Self-adaptive methods for general variational inequalities

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\textbf{A B S T R A C T}

It is well known that the general variational inequalities are equivalent to the fixed point problems and the Wiener–Hopf equations. In this paper, we use these alternative equivalent formulations to suggest and analyze some new self-adaptive iterative methods for solving the general variational inequalities. Our results can be viewed as a significant extension of the previously known results for variational inequalities. An example is given to illustrate the efficiency of the proposed method.

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\textbf{1. Introduction}

Variational inequalities introduced in the early sixties have played a critical and significant part in the study of several unrelated problems arising in finance, economics, network analysis, transportation, elasticity and optimization. Variational inequalities theory has witnessed an explosive growth in theoretical advances, algorithmic development and applications across all disciplines of pure and applied sciences, see [1–32]. It combines novel theoretical and algorithmic advances with a new domain of applications. In recent years, variational inequalities theory has seen a dramatic increase in its applications and numerical methods. As a result of these activities, variational inequalities have been extended in various directions using novel and innovative techniques. A useful and important generalization of variational inequalities is called the \textit{general variational inequality} involving the two nonlinear operators, see Noor [18]. We would like to remark that the general variational inequalities are also known as it Noor variational inequalities. Such types of general (Noor) variational inequalities arise in the study of elasticity with non-local friction laws, fluid flow through porous media and structural analysis, see [19–29]. It has been shown [11–15] that the general variational inequalities are equivalent to the fixed-point problems and the Wiener–Hopf equations using the projection operator technique. This equivalent formulation has been used to suggest and analyze some iterative methods. The convergence of these methods requires that the operator is both strongly monotone and Lipschitz continuous. Secondly, it is very difficult to evaluate the projection of the operator except for very simple cases. To overcome this disadvantage, He [14], Bnouhachem and Noor [3–8], Noor and Bnouhachem [27–29] and Noor [21–25] used these alternative equivalent formulations to suggest and analyze the modified projection iterative method for general variational inequalities.

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which is known as the general complementarity problem. For finding which is exactly the general variational inequality approach. To do so, we first define the set $K$ as

$$K = \{v : v \in H^2_0(\Omega) : v \geq \psi \text{ on } \Omega\},$$

which is a closed convex set in $H^2_0(\Omega)$, where $H^2_0(\Omega)$ is a Sobolev (Hilbert) space, see [1]. One can easily show that the energy functional associated with the problem (2.2) is

$$I[v] = -\int_0^1 \left( \frac{d^3v}{dx^3} \right) \left( \frac{dv}{dx} \right) dx - 2\int_0^1 f(x) \left( \frac{dv}{dx} \right) dx, \quad \text{for all } \frac{dv}{dx} \in K$$

$$= \int_0^1 \left( \frac{d^3v}{dx^3} \right)^2 dx - 2\int_0^1 f(x) \left( \frac{dv}{dx} \right) dx$$

$$= \langle Tu, g(v) \rangle - 2\langle f, g(v) \rangle$$

where

$$\langle Tu, g(v) \rangle = \int_0^1 \left( \frac{d^2u}{dx^2} \right) \left( \frac{d^2v}{dx^2} \right) dx$$

$$\langle f, g(v) \rangle = \int_0^1 f(x) \frac{dv}{dx} dx$$

and $g = \frac{d}{dx}$ is the linear operator. It is clear that the operator $T$ defined by (2.4) is linear, $g$-symmetric and $g$-positive. Using the technique of Noor [24], one can easily show that the minimum $u \in H$ of the functional $I[v]$ defined by (2.3) associated with the problem (2.2) on the closed convex set $K$ can be characterized by the inequality of type

$$\langle Tu, g(v) \rangle \geq \langle f, g(v) \rangle - g(u), \quad \forall g(v) \in K,$$

which is exactly the general variational inequality (2.1).

If $K^* = \{u \in H : (u, v) \geq 0, \forall v \in K\}$ is a polar (dual) cone of a convex cone $K$ in $H$, then problem (2.1) is equivalent to finding $u \in H$ such that

$$g(u) \in K, \quad Tu \in K^* \quad \text{and} \quad \langle Tu, g(u) \rangle = 0,$$

which is known as the general complementarity problem. For $g(u) = m(u) + K$, where $m$ is a point-to-point mapping, problem (2.5) is called the implicit (quasi) complementarity problem. If $g \equiv I$, then problem (2.5) is known as the
generalized complementarity problem. Such problems have been studied extensively in the literature — see the references. For suitable and appropriate choice of the operators and spaces, one can obtain several classes of variational inequalities and related optimization problems.

