New extragradient-type methods for general variational inequalities

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Abstract

In this paper, we consider and analyze a new class of extragradient-type methods for solving general variational inequalities. The modified methods converge for pseudomonotone operators which is weaker condition than monotonicity. Our proof of convergence is very simple as compared with other methods. The proposed methods include several new and known methods as special cases. Our results present a significant improvement of previously known methods for solving variational inequalities and related optimization problems.

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1. Introduction

Variational inequalities have been extended and generalized in several directions for studying a wide class of equilibrium problems arising in financial, economics, transportation, elasticity, optimization, pure and applied sciences. An important and useful generalization of variational inequalities is called the general variational inequality introduced by Noor [7] in 1988, which enables us to study the odd-order and nonsymmetric problems in a unified framework. This field is dynamic and is experiencing an explosive growth in both theory and applications: as a consequence, several numerical techniques including projection, the Wiener–Hopf equations, auxiliary principle, decomposition and descent are being developed for solving various classes of variational inequalities and
related optimization problems. Projection methods and its variants forms including the Wiener–Hopf equations represent important tools for finding the approximate solution of variational and quasi-variational inequalities, the origin of which can be traced back to Lions and Stampacchia [6]. The main idea in this technique is to establish the equivalence between the variational inequalities and the fixed-point problem by using the concept of projection. This alternative formulation has played a significant part in developing various projection-type methods for solving variational inequalities. It is well known that the convergence of the projection methods requires that the operator must be strongly monotone and Lipschitz continuous. Unfortunately these strict conditions rule out many applications of this method. This fact motivated to modify the projection method or to develop other methods. The extragradient method [1,4,17,29,33,34,36] overcome this difficulty by performing an additional forward step and a projection at each iteration according to the double projection. This method can be viewed as predictor–corrector method. Its convergence requires only that a solution exists and the monotone operator is Lipschitz continuous. When the operator is not Lipschitz continuous or when the Lipschitz continuous constant is not known, the extragradient method and its variant forms require an Armijo-like line search procedure to compute the step size with a new projection need for each trial, which leads to expansive computation. To overcome these difficulties, several modified projection and extragradient-type methods have been suggested and developed for solving variational inequalities. Wang et al. [33,34] have considered some classes of predictor–corrector extragradient type methods, which use better step size rule, whereas He and Liao [4] have improved the efficiency of the classical extragradient-type methods by using the Wiener–Hopf equations as step size. Noor [17] has suggested a unified extragradient-type method which combines both the modification of Wang et al. [33] and He and Liao [4] and its convergence requires only the pseudomonotonicity. In particular, Noor [17] has improved the convergence criteria of the method of He and Liao [4]. In passing, we would like to mention Sun [31] was the first to use the Wiener–Hopf equation as a step size.

Related to the variational inequalities, we have the concept of the Wiener–Hopf equations, which was introduced by Shi [27] and Robinson [26] in conjunction with variational inequalities from different point of views. Using the projection technique, one usually establishes the equivalence between the variational inequalities and the Wiener–Hopf equations. It turned out that the Wiener–Hopf equations are more general and flexible. This approach has played not only an important part in developing various efficient projection-type methods, but also in studying the sensitivity analysis as well as other concepts of variational inequalities. For recent applications and numerical methods, see [12–24] and references therein. Noor et al. [24] and Noor and Rassias [22] have suggested and analyzed some predictor–corrector-type projection methods by modifying the Wiener–Hopf equations. It has been shown in [22,24,33,34] that these predictor–corrector-type methods are efficient and robust. It shows that the Wiener–Hopf equation technique is a powerful tool for developing efficient methods. Inspired and motivated by this development, we suggest a new unified extragradient-type method for solving general variational inequalities and related problems. We prove that the convergence of the new method requires only the pseudomonotonicity, which is weaker condition than monotonicity. Since general variational inequalities include variational, quasi-variational inequalities and the comple-
mentality problems as special case, results obtained in this paper continue to hold for these problems. We would like to emphasize that almost all the extragradient and projection-type methods suggested in this paper can be considered as predictor–corrector-type methods. Our results can be viewed as significant and novel extension of the results of Wang et al. [33], Noor and Rassias [22], He and Liao [4] and Noor [17]. The comparison of these methods with the existing one is an interesting problem for future research work.

2. Preliminaries

Let $H$ be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let $K$ be a closed convex set in $H$ and $T, g : H \to H$ be a nonlinear operators. We now consider the problem of finding $u \in H$, $g(u) \in K$ such that

$$\langle Tu, g(v) - g(u) \rangle \geq 0, \quad \text{for all } g(v) \in K.$$  

(2.1)

Problem (2.1) is called the general variational inequality, which was introduced and studied by Noor [7] in 1988. It has been shown that a large class of unrelated odd-order and nonsymmetric obstacle, unilateral, contact, free, moving, and equilibrium problems arising in regional, physical, mathematical, engineering and applied sciences can be studied in the unified and general framework of the general variational inequalities (2.1); see [7–23,35] and references therein.

