Some New Iterative Methods for Solving Nonlinear Equations

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Abstract: In this paper, we suggest and analyze some new iterative methods for solving nonlinear equations by using a new series expansion of the nonlinear function. Some special cases are also discussed. These new methods can be viewed as significant modification and improvement of the Newton method. Several examples are given to illustrate the efficiency and robustness of these methods.

Key words: Householder method · Iterative method · Convergence · Nonlinear equation

INTRODUCTION

It is well known a wide class of linear and nonlinear problems which arise in different branches of mathematical such as physical, biomedical, regional, optimization, ecology, economics and engineering sciences can be formulated in terms of nonlinear equations. Iterative methods for finding the approximate solutions of the nonlinear equation \( f(x) = 0 \) are being developed using several different techniques including Taylor series, quadrature formulas, homotopy and decomposition techniques, see [1-5, 9-11, 13-17, 19, 21] and the references therein. Inspired and motivated by the ongoing research activities in this area, we suggest and analyze a new iterative method for solving nonlinear equations. To derive these iterative methods, we show that the nonlinear function can be approximated by a new series which can be obtained by using the trapezoidal rule and fundamental theorem of calculus. This new expansion is used to suggest these new iterative methods for solving nonlinear equations. We also consider the convergence analysis of these methods. Several examples are given to illustrate the efficiency and comparison with other methods.

Iterative Methods: It is well known that a wide class of problems, which arise in various fields of pure and applied sciences can be formulated in terms of nonlinear equations of the type.

\[
f(x) = 0
\] (1)

Various numerical methods have been developed using the Taylor series and other techniques. In this paper, we use another series of the nonlinear function \( f(x) \) which can be obtained by using the trapezoidal rule and the Fundamental Theorem of Calculus. To be more precise, we assume that \( \alpha \) is a simple root of (1) and \( \gamma \) is an initial guess sufficiently close to \( \alpha \). Now using the trapezoidal rule and fundamental theorem of calculus, one can show that the function \( f(x) \) can be approximated by the series

\[
f(x) = f(\gamma) + \frac{x-\gamma}{2}[f'(\gamma)]
\] (2)

where \( f(x) \) is the differential of \( f \).

From (1) and (2), we have

\[
x = \gamma - 2 \frac{f(\gamma)}{f'(\gamma)} - (x - \gamma) \frac{f'(x)}{f'(\gamma)}
\] (3)

Using (3), one can suggest the following iterative method for solving the nonlinear equations (1).

Algorithm 1: For a given initial choice \( x_n \), find the approximate solution \( x_{n+1} \) by the iterative scheme

\[
x_{n+1} = x_n - 2 \frac{f(x_n)}{f'(x_n)} -(x_{n+1} - x_n) \frac{f'(x_{n+1})}{f'(x_n)}, \quad n = 0,1,2,....
\]
Algorithm 1 is an implicit iterative method. To implement Algorithm 1, we use the predictor-corrector technique. Using the Newton method as a predictor and Algorithm 1, as a corrector, we suggest and analyze the following two-step iterative method for solving the nonlinear equation (1) and this is the main motivation of this note.

**Algorithm 2:** For a given initial choice $x_0$, find the approximate solution $x_{n+1}$ by the iterative schemes:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$x_{n+1} = x_n - 2 \frac{f(x_n)}{f'(x_n)} - (y_n - x_n) \frac{f'(y_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots$$

From Algorithm 2, we can deduce the following iterative method for solving the nonlinear equations $f(x) = 0$ which appears to be a new one.

**Algorithm 3:** For a given initial choice $x_0$, find the approximate solution $x_{n+1}$ by the iterative scheme

$$x_{n+1} = y_n - \frac{f(x_n)}{f'(x_n)} + \left[ \frac{f(x_n)}{f'(x_n)} \right] \frac{f'(y_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots$$

Algorithm 3 is called the modified Householder method for solving the nonlinear equations (1), which does not involve the second derivatives. From (1) and (2), we have

$$x = y - \frac{2(f(y))}{f'(y) + f'(y)}.$$

This fixed point formulation enables us to suggest the following iterative method for solving the nonlinear equation.