If \( g = I \), the identity operator, then problem (2.1) is equivalent to finding \( u \in K \) such that

\[
\langle Tu, v - u \rangle \geq 0, \quad \forall v \in K, \tag{2.6}
\]

which is called the classical variational inequality, introduced and studied by Stampacchia [31] in 1964. For the state-of-the-art, see [1–32].

We also need the following well known results and concepts.

**Definition 2.1.** The operator \( T : H \to H \) is said to be \( g \)-pseudomonotone, if

\[
\langle Tu, g(v) - g(u) \rangle \geq 0 \quad \text{implies} \quad \langle Tv, g(v) - g(u) \rangle \geq 0, \quad \forall u, v \in H.
\]

It is well known [11,24] that \( g \)-monotonicity implies \( g \)-pseudomonotonicity, but the converse is not true. This shows that pseudomonotonicity is a weaker condition than monotonicity.

**Lemma 2.1** ([10]). For a given \( z \in H, u \in K \) satisfies the inequality

\[
\langle u - z, v - u \rangle \geq 0, \quad \forall v \in K, \tag{2.7}
\]

if and only if

\[
u = P_K [z],
\]

where \( P_K \) is the projection of \( H \) onto the closed convex set \( K \).

It follows from **Lemma 2.1** that

\[
\langle P_K [z] - z, v - P_K [z] \rangle \geq 0, \quad \forall v \in K. \tag{2.8}
\]

The following result can be proved by using **Lemma 2.1**, see also Noor [18].

**Lemma 2.2.** \( u^* \) is a solution of Problem (2.1) if and only if \( u^* \in H \) satisfies the relation:

\[
g(u^*) = P_K [g(u^*) - \rho T (u^*)], \tag{2.9}
\]

where \( \rho > 0 \) is a constant and \( K \subset g(H) \).

From **Lemma 2.2**, it is clear that \( u \) is solution of (2.1) if and only if \( u \) is a zero point of the function

\[
r(u, \rho) := g(u) - P_K [g(u) - \rho T (u)]. \tag{2.10}
\]

Related to the general variational inequalities (2.1), we now consider the problem of solving the Wiener–Hopf equations. To be more precise, let \( Q_K = I - P_K \), where \( I \) is the identity operator and \( P_K \) is the projection operator. For given nonlinear continuous operators \( T, g : H \to H \), such that \( g^{-1} \) exists, consider the problem of finding \( z \in H \) such that

\[
\rho T g^{-1} P_K z + Q_K z = 0. \tag{2.11}
\]

which are known as the general Wiener–Hopf equations, introduced and studied by Noor [19]. It has been shown that the Wiener–Hopf equations are flexible and provide a unified framework to develop several efficient and powerful numerical techniques for solving variational inequalities and related optimization problems, see [18–25] and the references therein.

Using **Lemma 2.2**, it can be shown that the general variational inequalities (2.1) are equivalent to the Wiener–Hopf equations (2.11). This result is due to Noor [19].

**Lemma 2.3** (Noor [19]). The function \( u \in H : g(u) \in K \) is a solution of the general variational inequality (2.1) if and only if \( z \in H \) satisfies the Wiener–Hopf equation (2.11), provided

\[
g(u) = P_K z \tag{2.12}
\]

\[
z = g(u) - \rho Tu. \tag{2.13}
\]

**Lemma 2.3** implies that the general variational inequality (2.1) is equivalent to Wiener–Hopf equations (2.11). This equivalent formulation has been used by Noor [18–25] to suggest and analyze several iterative methods for solving the general variational inequalities and related optimization problems.

We use this useful equivalent formulation to suggest and analyze a self-adaptive method for solving the variational inequalities. This is the main motivation of this paper.

Using (2.10), (2.12) and (2.13), the Wiener–Hopf equations (2.11) can be written in the form:

\[
g(u) - P_K [g(u) - \rho Tu] - \rho Tu + \rho T g^{-1} P_K [g(u) - \rho Tu] = r(u, \rho) - \rho Tu + \rho T g^{-1} P_K [g(u) - \rho Tu] = 0. \tag{2.14}
\]
For a positive step $\alpha$, Eq. (2.14) can be written as

$$g(u) = (g) - \alpha d(u, \rho),$$

where

$$d(u, \rho) = r(u, \rho) - \rho Tu + \rho Tg^{-1}P_K[g(u) - \rho Tu].$$

This equivalent fixed-point formulation has been used by Noor [24,25] to suggest and analyze the following method for solving the variational inequalities (2.1).

**Algorithm 2.1.**

Step 0. Given $\varepsilon > 0, \gamma \in [1, 2), \mu \in (0, 1), \rho > 0, \delta_0, \delta \in (0, 1)$ and $u^0 \in H$, set $n = 0$.