For $g \equiv I$, where $I$ is the identity operator, problem (2.1) is equivalent to finding $u \in K$ such that

$$\langle Tu, v - u \rangle \geq 0 \quad \text{for all } v \in K,$$  

(2.2)

which is known as the classical variational inequality introduced and studied by Stampacchia [32] in 1964. For recent state-of-the-art, see [1–36] and references therein.

From now onward, we assume that $g$ is onto $K$ unless otherwise specified. If $N(u) = \{ w \in H : \langle w, v - u \rangle \leq 0, \text{ for all } v \in K \}$ is a normal cone to the convex set $K$ at $u$, then the general variational inequality (2.1) is equivalent to finding $u \in H$, $g(u) \in K$ such that

$$-T(u) \in N(g(u)),$$  

which are known as the general nonlinear equations.

If $T^{18}$ is the projection of $-Tu$ at $g(u) \in K$, then it has been shown that the general variational inequality problem (2.1) is equivalent to finding $u \in H$, $g(u) \in K$ such that

$$T^{18}(u) = 0,$$  

which are known as the tangent projection equations; see [35]. This equivalence has been used to discuss the local convergence analysis of a wide class of iterative methods for solving general variational inequalities (2.1).

If $K^* = \{ u \in H : \langle u, v \rangle \geq 0, \text{ for all } v \in K \}$ is a polar (dual) cone of a convex cone $K$ in $H$, then problem (2.1) is equivalent to finding $u \in H$ such that

$$g(u) \in K, \quad Tu \in K^* \quad \text{and} \quad \langle Tu, g(u) \rangle = 0,$$  

(2.3)
which is known as the general complementarity problem. For \( g(u) = m(u) + K \), where \( m \) is a point-to-point mapping, problem (2.3) is called the implicit (quasi) complementarity problem. If \( g \equiv I \), then problem (2.3) is known as the generalized complementarity problem. Such problems have been studied extensively in the literature; see the references.

For suitable and appropriate choice of the operators and spaces, one can obtain several classes of variational inequalities and related optimization problems.

We now recall the following well known result and concepts.

**Lemma 2.1.** For a given \( z \in H \), \( u \in K \) satisfies the inequality
\[
\langle u - z, v - u \rangle \geq 0, \quad \text{for all } v \in K,
\]
if and only if
\[
u = P_K[z],
\]
where \( P_K \) is the projection of \( H \) onto \( K \). Also, the projection operator \( P_K \) is nonexpansive and satisfies the inequality
\[
\| P_K[z] - u \|^2 \leq \| z - u \|^2 - \| z - P_K[z] \|^2.
\]

Related to the general variational inequalities, we now consider the problem of Wiener–Hopf equations. To be more precise, let \( Q_K = I - P_K \), where \( I \) is the identity operator and \( P_K \) is the projection of \( H \) onto \( K \). For given nonlinear operators \( T, g : H \to H \), consider the problem of finding \( z \in H \) such that
\[
\rho T g^{-1} P_K z + Q_K z = 0.
\]

Equations of the type (2.6) are called the general Wiener–Hopf equations, which were introduced and studied by Noor [8,11]. For \( g = I \), we obtain the original Wiener–Hopf equations, which were introduced and studied by Shi [27] and Robinson [26] in different settings independently. Using the projection operators technique one can show that the variational inequalities are equivalent to the Wiener–Hopf equations. This equivalent alternative formulation has played a fundamental and important role in studying various aspects of variational inequalities. It has been shown that Wiener–Hopf equations are more flexible and provide a unified framework to develop some efficient and powerful numerical technique for solving variational inequalities and related optimization problems; see, for example, [9–24] and references therein.

**Definition 2.1.** For all \( u, v \in H \), the operator \( T : H \to H \) is said to be

(i) **\( g \)-monotone**, if
\[
\langle Tu - Tv, g(u) - g(v) \rangle \geq 0;
\]

(ii) **\( g \)-pseudomonotone**, if
\[
\langle Tu, g(v) - g(u) \rangle \geq 0 \quad \text{implies} \quad \langle Tv, g(v) - g(u) \rangle \geq 0.
\]
For \( g \equiv I \), Definition 2.1 reduces to the usual definition of monotonicity, and pseudomonotonicity of the operator \( T \). Note that monotonicity implies pseudomonotonicity but the converse is not true; see [2].

3. Projection technique

In this section, we use the projection technique to suggest and analyze extragradient-type methods for solving general variational inequalities (2.1). For this purpose, we need the following result, which can be proved by invoking Lemma 2.1.

**Lemma 3.1** [7], The function \( u \in H, \ g(u) \in K \) is a solution of (2.1) if and only if \( u \in H \) satisfies the relation

\[
g(u) = P_K \left[ g(u) - \rho Tu \right],
\]

where \( \rho > 0 \) is a constant and \( g \) is onto \( K \).