**Algorithm 4:** For a given initial choice $x_0$, find the approximate solution $x_{n+1}$ by the iterative schemes:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$x_{n+1} = x_n - 2 \frac{f(x_n)}{f'(x_n)} + f'(y_n), \quad n = 0, 1, 2, \ldots$$

Using the Taylor series expansion of $f'(y_n)$, one obtains the following iterative method for solving the nonlinear equation $f(x) = 0$.

**Algorithm 5:** For a given initial choice $x_0$, find the approximate solution $x_{n+1}$ by the iterative schemes:

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2[f'(x_n)]^2 - f(x_n)f''(x_n)} \quad n = 0, 1, 2, \ldots$$

which is known as the Halley method. It can easily shown that Halley method has cubic convergence. In a similar way, one can obtain several known and new iterative methods from these algorithms.

We now consider the convergence analysis of Algorithm 2. In a similar way, one can prove the convergence of Algorithm 3 and algorithm 4.

**Theorem 2.1:** Let $\alpha \in I$ be a simple zero of sufficiently differentiable function $f: I \subseteq R \to R$ for an open interval $I$. If $x_0$ is sufficiently close to $\alpha$ then the iterative method defined by Algorithm 2 has second-order convergence.

**Proof:** Let $\alpha$ be a simple zero of $f$. Then by expanding $f(x)$ and $f'(x)$ about $\alpha$ we have

$$f(x_n) = f'(\alpha) \left[ c_n + 2c_n^2 + c_2\varepsilon_n e_n + c_4\varepsilon_n e_n^4 + O(e_n^5) \right], \quad (4)$$

and

$$f'(x_n) = f'(\alpha) \left[ 1 + 2c_2\varepsilon_n e_n + 3c_3\varepsilon_n e_n^3 + 4c_4\varepsilon_n e_n^4 + 5c_5\varepsilon_n e_n^5 + O(e_n^6) \right], \quad (5)$$

where $c_k = \frac{1}{k!} f^{(k)}(\alpha), k = 2, 3, \ldots$ and $e_n = x_n - \alpha$.

Now, from (4) and (5), we have

$$f(x_n) = e_n - 2c_2\varepsilon_n^2 + 2c_2\varepsilon_n^2 + (7c_3 - 4c_2\varepsilon_n) + O(e_n^5). \quad (6)$$

From (6), we have

$$y_n = \alpha + c_2\varepsilon_n^2 + 2c_2\varepsilon_n^2 - (7c_3 - 4c_2\varepsilon_n) + O(e_n^5). \quad (7)$$

From (7), we have
From (5) and (8), we have
\[ f'(y_n) = f'(x_n) \left[ 1 + \frac{2}{2} e_n^2 + 2c_3 - c_3^2 \right] + \left( -11c_5^2 - 6c_4^4 + 8c_2^4 - O(e_n^4) \right). \]  
\[ (8) \]

From (7), we have
\[ f'(y_n)/f'(x_n) = -2c_2 e_n + (-3c_3 + 6c_2^2) e_n^2 + (-16c_2^3 - 4c_4 + 16c_2 c_3) e_n^3 + O(e_n^4). \]  
\[ (9) \]

From (7), we have
\[ y_n - x_n = -e_n + c_2 e_n^2 + 2(c_3 - c_3^2) e_n^3 - (7c_2 c_3 - 4c_4 c_2^3 - 3c_4) e_n^4 + O(e_n^4). \]  
\[ (10) \]

From (9) and (10), we have
\[ (y_n - x_n)^2 = -e_n + 3c_2 e_n^2 + (5c_3 - 10c_2^2) e_n^3 - (30c_2 c_3 + 30c_2^3 + 7c_4) e_n^4 + O(e_n^4). \]  
\[ (11) \]

Thus, from (6) and (11), we have
\[ x_{n+1} = x - c_2 e_n^2 + (c_3 + 6c_2^3) e_n^3 + O(e_n^4), \]  
\[ (12) \]

which implies that
\[ e_{n+1} = -c_2 e_n^2 + (c_3 + 6c_2^3) e_n^3 + O(e_n^4). \]  
\[ (10) \]

This shows that Algorithm 2 is second-order convergent.