Step 1. Set $\rho_n = \rho$. If $\|r(u^n, \rho)\| < \varepsilon$, then stop; otherwise, find the smallest no-negative integer $m_n$, such that $\rho_n = \rho \mu^{m_n}$ satisfying

$$\rho_n(T(u_n) - Tg^{-1}P_K[g(u_n) - \rho_n T(u_n)], r(u_n, \rho_n)) \leq \delta_0 \|r(u_n, \rho_n)\|^2.$$

Step 2. Compute

$$d(u_n, \rho_n) = r(u_n, \rho_n) - \rho_n T(u_n) + \rho_n Tg^{-1}P_K[g(u_n) - \rho_n T(u_n)],$$

$$\alpha_n = \frac{(1 - \delta)_\|r(u_n, \rho_n)\|^2}{\|d(u_n, \rho_n)\|^2}.$$

Step 3. Get the next iterate

$$g(u_{n+1}) = g(u_n) - \alpha_n d(u_n, \rho_n).$$

Step 4. If

$$\rho_n(T(u_n) - Tg^{-1}P_K[g(u_n) - \rho_n T(u_n)], r(u_n, \rho_n)) \leq \delta_0 \|r(u_n, \rho_n)\|^2,$$

then set $\rho = \frac{\rho_n}{\mu}$, else set $\rho = \rho_n$. Set $n = n + 1$, and go to Step 1.

If $g = I$, the identity operator, then Algorithm 2.1 collapses to the following iterative method for solving variational inequalities (2.6).

**Algorithm 2.2.**

Step 0. Given $\varepsilon > 0, \gamma \in [1, 2), \mu \in (0, 1), \rho > 0, \delta_0, \delta \in (0, 1)$ and $u^0 \in H$, set $n = 0$.

Step 1. Set $\rho_n = \rho$. If $\|r(u^n, \rho)\| < \varepsilon$, then stop; otherwise, find the smallest no-negative integer $m_n$, such that $\rho_n = \rho \mu^{m_n}$ satisfying

$$\rho_n(T(u_n) - TP_K[u_n - \rho_n Tu_n], r(u_n, \rho_n)) \leq \delta_0 \|r(u_n, \rho_n)\|^2.$$

Step 2. Compute

$$d(u_n, \rho_n) = r(u_n, \rho_n) - \rho_n T(u_n) + \rho_n TP_K[u_n - \rho_n T(u_n)],$$

$$\alpha_n = \frac{(1 - \delta)_\|r(u_n, \rho_n)\|^2}{\|d(u_n, \rho_n)\|^2}.$$

Step 3. Get the next iterate

$$u_{n+1} = u_n - \alpha_n d(u_n, \rho_n).$$

Step 4. If

$$\rho_n(T(u_n) - TP_K[u_n - \rho_n T(u_n)], r(u_n, \rho_n)) \leq \delta_0 \|r(u_n, \rho_n)\|^2,$$

then set $\rho = \frac{\rho_n}{\mu}$, else set $\rho = \rho_n$. Set $n = n + 1$, and go to Step 1.

**Remark 2.1.** Algorithm 2.1 is obtained by using a self-adaptive technique to adjust parameter $\rho$ at each iteration in the original algorithm in [24,25].

Throughout this paper, we make the following assumptions.

**Assumptions:**

- $H$ is a finite dimension space.
- $T$ is $g$-continuous, $g$-pseudomonotone operator on $H$ and the operator $g$ is bijective and its inverse $g^{-1}$ exists.
- The solution set of problem (2.1) denoted by $S^*$ is nonempty.
3. Basic results

We prove some of the important results which will be required in our following analysis. The following lemma shows that \( \|r(u, \rho)\| \) is a non-decreasing function, while \( \frac{\|r(u, \rho)\|}{\rho} \) is a non-increasing one with respect to \( \rho \). This result is mainly due to Gafni and Bertsekas [9] and Toint [32]. We give the proof for the sake of completeness and to convey an idea of the technique involved.

