\]

\[
\leq L(X(u, \cdot), X(v, \cdot)),
\]
where \( L(p, q) \) is the logarithmic mean of strictly positive real numbers \( p, q \), i.e.,

\[
L(p, q) = \frac{p - q}{\ln p - \ln q} \quad \text{if} \quad p \neq q \quad \text{and} \quad L(p, p) = p.
\]

Also, note that the related results for convex stochastic processes and various types of convex stochastic processes can be seen in [8, 9, 13, 27, 22, 23, 24, 25].

The main subject of this paper is to introduce strongly-log-convex stochastic processes with modulus \( c > 0 \) and to give Hermite-Hadamard type inequalities for these processes, such as in (2.3).

\section{Hermite-Hadamard type inequalities for strongly log-convex stochastic processes}

\begin{definition}
Let \((\Omega, A, P)\) be a probability space and \(T \subset \mathbb{R}\) be an interval. We say that a stochastic process \(X : T \times \Omega \rightarrow [0, \infty)\) is strongly log-convex with modulus \( c > 0 \) if

\begin{equation}
X(\lambda u + (1 - \lambda) v, \cdot) \leq [X(u, \cdot)]^\lambda [X(v, \cdot)]^{1-\lambda} - c(1 - \lambda)(v - u)^2
\end{equation}

for all \( u, v \in T \) and \( \lambda \in (0, 1) \).

The following result offers the Hermite-Hadamard type inequalities for strongly log-convex stochastic process.

\begin{theorem}
If a stochastic process \(X : T \times \Omega \rightarrow (0, \infty)\) be a strongly log-convex with modulus \( c > 0 \) and integrable on \( T \times \Omega \), we have

\begin{equation}
X\left(\frac{u + v}{2}, \cdot\right) + \frac{(v - u)^2}{12} \leq \frac{1}{v - u} \int_u^v G(X(t, \cdot), X(u + v - t, \cdot)) \, dt \leq \frac{1}{v - u} \int_u^v X(t, \cdot) \, dt \leq L(X(u, \cdot), X(v, \cdot)) - c\frac{(v - u)^2}{6} \leq A(X(u, \cdot), X(v, \cdot)) - c\frac{(v - u)^2}{6}
\end{equation}

for all \( u, v \in I \) with \( u < v \).
\end{theorem}

\begin{proof}
From (3.1) and arithmetic-geometric mean, we have

\begin{equation}
X(\lambda s + (1 - \lambda) z, \cdot) \leq [X(s, \cdot)]^\alpha [X(z, \cdot)]^{1-\alpha} - c\alpha(1 - \alpha)(v - u)^2 \leq \frac{\alpha}{2} X(s, \cdot) + (1 - \alpha) X(z, \cdot) - c\alpha(1 - \alpha)(v - u)^2.
\end{equation}

If we take \( \alpha = \frac{1}{2} \) in (3.3), we have
\begin{equation}
X\left(\frac{s + z}{2}, \cdot \right) \leq \sqrt{X(s, \cdot)X(z, \cdot)} - c\frac{(z - s)^2}{4}
\leq \frac{X(s, \cdot) + X(z, \cdot)}{2} - c\frac{(z - s)^2}{4}.
\end{equation}

i.e., \( s = \lambda u + (1 - \lambda) v, \ z = (1 - \lambda) u + \lambda v, \)

\begin{equation}
X\left(\frac{u + v}{2}, \cdot \right)
\leq \sqrt{X(\lambda u + (1 - \lambda) v, \cdot)X((1 - \lambda) u + \lambda v, \cdot)} - c\frac{(v - u)^2(1 - 2\lambda)^2}{4}
\leq \frac{X(\lambda u + (1 - \lambda) v, \cdot) + X((1 - \lambda) u + \lambda v, \cdot)}{2} - c\frac{(v - u)^2(1 - 2\lambda)^2}{4}.
\end{equation}