**Numerical Results:** We present some examples to illustrate the efficiency of the new developed two-step iterative methods, see Table 1. We compare the Newton method (NM), Algorithm 2 (NR1) and Algorithm 3 (NR2). We used $\varepsilon = 10^{-10}$. The following stopping criteria is used for computer programs:

(i) $|x_{n+1} - x_n| < \varepsilon$,  
(ii) $|f(x_{n+1})| < \varepsilon$.

The examples are the same as in Chun [2].

\[ f_1(x) = \sin^2 x - x^2 + 1, \quad f_2(x) = x^2 - e^x - 3x + 2 \]
\[ f_3(x) = \cos x - x, \quad f_4(x) = (x - 1)^3 - 1 \]
\[ f_5(x) = x^3 - 10, \quad f_6(x) = xe^{x^2} - \sin^2 x + 3 \cos x + 5 \]
\[ f_7(x) = e^{x^2} + 7x^{10} - 1. \]

As for the convergence criteria, it was required that the distance of two consecutive approximations $\delta$ for the zero was less than $10^{-11}$. Also displayed is the number of iterations to approximate the zero (IT), the approximate zero $x_n$ and the value $f(x_n)$.

From the Table 1, we see that our method is comparable with the Newton Method. In fact, our methods can be considered as significant improvement of the Newton Method and can be considered as alternative method to other second order convergent methods of solving nonlinear equations.
Table 1: (Numerical Examples and Comparison)

<table>
<thead>
<tr>
<th>Method</th>
<th>IT</th>
<th>(x_i)</th>
<th>(f(x_i))</th>
<th>(\delta)</th>
</tr>
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<tbody>
<tr>
<td>(f_1, x_1 = -1)</td>
<td></td>
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<tr>
<td>NM</td>
<td>7</td>
<td>1.404491682153422260350868178</td>
<td>-1.04e-50</td>
<td>7.33e-26</td>
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<td>NR1</td>
<td>17</td>
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<td>6.77e-28</td>
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<tr>
<td>NR2</td>
<td>12</td>
<td>1.404491682153422260350868178</td>
<td>2.37e-45</td>
<td>3.49e-23</td>
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<tr>
<td>(f_2, x_1 = 1.7)</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>NM</td>
<td>6</td>
<td>0.25753028543986076045536730494</td>
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<td>9.10e-28</td>
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<tr>
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<td>0.25753028543986076045536730494</td>
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<td>(f_3, x_1 = 3.5)</td>
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<tr>
<td>NM</td>
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<td>2.06e-42</td>
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<tr>
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<td>3.54e-30</td>
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<tr>
<td>NR2</td>
<td>7</td>
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<td>(f_4, x_1 = 1.5)</td>
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<td></td>
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<tr>
<td>NM</td>
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<tr>
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<td>4.19e-19</td>
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<td>-7.57e-29</td>
<td>9.41e-16</td>
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</tbody>
</table>

**CONCLUSION**

In this paper, we have used a new series of the function \(f(x)\), which is obtained by using the trapezoidal rule and fundamental theorem of calculus. This series is used to suggest and analyze a new iterative method for solving the nonlinear equations. It is an interesting problem to use this expansion of the function to suggest and consider some new iterative methods for solving the variational inequalities and related problems, see [6-9, 17-20] and the reference therein. In our other papers, we will the homotopy perturbation method and some decompositions method to derive several iterative methods for solving the nonlinear equations. It is an interesting problem to derive the iterative methods for solving system of nonlinear equations.

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**REFERENCES**