**Lemma 3.1.** For all \( u \in H \) and \( \rho' \geq \rho > 0 \), it holds that
\[
\|r(u, \rho')\| \geq \|r(u, \rho)\| \tag{3.1}
\]
and
\[
\frac{\|r(u, \rho')\|}{\rho'} \leq \frac{\|r(u, \rho)\|}{\rho}. \tag{3.2}
\]

**Proof.** Let \( t := \frac{\|r(u, \rho')\|}{\|r(u, \rho)\|} \), we need only to prove that \( 1 \leq t \leq \rho'/\rho \). Note that its equivalent expression is
\[
(t - 1) \left( t - \frac{\rho'}{\rho} \right) \leq 0. \tag{3.3}
\]

Using inequality (2.4), we have
\[
\langle g(u) - \rho T(u) - P_K[g(u) - \rho T(u)], P_K[g(u) - \rho T(u)] - P_K[g(u) - \rho'T(u)] \rangle \geq 0 \tag{3.4}
\]
and
\[
\langle g(u) - \rho'T(u) - P_K[g(u) - \rho T(u)], P_K[g(u) - \rho'T(u)] - P_K[g(u) - \rho T(u)] \rangle \geq 0. \tag{3.5}
\]
From (3.4) and using
\[
P_K[g(u) - \rho T(u)] - P_K[g(u) - \rho'T(u)] = r(u, \rho') - r(u, \rho),
\]
we obtain
\[
\langle r(u, \rho), r(u, \rho') - r(u, \rho) \rangle \geq \rho \langle T(u), r(u, \rho') - r(u, \rho) \rangle. \tag{3.6}
\]
Similarly, we have
\[
\langle r(u, \rho'), r(u, \rho) - r(u, \rho') \rangle \geq \rho' \langle T(u), r(u, \rho) - r(u, \rho') \rangle. \tag{3.7}
\]
Multiplying (3.6) and (3.7) by \( \rho' \) and \( \rho \) respectively, and adding the resultant, we have
\[
\langle \rho' r(u, \rho) - \rho r(u, \rho'), r(u, \rho') - r(u, \rho) \rangle \geq 0 \tag{3.8}
\]
and consequently
\[
\rho' \|r(u, \rho)\|^2 + \rho \|r(u, \rho')\|^2 \leq (\rho + \rho') \langle r(u, \rho), r(u, \rho') \rangle. \tag{3.9}
\]
From Cauchy–Schwarz inequality, we have
\[
\langle r(u, \rho), r(u, \rho') \rangle \leq \|r(u, \rho)\| \cdot \|r(u, \rho')\|.
\]
Then
\[
\rho' \|r(u, \rho)\|^2 + \rho \|r(u, \rho')\|^2 \leq (\rho + \rho') \|r(u, \rho)\| \cdot \|r(u, \rho')\|. \tag{3.10}
\]
Dividing (3.10) by \( \|r(u, \rho)\|^2 \), we obtain
\[
\rho' + \rho t \leq (\rho + \rho') t
\]
and thus (3.3) holds and the lemma is proved. \( \Box \)

**Lemma 3.2.** Let \( T \) be a g-pseudomonotone. Then \( \forall u \in H, u^* \in S^* \) and \( \rho > 0 \), we have
\[
\langle g(u) - g(u^*), d(u, \rho) \rangle \geq \phi(u, \rho), \tag{3.11}
\]
where
\[
d(u, \rho) := r(u, \rho) - \rho T(u) + \rho T^{-1} P_K[g(u) - \rho T(u)] \tag{3.12}
\]
and
\[
\phi(u, \rho) := \|r(u, \rho)\|^2 - \rho \langle r(u, \rho), T(u) - T^{-1} P_K[g(u) - \rho T(u)] \rangle. \tag{3.13}
\]
Proof. Let $u^* \in S^*$ be a solution of problem (2.1). Then
\begin{equation}
\langle \rho T(u^*), g(v) - g(u^*) \rangle \geq 0, \quad \forall v \in g^{-1}(K), \rho > 0.
\end{equation}
Taking $g(v) = P_K[g(u) - \rho T(u)]$ in (3.14) and using the $g$-pseudo monotonicity of $T$, we obtain
\begin{equation}
\langle \rho g^{-1}P_K[g(u) - \rho T(u)], P_K[g(u) - \rho T(u)] - g(u^*) \rangle \geq 0.
\end{equation}
Substituting $z = g(u) - \rho T(u)$ and $v = g(u^*)$ into (2.4), and using the definition of $r(u, \rho)$, we get
\begin{equation}
\langle r(u, \rho) - \rho T(u), P_K[g(u) - \rho T(u)] - g(u^*) \rangle \geq 0.
\end{equation}
Adding (3.15) and (3.16), we have
\begin{equation}
\langle r(u, \rho) - r(u, \rho) - \rho[T(u) - g^{-1}P_K[g(u) - \rho T(u)]], P_K[g(u) - \rho T(u)] - g(u^*) \rangle \geq 0.
\end{equation}
which can be rewritten as
\begin{equation}
\langle r(u, \rho) - \rho[T(u) - g^{-1}P_K[g(u) - \rho T(u)]], g(u) - g(u^*) - r(u, \rho) \rangle \geq 0.
\end{equation}
then
\begin{equation}
\langle g(u) - g(u^*), d(u, \rho) \rangle \geq \|r(u, \rho)\|^2, \rho > 0.
\end{equation}
and the conclusion of Lemma 3.2 is proved. □