Lemma 3.1 implies that problems (2.1) and (3.1) are equivalent. This alternative formulation is very important from the numerical analysis point of view. This fixed-point formulation has been used to suggest and analyze the following method.

**Algorithm 3.1.** For a given \( u_0 \in H \), compute the approximate solution \( u_{n+1} \) by the iterative scheme

\[
g(u_{n+1}) = P_K \left[ g(u_n) - \rho Tu_n \right], \quad n = 0, 1, 2, \ldots.
\]

For the convergence analysis of Algorithm 3.1, see [7]. Xiu et al. [35] have proved that Algorithm 3.1 has the local convergence behaviour, which enables us to identify accurately the optimal constraint after finitely many iterations.

We now define the projection residue vector by the relation

\[
R(u) = g(u) - P_K \left[ g(u) - \rho Tu \right].
\]

From Lemma 3.1, it is clear the \( u \in H, \ g(u) \in K \) is a solution of (2.1) if and only if \( u \in H, \ g(u) \in K \) is a zero of the equation

\[
R(u) = 0.
\]

For a positive step size \( \gamma \), Eq. (3.3) can be written as

\[
g(u) = g(u) - \gamma R(u).
\]

This fixed-point formulation allows to suggest the following iterative method for solving the general variational inequalities (2.1).

**Algorithm 3.2.** For a given \( u_0 \in H \), compute the approximate solution \( u_{n+1} \) by the iterative schemes

\[
g(u_{n+1}) = P_K \left[ g(u_n) - \gamma_n R(u_n) \right], \quad n = 0, 1, 2, \ldots.
\]
Note that for $\gamma_n = 1$, Algorithm 3.2 coincides with Algorithm 3.1.

It is well known that the convergence analysis of Algorithm 3.1 requires that both the operators $T$ and $g$ must be strongly monotone and Lipschitz continuous. These strict conditions rule out many important applications of Algorithm 3.1. To overcome these drawbacks, we use the technique of updating the solution. Thus for a positive constant $\alpha$, we can rewrite Eq. (3.1) in the form

$$
g(u) = P_K\left[g(u) - \alpha T g^{-1} P_K\left[g(u) - \rho Tu\right]\right].$$

(3.5)

We use this fixed-point formulation to suggest the following extragradient-type method for solving general variational inequalities (2.1).

**Algorithm 3.3.** For a given $u_0 \in H$, compute the approximate solution $u_{n+1}$ by the iterative scheme:

*Predictor step.*

$$g(v_n) = P_K\left[g(u_n) - \rho_n Tu_n\right],$$

where $\rho_n$ satisfies

$$\rho_n \langle Tu_n - T^{-1} P_K\left[g(u_n) - \rho Tu_n\right], R(u_n) \rangle \leq \sigma \|R(u_n)\|^2, \quad \sigma \in (0, 1).$$

*Corrector step.*

$$g(u_{n+1}) = P_K\left[g(u_n) - \alpha_n T v_n\right], \quad n = 0, 1, 2, \ldots,$$

where

$$\alpha_n = \frac{(1 - \sigma)\|R(u_n)\|^2}{\|Tv_n\|^2},$$

$$Tv_n = T^{-1} P_K\left[g(u_n) - \rho_n Tu_n\right].$$

For $g \equiv I$, the identity operator, Algorithm 3.3 reduces to

**Algorithm 3.4.** For a given $u_0 \in K$, compute $u_{n+1}$ by the iterative schemes:

*Predictor step.*

$$v_n = P_K[u_n - \rho_n Tu_n],$$

where $\rho_n$ satisfies the relation

$$\rho_n \langle Tu_n - Tv_n, R(u_n) \rangle \leq \sigma \|R(u_n)\|^2, \quad \sigma \in (0, 1).$$

*Corrector step.*

$$g(u_{n+1}) = P_K\left[g(u_n) - \alpha_n T v_n\right],$$

where

$$\alpha_n = \frac{(1 - \sigma)\|R(u_n)\|^2}{\|Tv_n\|^2},$$

$$Tv_n = T P_K[u_n - \rho_n Tu_n].$$
Algorithm 3.4 is an improved version of the extragradient-type method. See He and Liao [4] with different predictor search line and corrector step size.

Since $K$ is convex set, for all $\eta \in [0, 1]$, $g(u), P_K[g(u) - \rho Tu] \in K$, we have

$$g(u) = (1 - \eta)g(u) + \eta P_K\left[g(u) - \rho Tu\right] = g(u) - \eta R(u) \in K.$$  \hspace{1cm} (3.6)

Using (3.6), we rewrite (3.1) in the form

$$g(u) = P_K\left[g(u) - \rho T g^{-1}(g(u) - \eta R(u))\right].$$  \hspace{1cm} (3.7)

This fixed-point formulation is used to suggest and analyze the following modified extragradient method for general variational inequalities (2.1).

