r-partite Covers of Graphs

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Abstract

We formulate three new NP-hard optimization problems, namely the Minimum $r$-partite Cover (MRC) problem, the Minimum Connected $r$-partite Cover (MCRC) problem and the Minimum Connected $r$-partite Decomposition (MCRD) problem. These problems generalize the various known problems of covering the edges of a graph with a minimum number of bipartite subgraphs. We define new graph parameters, namely $r$-particity, connected $r$-particity and $r$-decomposability of a graph as the minimum number of subgraphs in a solution of MRC, MCRC and MCRD problems respectively. We obtain integer programming formulations and heuristic algorithms for solving these optimization problems and study the relationship of the newly defined parameters with each other and with other well-known graph parameters like the chromatic number. Furthermore, Nordhaus-Gaddum type theorems are established for the $r$-particity to better understand the behavior of this parameter.

Key words and phrases: Edge covers and decompositions, $r$-partite cover, $r$-particity.

1 Introduction

Let $G$ be a graph with possible multiple edges, but no loops. A family $C$ of non-empty (not necessarily spanning) subgraphs of $G$ is said to be an edge cover, or simply a cover, of $G$ if every edge of $G$ is contained in at least one subgraph in $C$. Additionally, if every edge appears in exactly one subgraph in $C$, then $C$ is said to be an edge decomposition, or simply a decomposition, of $G$ [4].
The problem of covering the edges of a graph by bipartite subgraphs has generated a lot interest over the years in graph theory and operations research communities [1, 16, 20, 22, 23, 24]. The minimum number of bipartite subgraphs required to cover a graph $G$ is called the bipartiticy of $G$, denoted by $\beta(G)$ [4, 13] and is a measure of how far a graph is from being bipartite. Harary et al. [13] studied the relationship between biparticity and the chromatic number $\chi(G)$ of a graph $G$, showing that

$$\beta(G) = \lceil \log_2 \chi(G) \rceil$$

(1)

Many variants of the bipartite cover problem have been studied. Given a graph $G$, the Minimum Bipartite Graph Cover (MBGC) problem [16, 23] is to find a minimum cardinality family $C$ of connected bipartite subgraphs of $G$ such that every edge of $G$ is contained in at least one subgraph in $C$. If we require every edge of $G$ to be present in exactly one subgraph in $C$, the problem becomes the Minimum Bipartite Decomposition (MBD) [2] problem. Thus the subgraphs in a bipartite decomposition are pairwise edge-disjoint. It has been shown that the four color theorem is equivalent to the statements that every planar graph has a bipartite decomposition consisting of two subgraphs [18].

The interest in the above covering problems is both theoretical and practical. First of all, these problems are NP-hard, making them interesting from a theoretical point of view. Secondly, they find many practical applications (see [7, 8, 12]).

In this paper, we aim to generalize all the above problems and study the $r$-partite covers of a graph $G$, where $r \geq 2$ is an integer. We propose to formulate the Minimum $r$-partite Cover (MRC), the Minimum Connected $r$-partite Cover (MCRC) and the Minimum Connected $r$-partite Decomposition (MCRD) problems and define the $r$-particity $\beta_r(G)$, connected $r$-particity $\gamma_r(G)$ and $r$-decomposability $\delta_r(G)$ of a graph $G$, as the number of subgraphs in an optimal solution of the MRC, MCRC and MCRD problems respectively.

We show that all these problems are NP-hard. In Section 3, we provide integer linear programming (ILP) formulations for the MRC, MCRC and MCRD problems and also discuss the complete $r$-partite analogues of these problems. Moreover, we study the relationships between the newly defined parameters $\beta_r(G)$, $\gamma_r(G)$, $\delta_r(G)$ and other graph parameters such as the chromatic number. We also derive Nordhaus-Gaddum type inequalities to bound the $r$-particity of a graph.
2 Related Work

A biclique is a complete bipartite graph. The problem of covering the edges of a graph by bicliques, called the Minimum Biclique Cover (MBC) problem has attracted a lot of attention over the years [1, 20, 22, 24]. The problem is NP-hard in general but can be solved in polynomial time for bipartite domino-free graphs [2]. The MBC problem also has practical significance. Some of the important applications include continuous relaxation based methods [7], encoding of partial orders by bit vectors [11] and heuristic coloring algorithms [8].