Lemma 3.3. If $u$ is not a solution of problem (2.1), then there exist $\delta \in (0, 1)$ and $\epsilon' > 0$, such that for all $\rho \in (0, \epsilon')$,
\begin{equation}
\rho \|T(u) - g^{-1}P_K[g(u) - \rho T(u)]\| \leq \delta \|r(u, \rho)\|.
\end{equation}
Proof. Suppose that (3.17) is not true, that is,
\begin{equation}
\rho \|T(u) - g^{-1}P_K[g(u) - \rho T(u)]\| > \delta \|r(u, \rho)\|, \quad \forall \rho > 0.
\end{equation}
Since $T$ is $g$-continuous and $g^{-1}P_K[g(u) - \rho T(u)]$ is $T(u)$ as $\rho \to 0$. Let $\rho \to 0$ and taking the limit in the above inequality, we have
\begin{equation}
0 \geq \lim_{\rho \to 0} \delta \frac{\|r(u, \rho)\|}{\rho} \geq \delta \|r(u, 1)\|.
\end{equation}
where the second inequality follows from (3.2). Then $\|r(u, 1)\| = 0$, which contradicts the assumption of the lemma. □

From Lemmas 3.2 and 3.3 we have
\begin{equation}
\langle g(u) - g(u^*), d(u, \rho) \rangle \geq \phi(u, \rho) \geq (1 - \delta) \|r(u, \rho)\|^2.
\end{equation}
This fact has motivated us to construct the following iterative method for solving the general variational inequalities (2.1).

Algorithm 3.1.

Step 0. Given $\epsilon > 0$, $\gamma \in [1, 2)$, $\mu \in (0, 1)$, $\rho > 0$, $\delta_0$, $\delta \in (0, 1)$ and $u^0 \in H$, set $k = 0$.

Step 1. Set $\rho_k = \rho$. If $\|r(u^k, \rho_k)\| < \epsilon$, then stop; otherwise, find the smallest non-negative, see [12,13,17] integer $m_k$, such that $\rho_k = \rho \mu^{m_k}$ satisfying
\begin{equation}
\|\rho_k T(u^k) - T(u^k)\| \leq \delta \|r(u^k, \rho_k)\|,
\end{equation}
where
\begin{equation}
g(u^k) = P_K[g(u^k) - \rho_k T(u^k)].
\end{equation}

Step 2. Compute $d(u^k, \rho_k)$ and $\phi(u^k, \rho_k)$ from (3.12) and (3.13) respectively, and the stepsize
\begin{equation}
\alpha_k = \frac{\phi(u^k, \rho_k)}{\|d(u^k, \rho_k)\|^2}.
\end{equation}

Step 3. Get the next iterate
\begin{equation}
g(u^{k+1}) = g(u^k) - \gamma \alpha_k d(u^k, \rho_k).
\end{equation}

Step 4. If
\begin{equation}
\|\rho_k T(u^k) - T(u^k)\| \leq \delta_0 \|r(u^k, \rho_k)\|,
\end{equation}
then set $\rho = \frac{\rho_k}{\mu}$, else set $\rho = \rho_k$. Set $k := k + 1$, and go to Step 1.

If $g = 1$, the identity operator, then Algorithm 3.1 reduces to the following method for solving the variational inequalities (2.6) and appears to be new ones.
Algorithm 3.2.
Step 0. Given $\epsilon > 0$, $\gamma \in (1, 2)$, $\mu \in (0, 1)$, $\rho > 0$, $\delta \in (0, 1)$, $\delta_0 \in (0, 1)$ and $u^0 \in H$, set $k = 0$.
Step 1. Set $\rho_k = \rho$. If $\|r(u^k, \rho)\| < \epsilon$, then stop; otherwise, find the smallest non-negative integer $m_k$, such that $\rho_k = \rho \mu^{m_k}$ satisfying
\[
\|\rho_k(T(u^k) - T(u^k))\| \leq \delta \|r(u^k, \rho_k)\|,
\]
where
\[
w^k = P_K[u^k - \rho_k T(u^k)].
\]
Step 2. Compute
\[
d(u^k, \rho_k) := r(u^k, \rho_k) - \rho_k T(u^k) + \rho_k T_K[u^k - \rho_k T(u^k)],
\]
\[
\phi(u^k, \rho_k) := \|r(u^k, \rho_k)\|^2 - \rho_k \langle r(u^k, \rho_k), T(u^k) - TP_K[u^k - \rho_k T(u^k)] \rangle
\]
and the stepsize
\[
\alpha_k = \frac{\phi(u^k, \rho_k)}{\|d(u^k, \rho_k)\|^2}.
\]
Step 3. Get the next iterate
\[
u^{k+1} = u^k - \gamma \alpha_k d(u^k, \rho_k).
\]
Step 4. If
\[
\|\rho_k(T(u^k) - T(u^k))\| \leq \delta_0 \|r(u^k, \rho_k)\|,
\]
then set $\rho = \frac{\rho_k}{\mu}$, else set $\rho = \rho_k$. Set $k := k + 1$, and go to Step 1.