**Algorithm 3.5.** For a given $u_0 \in H$, compute the approximate solution $u_{n+1}$ by the iterative schemes:

*Predictor step.*

$$g(w_n) = g(u_n) - \eta_n R(u_n),$$  \hspace{1cm} (3.8)

where $\eta_n = a^m$, and $m_k$ is the smallest nonnegative integer $m$ such that

$$\rho_n \eta_n \langle Tu - T g^{-1}(g(u_n) - a^m R(u_n)), R(u_n) \rangle \leq \sigma \|R(u_n)\|^2,$$

$$\sigma \in (0, 1).$$  \hspace{1cm} (3.9)

*Corrector step.*

$$g(u_{n+1}) = P_K\left[g(u_n) - \alpha_n T g^{-1}(g(u_n) - \eta_n R(u_n))\right],$$

$$n = 0, 1, 2, \ldots,$$  \hspace{1cm} (3.10)

where

$$\alpha_n = \frac{(\eta_n - \sigma)\|R(u_n)\|^2}{\|T g^{-1}(g(u_n) - \eta_n R(u_n))\|^2}.$$  \hspace{1cm} (3.11)

For $g \equiv I$, where $I$ is the identity operator, we obtain a variant form of the modified extragradient-type methods for solving variational inequalities, which have been studied by Wang et al. [33] with different search line and step size.

For the convergence analysis of Algorithm 3.5, we need the following results.

**Lemma 3.2.** Let $\bar{u} \in H$ be a solution of (2.1). If the operator $T$ is pseudomonotone operator, then

$$\langle g(u) - g(\bar{u}), T g^{-1}(g(u) - \eta R(u)) \rangle \geq (\eta - \sigma) \|R(u)\|^2,$$

for all $u \in H$.  \hspace{1cm} (3.12)

**Proof.** Let $\bar{u} \in H$ be a solution of (2.1). Then

$$\langle T \bar{u}, g(v) - g(\bar{u}) \rangle \geq 0, \quad \text{for all } g(v) \in K,$$

implies
\[ \langle Tv, g(v) - g(\bar{u}) \rangle \geq 0, \]  
(3.13)

since \( T \) is pseudomonotone.

Now taking \( g(v) = g(u) - \eta R(u) \) in (3.13), we obtain
\[ \langle Tg^{-1}(g(u) - \eta R(u)), g(u) - \eta R(u) - g(\bar{u}) \rangle \geq 0, \]
from which we have
\[ \langle g(u) - g(\bar{u}), \rho Tg^{-1}(g(u) - \eta R(u)) \rangle \]
\[ \geq \eta \rho \| R(u) \| Tg^{-1}(g(u) - \eta R(u)) \|
\geq -\eta \rho \| R(u), Tu - Tg^{-1}(g(u) - \eta g(u)) \| + \rho \eta \| Tu, R(u) \|
\geq -\sigma \rho \| R(u) \|^2 + \rho \eta \| Tu, R(u) \|. \]  
(3.14)

Taking \( z = g(u) - \rho Tu, u = P_K[g(u) - \rho Tu], v = g(u) \) in (2.4), we obtain
\[ \langle P_K[g(u) - \rho Tu] - g(u) + \rho Tu, g(u) - P_K[g(u) - \rho Tu] \rangle \geq 0, \]
from which it follows that
\[ \langle \rho Tu, R(u) \rangle \geq \| R(u) \|^2. \]  
(3.15)

Combining (3.14) and (3.15), we have
\[ \langle g(u) - g(\bar{u}), \rho Tg^{-1}(g(u) - \eta R(u)) \rangle \geq (\eta - \sigma) \| R(u) \|^2, \]
the required results. \( \square \)

**Lemma 3.3.** Let \( \bar{u} \in H \) be a solution of (2.1) and let \( u_{n+1} \) be the approximate solution obtained from Algorithm 3.5. Then
\[ \| g(u_{n+1}) - g(\bar{u}) \|^2 \leq \| g(u_n) - g(\bar{u}) \|^2 - \frac{(\eta_n - \sigma)^2 \| R(u_n) \|^4}{\| Tg^{-1}(g(u_n) - \eta_n R(u_n)) \|^2}. \]  
(3.16)

**Proof.** From (3.10)–(3.12), we have
\[ \| g(u_{n+1}) - g(\bar{u}) \|^2 \leq \| g(u_n) - g(\bar{u}) \|^2
\leq \| g(u_n) - g(\bar{u}) \|^2
- 2\alpha_n \| g(u_n) - g(\bar{u}), Tg^{-1}(g(u_n) - \eta_n R(u_n)) \|
+ \alpha_n^2 \| Tg^{-1}(g(u_n) - \eta_n R(u_n)) \|^2
\leq \| g(u_n) - g(\bar{u}) \|^2
- 2\alpha_n (\eta_n - \sigma) \| R(u_n) \|^2
+ \alpha_n^2 \| Tg^{-1}(g(u_n) - \eta_n R(u_n)) \|^2
\leq \| g(u_n) - g(\bar{u}) \|^2
- \frac{(\eta_n - \sigma)^2 \| R(u_n) \|^4}{\| Tg^{-1}(g(u_n) - \eta_n R(u_n)) \|^2}, \]
the required result. \( \square \)
Theorem 3.1. Let $u_{n+1}$ be the approximate solution obtained from Algorithm 3.1 and $\bar{u} \in H$ be the solution of (2.1). If $H$ is a finite-dimensional subspace and $g$ is injective, then $\lim_{n \to \infty} u_n = \bar{u}$.