Amilhastre et al. [2] examined the relationship between the Minimum Biclique Cover (MBC) and Minimum Biclique Decomposition (MBD) problems, proving that the two problems coincide for bipartite domino-free graphs. It is still open to study the corresponding problems and their relationship for $r$-partite (not necessarily complete) subgraph covers.

The bipartite dimension $bc(G)$ of a graph $G$ is defined as the minimum number of bicliques in a biclique cover of $G$ [9]. Various authors have studied bounds on $bc(G)$. Most notably, Tuza [24] proved the upper bound $bc(G) < n - \log_2 \lceil n \rceil + 1$ for any graph $G$ with $n$ vertices. Another tight bound is given by $\lceil \log_2 \chi(G) \rceil < bc(G) < \tau(G)$, where $\tau(G)$ is the minimum number of stars required to cover the edge set of $G$ [13].

Despite the interest in biclique covers, there has been relatively less focus on covering the edges of a graph with bipartite, not necessarily complete, subgraphs. Harary et al. [13] studied the minimum bipartite cover problem and defined the biparticity of a graph as the minimum number of subgraphs in a bipartite cover. They further proved that the biparticity $\beta(G)$ of a graph $G$ is related to its chromatic number $\chi(G)$ by equation (1). The $r$-partite analogue of biparticity, which we refer to as $r$-particity, has so far not been studied.

In 1956, Nordhaus and Gaddum [21] proved their classical results bounding the sum and product of chromatic number of a graph and its complement by the number of vertices. Thereafter, similar results have been derived for a number of other graph parameters such as total chromatic number, fractional chromatic number, domination number, total domination number, independence number and matching number to name a few. A comprehensive survey of these results appears in [3]. However, currently no such relations exist for biparticity or its $r$-partite analogue.

The bipartite decompositions of planar graphs were considered by Hedetneimi [14] who proved that the edge set of every planar graph can be partitioned into
three bipartite subgraphs. The result was improved in [18], where it was shown that the four color theorem is equivalent to the statement that every planar graph has a decomposition into two bipartite, not necessarily spanning, subgraphs.

Plateau, Liberti and Alfandari [23] formulated the Minimum Bipartite Graph Cover (MBGC) problem as the problem of finding minimum cardinality covers of graphs consisting of connected bipartite subgraphs. The ILP formulation of the problem and heuristic algorithms were presented in [23]. Improved heuristics for the problem were discussed in [16]. The MBGC decision problem has been shown to be NP-Complete by reduction from the Minimum Cut Cover (MCC) problem [5] and therefore has the same applications such as electronic boards testing [17], fault analysis, medical diagnostics, and pattern recognition [12].

Gregory and Meulen [10] considered the problem of decomposing a graph into complete multipartite subgraphs with at most \( r \) partite sets. They established sharp bounds for the minimum number \( m_r(G) \) of complete (\( \leq r \))-partite subgraphs required to decompose a graph \( G \) with \( n \) vertices.

However, to our knowledge, no attempt has been made to study the \( r \)-partite edge covers of graphs, for a fixed value of \( r \). It is only natural to consider this generalization as \( r \)-partite covers inherit the computational complexity and practical applications of bipartite covers and have a scope for new applications. So far the authors have done some initial work in this direction [15].

3 Problem Formulations

In this section, we formally define three new optimization problems related to edge covers by \( r \)-partite subgraphs. Given a graph \( G \), the Minimum \( r \)-partite Cover (MRC) problem is to find a minimum cardinality family \( C \) of \( r \)-partite subgraphs of \( G \) such that every edge of \( G \) is contained in at least one subgraph in \( C \). If the subgraphs are required to be connected, the problem is said to be the Minimum Connected \( r \)-partite Cover (MCRC) problem. If we additionally require every edge of \( G \) to be present in exactly one subgraph in \( C \), the problem is called Minimum Connected \( r \)-partite Decomposition (MCRD) problem.