Integrating the inequality (3.5) on \((0, 1)\) over \( \lambda \), and taking into account,

\[
\int_0^1 X(\lambda u + (1 - \lambda) v, \cdot) d\lambda = \int_0^1 X((1 - \lambda) u + \lambda v, \cdot) d\lambda
\]

we obtain

\begin{equation}
X\left(\frac{u + v}{2}, \cdot \right) \leq \frac{1}{v - u} \int_u^v G(X(\lambda, \cdot), X(u + v - t, \cdot)) dt - c\frac{(v - u)^2}{12}
\leq \frac{1}{v - u} \int_u^v A(X(\lambda, \cdot), X(u + v - t, \cdot)) dt - c\frac{(v - u)^2}{12}.
\end{equation}

And so,

\begin{equation}
X\left(\frac{u + v}{2}, \cdot \right) + c\frac{(v - u)^2}{12}
\leq \frac{1}{v - u} \int_u^v G(X(\lambda, \cdot), X(u + v - t, \cdot)) dt
\leq \frac{1}{v - u} \int_u^v X(\lambda, \cdot) dt.
\end{equation}

Since \( X \) is a strongly log-convex function on \( T \times \Omega \), for \( s = u \) and \( z = v \), we get

\begin{equation}
X(\lambda u + (1 - \lambda) v, \cdot) \leq \left[ X(u, \cdot) \right]^\lambda \left[ X(v, \cdot) \right]^{1-\lambda} - c\lambda(1 - \lambda)(v - u)^2
\leq \lambda X(u, \cdot) + (1 - \lambda) X(v, \cdot) - c\lambda(1 - \lambda)(v - u)^2.
\end{equation}

Integrating the inequality (3.8) on \((0, 1)\) over \( \lambda \),

\[
\frac{1}{v - u} \int_u^v X(\lambda, \cdot) d\lambda \leq X(v, \cdot) \int_0^1 \frac{X(u, \cdot)^\lambda}{X(v, \cdot)^{1-\lambda}} d\lambda - c(v - u)^2 \int_0^1 \lambda(1 - \lambda) d\lambda
\leq X(u, \cdot) \int_0^1 \lambda d\lambda + X(v, \cdot) \int_0^1 (1 - \lambda) d\lambda - c(v - u)^2 \int_0^1 \lambda(1 - \lambda) d\lambda,
\]

and thereby
\[
\begin{align*}
(3.9) \quad & \frac{1}{v-u} \int_u^v X(t, \cdot) \, dt \\
& \leq L(X(u, \cdot), X(v, \cdot)) - c\frac{(v-u)^2}{6} \\
& \leq A(X(u, \cdot), X(v, \cdot)) - c\frac{(v-u)^2}{6}
\end{align*}
\]

So, from (3.7) and (3.9), the theorem is proved. \(\square\)

**Theorem 3.** If a stochastic process \(X : T \times \Omega \to (0, \infty)\) be a strongly log-convex with modulus \(c > 0\) and integrable on \(T \times \Omega\), we have

\[
(3.10) \quad \frac{1}{v-u} \int_u^v X(t, \cdot) X(u+v-t, \cdot) \, dt \\
\leq X(u, \cdot) X(v, \cdot) + \frac{c^2(v-u)^4}{30} \\
- \frac{4c(v-u)^2}{\ln[X(u, \cdot) - X(v, \cdot)]^2} \left[ A(X(u, \cdot), X(v, \cdot)) + L(X(u, \cdot), X(v, \cdot)) \right] \\
\leq \frac{2[A(X(u, \cdot), X(v, \cdot))]^2 + [G(X(u, \cdot), X(v, \cdot))]^2}{3} \\
- \frac{cA(X(u, \cdot), X(v, \cdot))(v-u)^2}{3} + \frac{c^2(v-u)^4}{30}
\]

for all \(u, v \in I\) with \(u < v\).