4. Global convergence

In this section, we consider the global convergence of Algorithm 3.1. For this purpose, we need the following result.

Theorem 4.1. Let $u^* \in H$ be a solution of problem (2.1) and let $u^{k+1}$ be the sequence obtained from Algorithm 3.1. Then $\{u^k\}$ is bounded and
\[
\|g(u^{k+1}) - g(u^*)\|^2 \leq \|g(u^k) - g(u^*)\|^2 - \frac{1}{2} \gamma(2 - \gamma)(1 - \delta) \|r(u^k, \rho_k)\|^2.
\]

Proof. Let $u^* \in H$ be a solution of problem (2.1). Then
\[
\|g(u^{k+1}) - g(u^*)\|^2 = \|g(u^k) - g(u^*) - \gamma \alpha_k d(u^k, \rho_k)\|^2
\]
\[
= \|g(u^k) - g(u^*)\|^2 - 2 \gamma \alpha_k \langle g(u^k) - g(u^*), d(u^k, \rho_k) \rangle
\]
\[
+ \gamma^2 \alpha_k^2 \|d(u^k, \rho_k)\|^2
\]
\[
\leq \|g(u^k) - g(u^*)\|^2 - 2 \gamma \alpha_k \phi(u^k, \rho_k) + \gamma^2 \alpha_k \phi^2(u^k, \rho_k)
\]
\[
\leq \|g(u^k) - g(u^*)\|^2 - \gamma(2 - \gamma)(1 - \delta) \alpha_k \|r(u^k, \rho_k)\|^2,
\]
where the first inequality follows from (3.18) and (3.20) and the second inequality follows from (3.18). Since $\gamma \in (1, 2)$ and $\delta \in (0, 1)$, we have
\[
\|g(u^{k+1}) - g(u^*)\| \leq \|g(u^k) - g(u^*)\| \leq \cdots \leq \|g(u^0) - g(u^*)\|.
\]
From the above inequality and by the assumptions on $g$, it follows that the sequence $\{u^k\}$ is bounded.
From Lemma 3.3 and (3.12), we have
\[
\phi(u, \rho) = \|r(u, \rho)\|^2 - \rho \langle r(u, \rho), T(u) - T^{-1} [g(u) - \rho T(u)] \rangle
\]
\[
\geq \frac{1}{2} \|r(u, \rho)\|^2 - \rho \langle r(u, \rho), T(u) - T^{-1} P_K [g(u) - \rho T(u)] \rangle
\]
\[
+ \frac{1}{2} \|\rho T(u) - \rho T^{-1} P_K [u - \rho T(u)]\|^2
\]
\[
= \frac{1}{2} \|d(u, \rho)\|^2.
\]
Thus we obtain
\[\alpha_k = \frac{\phi(u^k, \rho_k)}{\|d(u^k, \rho_k)\|^2} > \frac{1}{2}.\]
Combining (4.2) and the above inequality, we have the desired result. \(\square\)

Now we are ready to prove the convergence of Algorithm 3.1.

**Theorem 4.2.** The sequence \{\(u^k\)\} generated by Algorithm 3.1 converges to a solution point of problem (2.1).

**Proof.** It follows from (4.1) that
\[\sum_{k=0}^{\infty} \|r(u^k, \rho_k)\|^2 < \infty,\]
which means that
\[\lim_{k \to \infty} \|r(u^k, \rho_k)\| = 0,\] (4.3)
and it follows from Lemma 3.1 that
\[\min\{1, \rho_k\} \|r(u^k, 1)\| \leq \|r(u^k, \rho_k)\|.\] (4.4)
Combining (4.3) and (4.4), we get
\[\lim_{k \to \infty} \rho_k \|r(u^k, 1)\| = 0.\] (4.5)
We have two possible cases. Firstly, suppose that
\[\lim_{k \to \infty} \sup \rho_k > 0.\]
It follows from (4.5) that
\[\lim_{k \to \infty} \inf \|r(u^k, 1)\| = 0.\]
It follows from (4.1) that the sequence \{\(u^k\)\} is bounded and consequently, it has a cluster point \(\tilde{u}\) such that \(\|r(\tilde{u}, 1)\| = 0\), which implies \(\tilde{u}\) is a solution of problem (2.1).