We define the \( r \)-particity, \( \beta_r(G) \), of a graph \( G \) as the minimum number of \( r \)-partite subgraphs needed to cover the edges of \( G \), whereas the connected \( r \)-particity, \( \gamma_r(G) \), is defined as the minimum number of connected \( r \)-partite subgraphs in an edge
Figure 1: A minimal bipartite cover and a minimal bipartite connected cover of $2K_3$

cover of $G$. Finally, we define the $r$-decomposability, $\delta_r(G)$, as the minimum number of connected $r$-partite subgraphs in an edge decomposition of $G$. Thus $\beta_r(G)$, $\gamma_r(G)$ and $\delta_r(G)$ respectively equal the number of subgraphs in an optimal solution of MRC, MCRC and MCRD problems.

Note that these problems are different from one another and hence the parameters $\beta_r(G)$, $\gamma_r(G)$ and $\delta_r(G)$ have different values in general. To see this consider the following examples.

**Example 1.** Let $G = 2K_3$, that is, the disjoint union of two copies of $K_3$ (the complete graph on three vertices). Then $\beta_2(G) = 2$ and $\gamma_2(G) = 4$ and $\delta_r(G) = 4$ as shown in Figure 1. Thus in general, $\beta_r(G) \neq \gamma_r(G)$.

**Example 2.** Let $H$ be the graph obtained by joining two copies of $K_3$ by a single edge. Then $\beta_2(G) = 2$, $\gamma_2(G) = 2$ and $\delta_r(G) = 3$ as shown in Figure 2. This shows that in general, $\beta_r(G) \neq \delta_r(G)$.

The following result shows that all these problems are NP-hard. Recall that the Graph $k$-Colorability Problem (GCP) is the decision problem that asks whether the vertices of a graph can be colored by $k$ colors. It is well-known that for $k \geq 3$, GCP
A minimum (connected) bipartite cover of $H$

A minimum (connected) bipartite cover of $H$

A minimum connected bipartite decomposition of $H$

Figure 2: A minimal bipartite cover and a minimal bipartite connected cover of $H$

is NP-complete. An instance of GCP consists of a graph $G$ and a positive integer $c$ and we denote it by $GCP(G, c)$.

**Theorem 3.1.** The problems MRC, MCRC and MCRD are NP-hard.

**Proof:** We prove the assertion about MRC by showing that the corresponding decision problem is NP-complete. Consider the MRC decision problem, that is, given a graph $G$ and a positive integer $k$, decide if $G$ is the union of $k$ subgraphs that are $r$-partite. We demonstrate the NP-completeness of the MRC decision problem by reduction from GCP. Given an instance of $GCP(G, c)$ GCP, Theorem 5.5 of Section 4 shows that we can transform it into an MRC instance $MRC(G, \lceil \log_r c \rceil)$ in polynomial time. Thus any algorithm that solves MRC decision problem in polynomial time also solves GCP in polynomial time.

Since MBGC, which is a special case of MCRC (case $r = 2$) is NP-hard, it follows that MCRC is NP-hard. Similar arguments prove that MCRD is NP-hard.

We now provide integer programming formulations of MRC, MCRC and MCRD problems. Let $C = \{G_i\}_{i=1}^n$ be a family of $r$-partite subgraphs of a graph $G$. If $\bigcup_{i=1}^n G_i = G$, we say that $C$ is a cover of $G$. For the MRC problem we want to find a cover $C$ with minimum cardinality. Let $n'$ be an upper bound for $n$ and let $V(G)$ and
respectively denote the vertex and edge sets of \( G \). We introduce the following sets of binary variables for \( k = 1, \ldots, n' \), \( i \in V(G) \) and \( \{i, j\} \in E(G) \).

\[
x_k = \begin{cases} 
1 & \text{if the } k^{th} \text{ } r - \text{partite subgraph lies in } C \\
0 & \text{otherwise} 
\end{cases} \quad (2)
\]

\[
e_{ij}^k = \begin{cases} 
1 & \text{if edge } \{i, j\} \text{ lies in the } k^{th} \text{ } r - \text{partite subgraph} \\
0 & \text{otherwise} 
\end{cases} \quad (3)
\]

\[
y_{i,l}^k = \begin{cases} 
1 & \text{if vertex } i \text{ is in the } l^{th} \text{ partite set of the } k^{th} \text{ subgraph} \\
0 & \text{otherwise} 
\end{cases} \quad (4)
\]

The MRC problem can now be expressed as the following integer linear program.