Proof. Let $u^* \in H$ be a solution of (2.1). Then, from (3.16), it follows that the sequence $\{u_n\}$ is bounded and

$$\sum_{n=0}^{\infty} \frac{(\eta_n - \sigma)^2 \| R(u_n) \|^4}{\| T g^{-1} (g(u_n) - \eta_n R(u_n)) \|^2} \leq \left\| g(u_0) - g(u^*) \right\|^2,$$

which implies that either

$$\lim_{n \to \infty} R(u_n) = 0 \quad (3.17)$$

or

$$\lim_{n \to \infty} \eta_n = 0. \quad (3.18)$$

Assume that (3.17) holds. Let $\bar{u}$ be the cluster point of $\{u_n\}$ and the subsequence $\{u_{n_i}\}$ of the sequence $\{u_n\}$ converge to $\bar{u}$. Since $R$ is continuous, it follows that

$$R(\bar{u}) = \lim_{i \to \infty} R(u_{n_i}) = 0,$$

which implies that $\bar{u}$ is a solution of (2.1) by invoking Lemma 3.1 and

$$\left\| g(u_{n+1}) - g(\bar{u}) \right\|^2 \leq \left\| g(u_n) - g(\bar{u}) \right\|^2. \quad (3.19)$$

Thus the sequence $\{u_n\}$ has exactly one cluster point and consequently

$$\lim_{n \to \infty} g(u_n) = g(\bar{u}).$$

Since $g$ is injective, it follows that $\lim_{n \to \infty} u_n = \bar{u} \in H$ satisfying the general variational inequality (2.1).

Assume that (3.18) holds, that is, $\lim_{n \to \infty} \eta_n = 0$. If (3.9) does not hold, then by a choice of $\eta_n$, we obtain

$$\sigma \left\| R(u_n) \right\|^2 \leq \rho_n \eta_n \langle T u_n - T g^{-1}(g(u_n) - \eta_n R(u_n)), R(u_n) \rangle. \quad (3.20)$$

Let $\bar{u}$ be a cluster point of $\{u_n\}$ and let $\{u_{n_i}\}$ be the corresponding subsequence of $\{u_n\}$ converging to $\bar{u}$. Taking the limit in (3.20), we have

$$\sigma \left\| R(\bar{u}) \right\|^2 \leq 0,$$

which implies that $R(\bar{u}) = 0$, that is, $\bar{u} \in H$ is solution of (2.1) by invoking Lemma 3.1 and (3.20) holds. Repeating the above arguments, we conclude that $\lim_{n \to \infty} u_n = \bar{u}$. \qed

4. Wiener–Hopf equations technique

In this section, we suggest another class of modified extragradient-type method for solving general variational inequalities (2.1) using the Wiener–Hopf equations technique. For this purpose, we need the following result, which is mainly due to Noor [9,11].
Lemma 4.1. The general variational inequality (2.1) has a unique solution \( u \in H \), \( g(u) \in K \) if and only if the Wiener–Hopf equation (2.6) has a unique solution \( z \in H \), where

\[
g(u) = P_K z \quad \text{and} \quad z = g(u) - \rho T u.
\] (4.1)

From Lemma 4.1, we see that both the problems (2.1) and (2.6) are equivalent. Using (4.1), we can rewrite the Wiener–Hopf equations (2.6) in the form

\[
g(u) - P_K [g(u) - \rho T u] - \rho T u + \rho T g^{-1} P_K [g(u) - \rho T u] = R(u) - \rho T u + \rho T g^{-1} P_K [g(u) - \rho T u] = 0.
\] (4.2)

Invoking Lemmas 2.1 and 3.1, one can easily show that \( u \in H \), \( g(u) \in K \) is solution of (2.1) if and only if \( u \in H \), \( g(u) \in K \) is a zero of Eq. (4.2).

For a positive step \( \alpha \), Eq. (4.2) can be written as

\[
g(u) = g(u) - \alpha d_1(u).
\] (4.3)

where

\[
d_1(u) = T(u) - \rho T u + \rho T g^{-1} P_K [g(u) - \rho T u],
\] (4.4)

This fixed-point formulation has been used to develop some very effective and efficient iterative projection methods for solving various classes of variational inequalities and complementarity problems; see, for example, [3,10,13,20,21,29] and references therein.