**Proof.** Since \(X\) is strongly log-convex stochastic process with modulus \(c > 0\), we have that for all \(\lambda \in (0, 1)\),

\[
(3.11) X(\lambda u + (1 - \lambda) v, \cdot) \leq [X(u, \cdot)]^\lambda [X(v, \cdot)]^{1-\lambda} - c\lambda(1-\lambda)(v-u)^2 \\
\leq \lambda X(u, \cdot) + (1-\lambda) X(v, \cdot) - c\lambda(1-\lambda)(v-u)^2
\]

and

\[
(3.12) X((1-\lambda) u + \lambda v, \cdot) \leq [X(u, \cdot)]^{1-\lambda} [X(v, \cdot)]^\lambda - c\lambda(1-\lambda)(v-u)^2 \\
\leq (1-\lambda) X(u, \cdot) + \lambda X(v, \cdot) - c\lambda(1-\lambda)(v-u)^2
\]

Multiplying both sides of (3.11) by (3.12), it follows that

\[
(3.13) \quad X(\lambda u + (1 - \lambda) v, \cdot) X((1-\lambda) u + \lambda v, \cdot) \\
\leq X(u, \cdot) X(v, \cdot) + c^2\lambda^2(1-\lambda)^2(v-u)^4 \\
- c\lambda(1-\lambda)(v-u)^2 \left( X(v, \cdot) \left[ \frac{X(u, \cdot)}{X(v, \cdot)} \right]^\lambda + X(u, \cdot) \left[ \frac{X(v, \cdot)}{X(u, \cdot)} \right]^\lambda \right) \\
\leq \lambda(1-\lambda) \left( [X(u, \cdot)]^2 + [X(v, \cdot)]^2 \right) + \lambda^2(1-\lambda)^2 X(u, \cdot) X(v, \cdot) \\
- c(v-u)^2 \lambda(1-\lambda) [X(u, \cdot) + X(v, \cdot)] + c^2\lambda^2(1-\lambda)^2(v-u)^4
\]
Integrating the inequality (3.13) with respect to \( \lambda \) over \((0,1)\) and , we obtain

\[
(3.14) \quad \int_0^1 X(\lambda u + (1-\lambda)v, \cdot) X((1-\lambda)u + \lambda v, \cdot) d\lambda \\
\leq \int_0^1 X(u, \cdot) X(v, \cdot) d\lambda + c^2 (v-u)^4 \int_0^1 \lambda^2 (1-\lambda)^2 d\lambda \\
- c(v-u)^2 X(v, \cdot) \int_0^1 \lambda (1-\lambda) \left[ \frac{X(u, \cdot)}{X(v, \cdot)} \right]^\lambda d\lambda \\
- c(v-u)^2 X(u, \cdot) \int_0^1 \lambda (1-\lambda) \left[ \frac{X(v, \cdot)}{X(u, \cdot)} \right]^\lambda d\lambda \\
\leq \left( [X(u, \cdot)]^2 + [X(v, \cdot)]^2 \right) \int_0^1 \lambda (1-\lambda) d\lambda + X(u, \cdot) X(v, \cdot) \int_0^1 \lambda^2 (1-\lambda)^2 d\lambda \\
- c(v-u)^2 [X(u, \cdot) + X(v, \cdot)] \int_0^1 \lambda (1-\lambda) d\lambda + c^2 (v-u)^4 \int_0^1 \lambda^2 (1-\lambda)^2 d\lambda
\]