**MRC Problem:**

\[
\min \sum_{k=1}^{n'} x_k
\]

subject to

\[
\sum_{k=1}^{n'} e_{ij}^k \geq 1 \quad (6)
\]

\[
\sum_{l=1}^{r} y_{i,l}^k = 1 \quad (7)
\]

\[
y_{i,l}^k + y_{j,l}^k \leq 1 \quad (8)
\]

for all \( i \in V(G), \{i, j\} \in E(G), k = 1, \ldots, n' \) and \( l = 1, \ldots, r \).

It can be readily seen that constraints (6) ensure an edge covering, while constraints (7) and (8) ensure that each subgraph in the cover is indeed \( r \)-partite. We can now formulate the other problems by modifying the above formulation of the MRC problem.

**MCRC Problem:**

The only additional requirement is that the subgraphs should be connected. The connectedness of subgraphs can be guaranteed by using additional flow variables and
flow constraints as discussed in [23] (constraints (12)-(16)) for the MBGC problem. Since the flow constraints are identical for MCRC we omit them here.

**MCRD Problem:**
The MCRD formulation can be obtained from the MRC formulation by modifying the covering constraints (6) to the decomposition constraints

\[ \sum_{k=1}^{n'} e^k_{ij} = 1 \]  

(9)

for all \( \{i, j\} \in E(G) \) and \( k = 1, \ldots, n' \); and adding the flow constraints of the MCRC problem.

**Complete r-partite MRC, MCRC and MCRD:**
Since there has been significantly more interest in Minimum Biclique Problem compared to Minimum Bipartite Graph Cover Problem, we also consider the MRC, MCRC and MCRD problems under the restriction that the subgraphs in the cover should be complete r-partite. This can be achieved by augmenting the following completeness constraint to either problem formulation.

\[ y^k_{i,l} + y^k_{j,m} \leq e^k_{ij} + 1 \]  

(10)

for all \( \{i, j\} \in E(G) \); \( k = 1, \ldots, n' \); \( l, m = 1, \ldots, r \) and \( l \neq m \).

### 4 Heuristic Algorithms

In this section, we present heuristics for the MRC, MCRC and MRD problems. First we need to introduce some notation. Throughout this section, \( G(V, E) \) denotes an input graph, whereas \( G_k(V^1_k, \ldots, V^r_k; E_k) \) denotes the \( k^{th} \) r-partite subgraph in an output of any of our heuristic algorithms. Here \( V^1_k, \ldots, V^r_k \) denote the vertex partite sets of \( G_k \) and \( E_k \) denotes the edge set of \( G_k \). An output will consist of \( n \) such graphs \( (k=1, \ldots, n) \) and as in Section 3, \( n' \) is a known upper bound on \( n \). For any \( v \in V \), we denote by \( N_E(v) \) the vertex neighbourhood of \( v \) in the graph whose vertes set is \( V \) and edge set is \( E \), that is \( N_E(v) \) is the set of vertices \( u \in V \) such that \( \{u, v\} \in E \). Similarly, let \( M_E(v) \) represent the edge neighbourhood of \( v \), that is the edges in \( E \) that are incident to \( v \). Finally, for any set \( S \), \(|S|\) stands for the cardinality of \( S \), while for any two sets \( S, T \), let \( S - T \) denote the set difference of \( S \) and \( T \).
The following is an efficient heuristic for determining a minimum connected \( r \)-partite cover of an input graph \( G(V,E) \).

**Algorithm 1** Heuristic for MCRC

Initialize \( G_k = \emptyset, k = 1, \ldots, n' \), and \( U_j(v) = \emptyset, j = 1, \ldots, r, v \in V \).