We here use the fixed-point formulation (4.3) to suggest the following modified projection-type method for general variational inequalities (2.1).

Algorithm 4.1. For a given \( u_0 \in H \), compute the approximate solution \( u_{n+1} \) by the iterative schemes

\[
g(u_{n+1}) = P_K [g(u_n) - \alpha_n d_1(u_n)], \quad n = 0, 1, 2, \ldots,
\]

where \( \rho_n \) (prediction) satisfies

\[
\rho_n [T u_n - T g^{-1} P_K [g(u_n) - \rho_n T u_n], R(u_n)] \leq \sigma \left\| R(u_n) \right\|^2, \quad \sigma \in (0, 1),
\]

and

\[
d_1(u_n) = R(u_n) - \rho_n T u_n + \rho_n T g^{-1} P_K [g(u_n) - \rho_n T u_n],
\]

\[
\alpha_n = \frac{(1 - \sigma) \left\| R(u_n) \right\|^2}{\left\| d_1(u_n) \right\|^2}
\]

is the corrector step size.

For \( g = I \), where \( I \) is the identity operator, Algorithm 4.1 reduces to
Algorithm 4.2. For a given $u_n \in H$, compute the approximate solution $u_{n+1}$ by the iterative schemes

$$ u_{n+1} = P_K\left[u_n - \alpha_n d_2(u_n)\right], \quad n = 0, 1, 2, \ldots, $$

where $\rho_n$ (prediction) satisfies

$$ \rho_n \langle Tu_n - TP_K [u_n - \rho_n Tu_n], R(u_n) \rangle \leq \sigma \|R(u_n)\|^2, \quad \sigma \in (0, 1), $$

and

$$ d_2(u_n) = R(u_n) - \rho_n Tu_n + \rho_n TP_K [u_n - \rho_n Tu_n]. $$

$$ \alpha_n = \frac{(1 - \sigma)\|R(u_n)\|^2}{\|d_2(u_n)\|^2} $$

is the corrector step size.

Algorithm 4.2 can be viewed as an improvement of the modified projection-type methods of He [3], Solodov and Tseng [29] and Noor [13] for solving classical variational inequalities (2.2) with different (predictor, corrector) step size.

Now we suggest an improved version of an extragradient-type method, which involves the Wiener–Hopf equation as a step size.

Algorithm 4.3. For a given $u_0 \in H$, compute $u_{n+1}$ by the iterative schemes:

**Predictor step.**

$$ g(v_n) = P_K\left[g(u_n) - \rho_n Tu_n\right], \quad n = 0, 1, 2, \ldots, $$

where $\rho_n$ satisfies

$$ \rho_n \langle Tu_n - T^{-1}P_K [g(u_n) - \rho_n Tu_n], R(u_n) \rangle \leq \sigma \|R(u_n)\|^2, \quad \sigma \in (0, 1). $$

**Corrector step.**

$$ g(u_{n+1}) = P_K\left[g(u_n) - \rho_n Tv_n\right], \quad n = 0, 1, 2, \ldots, $$

where

$$ d_1(u_n) = R(u_n) - \rho_n Tu_n + \rho_n T^{-1}P_K [g(u_n) - \rho Tu_n]. $$

$$ \alpha_n = \frac{(1 - \sigma)\|R(u_n)\|^2}{\|d_1(u_n)\|^2} $$

is the corrector step size.

For $g \equiv I$, where $I$ is the identity operator, Algorithm 4.3 is due to He and Liao [4] for the classical variational inequalities (2.2).

We now modify the Wiener–Hopf equation to suggest some modified projection-type methods. Using (3.4), we can rewrite (4.3) in the form

$$ g(u) = g(u) - \alpha d(u), \quad (4.5) $$
where
\[ d(u) = \eta R(u) - \eta \rho Tu + \rho T g^{-1}\left(g(u) - \eta R(u)\right), \]  
(4.6)
where \( \eta \) and \( \rho \) are positive constants.

Noor and Rassias [22] used the fixed-point formulation (4.5) to suggest and analyze the following modified projection method for solving general variational inequalities (2.1).

**Algorithm 4.4.** For a given \( u_0 \in H \), compute \( u_{n+1} \) by the iterative schemes:

**Predictor step.**
\[ g(u_n) = g(u_n) - \eta_n R(u_n), \]
where \( \eta_n = a^{m_n} \) and \( m_n \) is the nonnegative integer such that
\[ \rho_n \eta_n \left(T(u_n) - \rho_n T g^{-1}\left(g(u_n) - \eta_n R(u_n)\right), R(u_n)\right) \leq \sigma \| R(u_n) \|^2, \quad \sigma \in (0, 1). \]  
(4.8)

**Corrector step.**
\[ g(u_{n+1}) = P_K\left[g(u_n) - \alpha_n d(u_n)\right], \quad n = 0, 1, 2, \ldots, \]
\[ d(u_n) = \eta_n R(u_n) - \eta_n \rho_n T u_n + \rho_n T g^{-1}\left(g(u_n) - \eta_n R(u_n)\right), \]
\[ \alpha_n = \eta_n (1 - \sigma) \| R(u_n) \|^2 / \| d(u_n) \|^2. \]
(4.9)

For the convergence analysis of Algorithm 4.4, see [22]. For \( g = I \), the identity operator, algorithm is due to Noor et al. [24], where they gave some examples to illustrate the efficiency of the method.