Now, we consider the second possible case
\[\lim_{k \to \infty} \rho_k = 0.\]

By the choice of \(\rho_k\) we know that (3.19) was not satisfied for \(m_k - 1\). Then for \(k\) large enough such that \(\rho_k < \mu\), we obtain
\[\|T(u^k) - Tg^{-1}P_k[g(u^k) - (\rho_k/\mu)T(u^k)]\| > \delta\mu \|r(u^k, \rho_k/\mu)\|/\|\rho_k\| \geq \delta \|r(u^k, 1)\|,\]
where the second inequality follows from (3.2).

Let \(\tilde{u}\) be a cluster point of \{\(u^k\)\} and the subsequence \(\{u^j\}\) converges to \(\tilde{u}\). Then, we have
\[\|r(\tilde{u}, 1)\| = \lim_{j \to \infty} \|r(u^j, 1)\| \leq \lim_{j \to \infty} \|T(u^j) - Tg^{-1}P_k[g(u^j) - (\rho_k/\mu)T(u^j)]\|/\delta = 0,\]
which means that \(\tilde{u}\) is a solution of problem (2.1). In the following, we prove that the sequence \{\(u^k\)\} has exactly one cluster point. Assume that \(\tilde{u}\) is another cluster point and satisfies
\[\delta := \|g(\tilde{u}) - g(\tilde{u})\| > 0.\]
Since \(\tilde{u}\) is a cluster point of the sequence \(\{u^k\}\), there is a \(k_0 > 0\) such that
\[\|g(u^{k_0}) - g(\tilde{u})\| \leq \frac{\delta}{2}.\]
On the other hand, since \(\tilde{u} \in S^*\) and from (4.1), we have
\[\|g(u^k) - g(\tilde{u})\| \leq \|g(u^{k_0}) - g(\tilde{u})\| \quad \text{for all} \quad k \geq k_0,\]
it follows that
\[\|g(u^k) - g(\tilde{u})\| \geq \|g(\tilde{u}) - g(\tilde{u})\| - \|g(u^k) - g(\tilde{u})\| \geq \frac{\delta}{2} \quad \forall k \geq k_0.\]
This contradicts the assumption, thus the sequence \(\{u^k\}\) converges to \(\tilde{u} \in S^*\). \(\square\)
In the section, we presented some numerical results for the proposed method. We consider the nonlinear complementarity problems:

\[ u \geq 0, \quad T(u) \geq 0, \quad \langle u, T(u) \rangle = 0, \quad (5.1) \]

where \( T(u) = D(u) + Mu + q, D(u) \) and \( Mu + q \) are the nonlinear part and linear parts of \( T(u) \) respectively.

Problem (5.1) is a special case of Problem (2.1), by taking \( g = I \), the identity operator. In this case Algorithms 2.1 and 3.1 collapse to Algorithms 2.2 and 3.2 respectively.

We form the test problems similarly as in Harker and Pang [13]. The matrix \( M = A^T A + B \), where \( A \) is an \( n \times n \) matrix whose entries are randomly generated in the interval \((-5, 5)\) and a skew-symmetric matrix \( B \) is generated in the same way. The vector \( q \) is generated from a uniform distribution in the interval \((-500, 500)\) (easy problems and \((-500, 0)\) (hard problems), respectively. In \( D(u) \), the nonlinear part of \( T(u) \), the components are \( d_j(u) = d_j * \arctan(u_i) \) and \( d_j \) is a random variable in \((0, 1)\).

In all tests we took \( \mu = 2/3, \delta = 0.95, \delta_0 = 0.2, \) and \( \gamma = 1.95 \). The starting point \( u^0 = (0, \ldots, 0)^T \). All codes are written in Matlab and run on a P4-2.00G notebook computer. The computation begins with \( \rho_0 = 1 \) and stops as soon as \( \|r(u^k, \rho_k)\|_\infty \leq 10^{-7} \). The test results for easy problems (\( q \in (-500, 500) \)) and hard problems (\( q \in (-500, 0) \)) are reported in Tables 1 and 2.

Tables 1 and 2 show that Algorithm 3.2 converges quicker than Algorithm 2.2. In addition, for Algorithm 3.2, it seems that the CPU are not very sensitive to the problem size compared with Algorithm 2.2.