Let \( k = 1 \)

while \( E \neq \emptyset \) do

\[
\begin{align*}
\text{if } V_k^1 = V_k^2 = \cdots = V_k^r = \emptyset & \text{ then} \\
& \quad \text{Let } U = W \leftarrow V \\
\text{else} & \\
& \quad \text{Let } U = \{ v \in W : \text{ for all } u \in V_k^i, \{u,v\} \notin E, \text{ for some } j, \} \\
& \quad \quad \text{for } j = 1, \ldots, r, \text{ } v \in V \text{ do} \\
& \quad \quad \quad \{ U_j(v) = \{ x \in N_E(v) : j \text{ smallest with } N_E(v) \cap V_k^j \neq \emptyset \} \}
\end{align*}
\]

end if

If \( U = \emptyset \) then

Set \( k \leftarrow k + 1 \)

else

Let \( v \in U \) such that \( |N_E(v)| \) is minimum

Set \( V_k^1 \leftarrow V_k^1 \cup \{ v \} \)

\( W \leftarrow W - (N_E(v) \cup \{ v \}) \)

\( E \leftarrow E - ME(v) \)

\( E_k \leftarrow E_k \cup ME(v) \)

for \( j = 1, \ldots, r \) do

\( \{ V_k^j \leftarrow V_k^j \cup U_j(v) \} \)

end if

\}

Let \( n = k \)

Clearly, Algorithm 1 can be implemented in \( O(r |V|^4 |E|) \) time. Algorithm 1 also provides a basis for MRC and MRD heuristics. For instance, the MRC heuristic can be obtained from Algorithm 1 by removing connectivity requirements throughout. Algorithm 2 provides such a heuristic.

We conclude this section with a heuristic algorithm for the MRD problem. We first use Algorithm 1 to obtain a connected \( r \)-partite cover \( C = \{G_k\}_{k=1}^n \) of the input graph \( G \). We then apply the following algorithm to generate an \( r \)-partite decomposition of
Algorithm 2 Heuristic for MRC

Initialize $G_k = \emptyset$, $k = 1, \ldots, n'$, and $U_j(v) = \emptyset$, $j = 1, \ldots, r$, $v \in V$.

Let $k = 1$

while $E \neq \emptyset$ do

\{ if $V_k^1 = V_k^2 = \cdots = V_k^r = \emptyset$ then

Let $U = W \leftarrow V$

else

Let $U = \{v \in W : \text{for all } u \in V_k^1, \{u, v\} \notin E\}$

for $j = 1, \ldots, r$, $v \in V$ do

\{Distribute elements of $N_E(v)$ among $U_j(v)$'s as equally as possible\}

end if

If $U = \emptyset$ then

Set $k \leftarrow k + 1$

else

Let $v \in U$ such that $|N_E(v)|$ is minimum

Set $V_k^1 \leftarrow V_k^1 \cup \{v\}$

$W \leftarrow W - (N_E(v) \cup \{v\})$

$E \leftarrow E - M_E(v)$

$E_k \leftarrow E_k \cup M_E(v)$

for $j = 1, \ldots, r$ do

\{$V_k^j \leftarrow V_k^j \cup U_j(v)$\}

end if

\}

Let $n = k$
Algorithm 3 Heuristic for MRD

Let \( m \leftarrow n \)

for \( i = 1, \ldots, m \) do

{ for \( j = 1, \ldots, m \) do

{ (1) if \( E_i \cap E_j \neq \emptyset \) then

Select \( e \in E_i \cap E_j \), remove \( e \) from \( E_j \)

if \( G_j \) is disconnected then

In \( C \) replace \( G_j \) by its connected components

\( \{G_j, G_{j+1}, \ldots, G_{j+c}\} \) giving \( C = \{G_1, \ldots, G_j, G_{j+1}, \ldots, G_{j+c}, \ldots, G_{n+c}\} \)

Set \( m \leftarrow |C| \)

\( j \leftarrow j + 1 \)

Go to (1)