We now suggest a new unified extragradient methods which combines the main feature of Algorithms 4.4 and 4.3 and is the main motivation of this paper.

**Algorithm 4.5.** For a given \( u_0 \in H \), compute the approximate solution \( u_{n+1} \) by the iterative schemes:

**Predictor step.**
\[ g(u_n) = g(u_n) - \eta_n R(u_n), \quad n = 0, 1, 2, \ldots, \]  
(4.7)
where \( \eta_n \) satisfies
\[ \rho_n \eta_n \left(T(u_n) - \rho_n T g^{-1}\left(g(u_n) - \eta_n R(u_n)\right), R(u_n)\right) \leq \sigma \| R(u_n) \|^2, \quad \sigma \in (0, 1). \]  
(4.8)

**Corrector step.**
\[ g(u_{n+1}) = P_K\left[g(u_n) - \alpha_n T g^{-1}\left(g(u_n) - \eta_n R(u_n)\right)\right], \quad n = 0, 1, 2, \ldots, \]
(4.9)
where
\[ D(u_n) = R(u_n) - \rho_n T u_n + \rho_n \eta_n T g^{-1}\left(g(u_n) - \eta_n R(u_n)\right), \]
\[ \alpha_n = \frac{(1 - \sigma) \| R(u_n) \|^2}{\| D(u_n) \|^2}. \]  
(4.11)
For $g \equiv I$, the identity operator, Algorithm 4.5 is exactly the same as in Noor [17] for classical variational inequalities (2.2).

For the sake of simplicity and without loss of generality, we take $\rho_n = 1$, we denote $u_n$ by $u$, $\eta_n$ by $\eta$, $\alpha_n$ by $\alpha$, and

$$M(u) = Tg^{-1}(g(u) - \eta R(u)), \quad (4.12)$$

$$D(u) = R(u) - Tu + \eta M(u), \quad (4.13)$$

$$u(\alpha) = P_K[g(u) - \alpha M(u)], \quad (4.14)$$

From (4.13) and (4.8), one can easily obtain

$$\langle R(u), D(u) \rangle = \lVert R(u) \rVert^2 - \langle R(u), Tu - \eta M(u) \rangle \geq (1 - \sigma) \lVert R(u) \rVert^2 \quad (4.15)$$

and

$$\langle R(u), D(u) \rangle = \lVert R(u) \rVert^2 - \langle R(u), Tu - \eta M(u) \rangle$$

$$\quad = \frac{1}{2} \lVert R(u) \rVert^2 - \langle R(u), Tu - \eta M(u) \rangle + \frac{1}{2} \lVert Tu - \eta M(u) \rVert^2$$

$$\quad \geq \frac{1}{2} \lVert D(u) \rVert^2. \quad (4.16)$$

We now study the convergence of Algorithm 4.5 and show that its convergence requires only the pseudomonotonicity, which is weaker condition than monotonicity.

**Theorem 4.1.** Let $\bar{u} \in H$ be a solution of (2.1). If the operator $T : H \to H$ is a pseudomonotone operator, then

$$\lVert \bar{u} - u(\alpha) \rVert \leq \lVert \bar{u} - u \rVert - \frac{(1 - \sigma)}{2} \lVert R(u) \rVert^2, \quad \text{for all } v \in H. \quad (4.17)$$

**Proof.** Let $\bar{u} \in H$ be a solution of (2.1). Then, as in Lemma 3.2, we obtain

$$\langle g(u) - g(\bar{u}), M(u) \rangle \geq \eta \{ R(u), M(u) \} = \{ R(u), \eta M(u) \}. \quad (4.18)$$

Setting $z = g(u) - Tu$, $v = u(\alpha)$, $u = P_K[g(u) - Tu]$ in (2.4), we have

$$\{ P_K[g(u) - Tu] - g(u) + Tu, u(\alpha) - P_K[g(u) - Tu] \} \geq 0,$$

which implies

$$\{ g(u) - u(\alpha), R(u) - Tu \} \geq \{ R(u), R(u) - Tu \}. \quad (4.19)$$

From (2.5) and (4.18), we have

$$\lVert u(\alpha) - g(u) \rVert^2 \leq \lVert g(u) - \alpha M(u) - g(\bar{u}) \rVert^2 - \lVert g(u) - \alpha M(u) - u(\alpha) \rVert^2$$