### 6. Applications

In this section we show that the results obtained in Sections 3–5 can be extended for a class of quasi variational inequalities. If the convex set \( K \) depends upon the solution explicitly or implicitly, then variational inequality problem is known as the quasi variational inequality. For a given operator \( T : H \rightarrow H \), and a point-to-set mapping \( K : u \rightarrow K(u) \), which associates a closed convex-valued set \( K(u) \) with any element \( u \) of \( H \), we consider the problem of finding \( u \in K(u) \) such that

\[ \langle Tu, v - u \rangle \geq 0, \quad \forall v \in K(u). \quad (6.1) \]

Inequality of type (6.1) is called the quasi variational inequality. To convey an idea of the applications of the quasi variational inequalities, we consider the second-order implicit obstacle boundary value problem of finding \( u \) such that

\[
\begin{align*}
-u'' &\geq f(x) & \text{on } \Omega = [a, b] \\
u &\geq M(u) & \text{on } \Omega = [a, b] \\
[-u'' - f(x)][u - M(u)] &= 0 & \text{on } \Omega = [a, b] \\
u(a) &= 0, & u(b) = 0 \\
\end{align*}
\]

where \( f(x) \) is a continuous function and \( M(u) \) is the cost (obstacle) function. The prototype encountered is

\[ M(u) = k + \inf \{u_i \}. \quad (6.3) \]
In (6.3), \( k \) represents the switching cost. It is positive when the unit is turned on and equal to zero when the unit is turned off. Note that the operator \( M \) provides the coupling between the unknowns \( u = (u^1, u^2, \ldots, u^i) \), see [2]. We study the problem (6.2) in the framework of variational inequality approach. To do so, we first define the set \( K \) as

\[
K(u) = \{v : v \in H^1_0(\Omega) : v \geq M(u), \text{ on } \Omega\},
\]

which is a closed convex-valued set in \( H^1_0(\Omega) \), where \( H^1_0(\Omega) \) is a Sobolev (Hilbert) space. One can easily show that the energy functional associated with the problem (6.2) is

\[
I[v] = -\int_a^b \left( \frac{d^2 v}{dx^2} \right) v \, dx - 2\int_a^b f(x) \, (v) \, dx, \quad \forall v \in K(u)
\]

\[
= \int_a^b \left( \frac{dv}{dx} \right)^2 \, dx - 2\int_a^b f(x) \, (v) \, dx
\]

\[
= \langle Tu, v \rangle - 2 \langle f, v \rangle
\]

where

\[
\langle Tu, v \rangle = \int_a^b \left( \frac{d^2 u}{dx^2} \right)(v) \, dx = \int_a^b \frac{du}{dx} \, \frac{dv}{dx} \, dx
\]

\[
\langle f, v \rangle = \int_a^b f(x)(v) \, dx.
\]

It is clear that the operator \( T \) defined by (6.5) is linear, symmetric and positive. Using the technique of Noor [24], one can show that the minimum of the functional \( I[v] \) defined by (6.4) associated with the problem (6.2) on the closed convex-valued set \( K(u) \) can be characterized by the inequality of type

\[
\langle Tu, v - u \rangle \geq \langle f, v - u \rangle, \quad \forall v \in K(u),
\]

which is exactly the quasi variational inequality (6.1). See also [1,2,11,23,24,26] for the formulation, applications, numerical methods and sensitivity analysis of the quasi variational inequalities.

Using Lemma 2.1, one can show that the quasi variational inequality (6.1) is equivalent to finding \( u \in K(u) \) such that

\[
u = P_{K(u)}[u - \rho Tu].
\]

In many important applications [2], the convex-valued set \( K(u) \) is of the form

\[
K(u) = m(u) + K,
\]

where \( m \) is a point-to-point mapping and \( K \) is a closed convex set.

From (6.7) and (6.8), we see that problem (6.1) is equivalent to

\[
u = P_{K(u)}[u - \rho Tu] = P_{m(u) + K}[u - \rho Tu]
\]

\[
n = m(u) + P_K[u - m(u) - \rho Tu]
\]

which implies that

\[
g(u) = P_K[g(u) - \rho Tu] \quad \text{with} \quad g(u) = u - m(u).
\]

which is equivalent to the general variational inequality (2.1) by an application of Lemma 3.1. We have shown that the quasi variational inequalities (6.1) with the convex-valued set \( K(u) \) defined by (6.8) are equivalent to the general variational inequalities (2.1). Thus all the results obtained in this paper continue to hold for quasi variational inequalities (6.1) with \( K(u) \) defined by (6.8).

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