} } = 0

5 The \( r \)-particity and Other Parameters

In this section we study \( r \)-particity, \( \beta_r(G) \), its relationship with parameters \( \gamma_r(G) \) and \( \delta_r(G) \) defined in this paper and with other graph theoretic parameters like the chromatic number and genus. We also obtain Nordhaus-Gaddum type results that relate the \( r \)-particity of a graph and its complement. We first note that

**Theorem 5.1.** Given a graph \( G \), \( \beta_r(G) \leq \gamma_r(G) \leq \delta_r(G) \).

**Proof:** The \( r \)-particity \( \beta_r(G) \) of a graph \( G \) is the minimum number of, not necessarily connected, \( r \)-partite subgraphs needed to cover the edges of \( G \), whereas the connected \( r \)-particity \( \gamma_r(G) \) is the minimum number of connected \( r \)-partite subgraphs in an edge cover of \( G \). Hence \( \beta_r(G) \leq \gamma_r(G) \). Furthermore, the \( r \)-decomposability is the minimum number of connected \( r \)-partite subgraphs in an edge-disjoint cover of \( G \). Thus \( \gamma_r(G) \leq \delta_r(G) \).

Since we do not require connectedness, such a minimum edge cover can be easily transformed into an edge decomposition simply by deleting any repeated edges. Thus in the absence of any connectedness requirement, the minimum number of \( r \)-partite
subgraphs in a cover of $G$ is the same as the minimum number of $r$-partite subgraphs in a decomposition. Therefore, in our discussion of $r$-particity we can consider edge covers or edge decompositions as convenient.

Furthermore, all graphs considered in this section are simple. This will simplify the presentation of proofs, but we note that the results and proofs can be easily mirrored for multigraphs. We can assume all subgraphs in a covering of $G$ to be spanning subgraphs. Since we make no assumptions regarding the connectedness of subgraphs, any covering can be extended to a covering with spanning subgraphs by the addition of isolated vertices. Thus a decomposition of a graph with vertex set $V(G)$ and edge set $E(G)$, is a collection of subgraphs $\{G_i\}_{i=1}^{k}$ such that $V(G_i) = V(G)$ and $E(G_i) \subseteq E(G)$, for all $i = 1, 2, ..., k$. Additionally, we write $G = \bigoplus_{i=1}^{k} G_i$ to denote that $\{G_i\}_{i=1}^{k}$ is a decomposition of $G$.

Hedetniemi [3] proved that a planar graph has a decomposition into three bipartite subgraphs. Mabry [4] used a coloring argument to improve the result to two bipartite subgraphs. He describes a method to arrange a four coloring of a planar graph to obtain a bipartite decomposition. In fact, he showed that the four color theorem is equivalent to the statement that a planar graph has a decomposition into two bipartite subgraphs. We begin by generalizing this color decomposition idea to include all graphs and their multipartite subgraphs.

**Theorem 5.2.** Let $G$ be a graph and $r_1, r_2, \ldots, r_n$ be positive integers and $k = r_1 r_2 \cdots r_n$, then $G$ is $k$-colorable if and only if there exists a family $\{G_i\}_{i=1}^{n}$ of subgraphs of $G$, such that $G = \bigoplus_{i=1}^{n} G_i$ and for $i = 1, \ldots, n$ $G_i$ is an $r_i$-partite graph.

**Proof:** Let $V = V(G)$ and $E = E(G)$ and assume that $G$ is $k$-colorable, where $k = r_1 r_2 \cdots r_n$. Given any $k$-coloring $c$ of $G$, we can represent each color as an $n$-tuple $(j_1, j_2, \ldots, j_n)$, where each component $j_i$ takes values from $\{1, 2, \ldots, r_i\}$. We now construct the subgraphs $G_i$ recursively in the following way. Let $V(G_i) = V(G)$ and $E_i = E(G_i)$ consists of all edges of $E - \bigcup_{l=1}^{i-1} E_l$ (with the provision that $E_0 = \phi$) that join vertices whose colors differ in the $i^{th}$ component. For any $i$, $G_i$ is $r_i$-partite as its adjacent vertices can be properly $r_i$-colored by the $i^{th}$ components of the coloring $c$. Moreover, the $G_i$’s are edge-disjoint as each $E_i$ is a subset of $E - \bigcup_{l=1}^{i-1} E_l$ and $E = \bigcup_{i=1}^{n} E_i$.