Integrating by parts for \( I_1 \) and \( I_2 \) integrals, we obtain

\[
I_1 = \left( \int_0^1 \lambda (1-\lambda) \left[ \frac{X(u, \cdot)}{X(v, \cdot)} \right]^\lambda d\lambda \\
= \lambda(1-\lambda) \left[ \frac{1}{\ln \left[ \frac{X(u, \cdot)}{X(v, \cdot)} \right]} \left[ \frac{X(u, \cdot)}{X(v, \cdot)} \right]^\lambda \right] \bigg|_0^1 - \frac{1}{\ln \left[ \frac{X(u, \cdot)}{X(v, \cdot)} \right]} \left[ \frac{X(u, \cdot)}{X(v, \cdot)} \right]^\lambda \bigg|_0^1 - \ln \left[ \frac{X(u, \cdot)}{X(v, \cdot)} \right] \left[ \frac{X(u, \cdot)}{X(v, \cdot)} \right]^\lambda \bigg|_0^1 \\
= - \frac{1}{\ln \left[ \frac{X(u, \cdot)}{X(v, \cdot)} \right]} \left( 1-2\lambda \right) \left[ \ln \left[ \frac{X(u, \cdot)}{X(v, \cdot)} \right] \right] \left[ \frac{X(u, \cdot)}{X(v, \cdot)} \right]^\lambda \bigg|_0^1 + \frac{2}{\ln \left[ \frac{X(u, \cdot)}{X(v, \cdot)} \right]} \left[ \frac{X(u, \cdot)}{X(v, \cdot)} \right]^\lambda \bigg|_0^1 \\
= \frac{1}{X(v, \cdot) (\ln X(u, \cdot) - \ln X(v, \cdot))^2} + \frac{2X(u, \cdot) - 2X(v, \cdot)}{[\ln X(u, \cdot) - \ln X(v, \cdot)]^2}
\]

and similarly we get,

\[
(3.16) \quad I_2 = \int_0^1 \lambda (1-\lambda) \left[ \frac{X(v, \cdot)}{X(u, \cdot)} \right]^\lambda d\lambda \\
= \frac{1}{X(u, \cdot)} \left[ \frac{X(u, \cdot) + X(v, \cdot)}{\ln X(u, \cdot) - \ln X(v, \cdot)} \right]^2 + \frac{2X(v, \cdot) - 2X(u, \cdot)}{[\ln X(u, \cdot) - \ln X(v, \cdot)]^2}.
\]
And also we get,

\[
(3.17) \quad \left( [X(u, \cdot)]^2 + [X(v, \cdot)]^2 \right) \int_0^1 \lambda (1 - \lambda) d\lambda + X(u, \cdot) X(v, \cdot) \int_0^1 \lambda^2 (1 - \lambda)^2 d\lambda \\
- c(v - u)^2 [X(u, \cdot) + X(v, \cdot)] \int_0^1 \lambda (1 - \lambda) d\lambda + c^2 (v - u)^4 \int_0^1 \lambda^2 (1 - \lambda)^2 d\lambda \\
= \frac{[X(u, \cdot)]^2 + [X(v, \cdot)]^2}{6} + \frac{2X(u, \cdot) X(v, \cdot)}{3} - \frac{c(v - u)^2 [X(u, \cdot) + X(v, \cdot)]}{6} + \frac{c^2 (v - u)^4}{6} \\
= \frac{2 [A(X(u, \cdot), X(v, \cdot))]^2 + [G(X(u, \cdot), X(v, \cdot))]^2}{3} - \frac{cA(X(u, \cdot), X(v, \cdot))(v - u)^2}{3} + \frac{c^2 (v - u)^4}{30}.
\]

Putting (3.15), (3.16) and (3.17), and if we change the variable \( t := \lambda u + (1 - \lambda)v \), \( \lambda \in (0, 1) \), we get the required inequality in (3.10). This proves the theorem. \( \square \)

**References**


**Department of Mathematics, Faculty of Science and Arts, Ordu University, 52200 Ordu, Turkey**

*E-mail address: mularreptomar@gmail.com*

**Current address:** Department of Mathematics, Faculty of Science and Arts, Ordu University, 52200 Ordu, Turkey

*E-mail address: erhanset@yahoo.com*

**Department of Statistics, Faculty of Science and Arts, Giresun University, Giresun, Turkey**