$$= \lVert g(u) - g(\bar{u}) \rVert^2 - \lVert g(u) - u(\alpha) \rVert^2$$

$$+ 2\alpha \{ g(u) - u(\alpha), M(u) \} - 2\alpha \{ g(u) - g(\bar{u}), M(u) \}$$

$$\leq \lVert g(u) \rVert^2 - \lVert g(u) - u(\alpha) \rVert^2 - 2\alpha \{ R(u), \eta M(u) \}$$

$$- 2\alpha \{ g(u) - u(\alpha), M(u) \},$$
from which it follows

\[ \| g(u) - g(\bar{u}) \|^2 - \| g(u) - u(\alpha) \|^2 \]

\[ \geq 2\alpha \langle R(u), \eta M(u) \rangle + \| g(u) - u(\alpha) \|^2 - 2\alpha \| g(u) - u(\alpha) \|^2 \]

\[ = 2\alpha \langle R(u), \eta M(u) \rangle + \| g(u) - u(\alpha) - \alpha D(u) \|^2 - \alpha^2 \| D(u) \|^2 \]

\[ + 2\alpha \| g(u) - u(\alpha), D(u) - \eta M(u) \| \]

\[ \geq 2\alpha \langle R(u), \eta M(u) \rangle + \| g(u) - u(\alpha) - \alpha D(u) \|^2 - \alpha^2 \| D(u) \|^2 \]

\[ + 2\alpha \langle R(u), R(u) - Tu \rangle \quad (\text{using } (4.19)) \]

\[ = 2\alpha \langle R(u), R(u) - Tu + \eta M(u) \rangle - \alpha^2 \| D(u) \|^2, \quad (4.20) \]

which is a quadratic in \( \alpha \) and has a maximum at

\[ \alpha^* = \frac{\langle R(u), R(u) - Tu + \eta M(u) \rangle}{\| D(u) \|^2} = \frac{\langle R(u), D(u) \rangle}{\| D(u) \|^2} = h(u). \quad (4.21) \]

From (4.15), (4.16), (4.20) and (4.21), we have

\[ \| g(u) - g(\bar{u}) \|^2 - \| g(u) - u(\alpha) \|^2 \]

\[ \geq \alpha^* \| R(u), D(u) \| = h(u) \| R(u), D(u) \| = \frac{1}{2} h(u) \| D(u) \|^2 \]

\[ = \frac{1}{2} \| R(u), D(u) \| \geq \frac{(1 - \sigma)}{2} \| R(u) \|^2, \]

that is,

\[ \| g(\bar{u}) - u(\alpha) \|^2 \leq \| g(u) - g(\bar{u}) \|^2 - \frac{(1 - \sigma)}{2} \| R(u) \|^2, \]

the required result. □

**Theorem 4.2.** Let \( u_{n+1} \) be the approximate solution obtained from Algorithm 4.5 and let \( \bar{u} \) be a solution of (2.1). Then

\[ \lim_{n \to \infty} (u_n) = \bar{u}. \]

**Proof.** Its proofs is similar to that of Theorem 3.1. □

### 5. Applications

In this section we show that the results derived in the previous section can be extended for a class of quasi-variational inequalities. If the convex set \( K \) depends upon the solution explicitly or implicitly, then variational inequality problem is known as the quasi-variational inequality. For a given operator \( T : H \to H \), and a point-to-point mapping \( K : u \to K(u) \), which associates a closed convex-valued set \( K(u) \) with any element \( u \) of \( H \), we consider the problem of finding \( u \in K(u) \) such that
\[ \langle Tu, v - u \rangle \geq 0, \quad \text{for all } v \in K(u). \] (5.1)

Inequality of type (5.1) is called the quasi-variational inequality. For the formulation, applications, numerical methods and sensitivity analysis of the quasi-variational inequalities, see [2,11,13–15,23,25] and references therein.

Using Lemma 2.1, one can show that the quasi-variational inequality (5.1) is equivalent to finding \( u \in K(u) \) such that

\[ u = P_{K(u)}[u - \rho Tu]. \] (5.2)

In many important applications, the convex-valued set \( K(u) \) is of the form

\[ K(u) = m(u) + K, \] (5.3)

where \( m \) is a point-to-set mapping and \( K \) is a closed convex set.

From (5.3) and (5.2), we see that problem (5.1) is equivalent to

\[ u = P_{K(u)}[u - \rho Tu] = P_{m(u) + K}[u - \rho Tu] = m(u) + P_K[u - m(u) - \rho Tu] \]

which implies that

\[ g(u) = P_K[g(u) - \rho Tu] \quad \text{with } g(u) = u - m(u), \]

which is equivalent to the general variational inequality (2.1) by an application of Lemma 3.1. We have shown that the quasi-variational inequalities (5.1) with the convex-valued set \( K(u) \) defined by (5.3) are equivalent to the general variational inequalities (2.1). Thus all the results obtained in this paper continue to hold for quasi-variational inequalities (5.1) with \( K(u) \) defined by (5.3).

References