Conversely, assume that $G = \bigoplus_{i=1}^{n} G_i$, where $G_i$ is an $r_i$-partite subgraph of $G$ for $i = 1, \ldots, n$. Also let $k = r_1 r_2 \cdots r_n$. It is clear that any $G_i$ is $r_i$-colorable. Let $c_i : V(G_i) \rightarrow \{1, 2, \ldots, r_i\}$ be a proper coloring of $G_i$. Then the function $c : V(G) \rightarrow \{1, 2, \ldots, r_i\}$ is a proper coloring of $G$. Thus, $G$ is $k$-colorable and we are done.
\{1, 2, \ldots, k\} defined by \(c(v) = (c_1(v), c_2(v), \ldots, c_n(v))\) is a proper \(k\)-coloring of \(G\) and hence \(G\) is \(k\)-colorable.

**Corollary 5.3.** Given a graph \(G\), there exists positive integers \(k\) and \(n\), positive integers \(r_1, \ldots, r_n\) all less than \(k\), positive integers \(\nu_1, \ldots, \nu_n\) and \(r\)-partite graphs \(\{G_i^{(j)}\}_{j=1}^{\nu_i}\), such that \(G = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{\nu_i} G_i^{(j)}\).

**Proof:** Since \(\chi(G) \leq V(G)\), there always exists at least one composite natural number \(k\) satisfying \(\chi(G) \leq k \leq V(G)\). Let \(k = r_1^{\nu_1}, r_2^{\nu_2}, \ldots, r_n^{\nu_n}\) be a factoring of \(k\). Since \(G\) is \(k\)-colorable, by Theorem 5.2 \(G\) can be decomposed into \(\sum_{i=1}^{n} \nu_i\) subgraphs \(\{G_i^{(j)}\}_{i=1}^{\nu_i}\), where the graphs \(\{G_i^{(j)}\}_{i=1}^{\nu_i}\) are \(r\)-partite.

Clearly, the decomposition in Corollary 5.3 is not unique as there can be multiple possible values of \(k\) and \(n\). The result has some similarity with results of elementary number theory and of abelian group theory.

It was proved in [18] that any 4-colorable graph can be decomposed into two bipartite subgraphs. The following result generalizes this assertion to decompositions of \(k\)-colorable graphs into \(r\)-partite subgraphs for any \(k \geq 4\) and \(r \geq 2\).

**Corollary 5.4.** If \(G\) is \(k\)-colorable and \(r\) and \(n\) are any positive integers such that \(r^{n-1} \leq k \leq r^n\), then there exist \(r\)-partite graphs \(\{G_i\}_{i=1}^{n}\) such that \(G = \bigoplus_{i=1}^{n} G_i\).

**Proof:** Clearly, \(k = r^{n-1} + l\) for some integer \(1 < l \leq r\). We can color \(G\) by \(r^n\) ordered \(n\)-tuples \((j_1, j_2, \ldots, j_n)\), where \(j_i = 1, \ldots, r\) for \(i = 1, \ldots, n\). Now by Theorem 5.2 \(G\) can be decomposed into \(n\) \(r\)-partite subgraphs.

We can now obtain a relationship between \(r\)-particity and the chromatic number of a graph. This extends equation (1) that expresses the biparticity of a graph in terms of its chromatic number.

**Theorem 5.5.** Let \(G\) be a graph with chromatic number \(\chi(G)\), then \(\beta_r(G) = \lceil \log_r(\chi(G)) \rceil\).

**Proof:** There is a positive integer \(n\) such that \(r^{n-1} < \chi(G) \leq r^n\), hence by Corollary 5.4, \(G\) has a decomposition into \(n\) \(r\)-partite graphs and so \(\beta_r(G) \leq n = \log_r(r^n) = \lceil \log_r(\chi(G)) \rceil\).

For the reverse inequality, let \(\{H_i\}_{i=1}^{m}\) be a minimal \(r\)-partite cover of \(G\), that is \(\beta_r(G) = m\). Let \((H_1^r, \ldots, H_m^r)\) be an \(r\)-partition of \(H_i\), we define a function \(c: V(G) \to \{(j_1, \ldots, j_m) : j_i = 1, \ldots, r\} \)
\[ c_i(v) = \begin{cases} 
1 & \text{if } v \in H_i^1 \\
2 & \text{if } v \in H_i^2 \\
\vdots & \\
r & \text{if } v \in H_i^r 
\end{cases} \]

So if \( u \) and \( v \) are two vertices such that \( c(u) = c(v) \), then \( u \) and \( v \) are not adjacent as they are in the same color class for every \( r \)-partite \( H_i \). Hence \( \chi(G) \leq r^n \) and therefore \([\log_r(\chi(G))] \leq m = \log_r(r^m = \beta_r(G))\). Hence the equality. 

Let \( S_g \) be a closed orientable surfaces of genus \( g \). The chromatic number of \( S_g \) is the maximum chromatic number of any graph embedded in \( S_g \). A graph is said to be of genus \( g \) if it cannot be embedded in a surface of genus less than \( g \). The Heawood theorem states that for any positive integer \( g \), the chromatic number \( S_g \) is equal to the Heawood number \( H(g) = \left\lfloor \frac{7 + \sqrt{1 + 48g^2}}{2} \right\rfloor \).

**Corollary 5.6.** The Heawood theorem is equivalent to the statement that any graph \( G \) of genus \( g \) can be decomposed into subgraphs \( \{G_i\}_{i=1}^n \), where \( G_i \) is an \( r_i \)-partite graph for each \( i \), and \( r_1r_2\cdots r_n \leq H(g) \).

The above result generalizes Mabry’s result that the four color theorem is equivalent to the fact that a planar graph has decomposition in two bipartite graphs. The following result relates the \( \beta_r(G) \) and the genus \( g \).

**Corollary 5.7.** \( \beta_r(G) \leq \left\lceil \log_r \left( \frac{7 + \sqrt{1 + 48g^2}}{2} \right) \right\rceil \).

A Nordhaus-Gaddum type result is a lower or upper bound on the sum or product of a parameter of a graph and its complement. The classical result of Nordhaus and Gaddum concerns chromatic number.

**Theorem 5.8.** If \( \bar{G} \) denotes the complement of a graph \( G \) then

\[ 2\sqrt{n} \leq \chi(G) + \chi(\bar{G}) \leq n + 1 \quad (11) \]

\[ n \leq \chi(G)\chi(\bar{G}) \leq \left( \frac{n + 1}{2} \right)^2 \quad (12) \]

By combining Theorem 5.5 and Theorem 5.8 we can obtain the following Nordhaus-Gaddum type inequalities for \( r \)-particity.
Theorem 5.9. Given a graph $G$

$$\beta_r(G) + \beta_r(\overline{G}) \leq 2 \log_r\left(\frac{n+1}{2}\right)$$  \hspace{1cm} (13)

$$r^{\beta_r(G)} + r^{\beta_r(\overline{G})} \geq 2\sqrt{n}$$ \hspace{1cm} (14)

Proof: We have from equation (12)

$$\beta_r(G) + \beta_r(\overline{G}) \leq \log_r(\chi(G)) + \log_r(\chi(\overline{G})) = \log_r(\chi(G)\chi(\overline{G})) \leq 2 \log_r\left(\frac{n+1}{2}\right).$$

Similarly, from equation (11) we have

$$2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq r^{\beta_r(G)} + r^{\beta_r(\overline{G})}.$$

\[\blacksquare\]

Acknowledgement

The authors thank King Fahd University of Petroleum and Minerals (KFUPM), Dhahran, Saudi Arabia for its continuous support of their research. This research was funded by the Deanship of Scientific Research at KFUPM under Research Grant IN111027.

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