NUMERICAL ANALYSIS AND VARIATIONAL INEQUALITIES

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Abstract. Variational inequalities provide us with a unified, natural, novel and simple framework to study a wide class of problems arising in pure and applied sciences. In this paper, we present a number of new and known numerical techniques for solving variational inequalities using various techniques including projection, Wiener-Hopf equations, updating the solution, auxiliary principle, inertial proximal, penalty function and dynamical system. We also consider sensitivity analysis of the variational inequalities as well as the finite convergence of the projection-type algorithms. Variational-like inequalities, regularized inequalities inequalities and equilibrium problems are also investigated. Our proofs of convergence are very simple as compared with other methods. Our results present a significant improvement of previously known results for solving variational inequalities and related optimization problems. Several open problems have been suggested for further research in these areas.
Keywords: Variational inequalities, Wiener-Hopf equations, extragradient methods, auxiliary principle, updating the technique, splitting methods, predictor-corrector methods, inertial proximal methods, dynamical systems, well-posedness, sensitivity analysis, penalty function method, well-posedness, globally stable, fixed-point, equilibrium problems, convergence.
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1. Introduction

Variational inequalities theory, which was introduced in the sixties, has emerged as an interesting and fascinating branch of applicable mathematics with a wide range of applications in industry, finance, economics, optimization, social, regional, pure and applied sciences. This field is dynamic and is experiencing an explosive growth in both theory and applications; as a consequence, research techniques and problems are drawn from various fields. The ideas and techniques of variational inequalities are being applied in a variety of diverse areas of sciences and prove to be productive and innovative. It has been shown that this theory provides the most natural, direct, simple, unified and efficient framework for a general treatment of a wide class of unrelated linear and nonlinear problems, see, for example, [1-164] and the references therein.

During the years which have been elapsed since its discovery, a number of numerical methods including projection method and its variant forms, Wiener-Hopf equations, auxiliary principle, decomposition, dynamical systems have been developed for solving the variational inequalities and related optimization problems. Projection method and its variants forms including the Wiener-Hopf equations represent important tools for finding the approximate solution of variational inequalities, the origin of which can be traced back to Lions and Stampacchia [56]. The main idea in this technique is to establish the equivalence between the variational inequalities and the fixed-point problem by using the concept of projection. This alternative formulation has played a significant part in developing

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various projection-type methods for solving variational inequalities. It is well known that the convergence of the projection methods requires that the operator must be strongly monotone and Lipschitz continuous. Unfortunately these strict conditions rule out many applications of this method. This fact motivated to modify the projection method or to develop other methods. The extragradient-type methods [11,41,45-48,51,94,98-101,115,118,140,148,149] overcome this difficulty by performing an additional forward step and a projection at each iteration according to the double projection. These methods can be viewed as predictor-corrector methods. Their convergence requires only that a solution exists and the monotone operator is Lipschitz continuous. When the operator is not Lipschitz continuous or when the Lipschitz continuous constant is not known, the extragradient method and its variant forms require an Armijo-like line search procedure to compute the step size with a new projection needed for each trial, which leads to expansive computation. To overcomes these difficulties, several modified projection and extragradient-type methods have been suggested and developed for solving variational inequalities, see [41,45-48,94,98-101,118,139,148,149].

Related to the variational inequalities, we have the concept of the Wiener-Hopf equations, which was introduced by Shi [136] and Robinson [134] in conjunction with variational inequalities from different point of views. Using the projection technique, one usually establishes the equivalence between the variational inequalities and the Wiener-Hopf equations. It turned out that the Wiener-Hopf equations are more general and flexible. This approach has played not only an important part in developing various efficient projection-type methods, but also in studying the sensitivity analysis as well as other concepts of variational inequalities. Noor, Wang and Xiu [125] and Noor and Rassias [119] have suggested and analyzed some predictor-corrector type projection methods by modifying the Wiener-Hopf equations. It has been shown in [94,98,119,125] that these predictor-corrector-type methods are efficient and robust. It shows that the Wiener-Hopf equation technique is a powerful tool for developing efficient methods. Inspired and motivated by this development, we suggest a new unified extragradient-type method for solving variational inequalities and related problems. We prove that the convergence of the new method requires only the pseudomonotonicity, which is a weaker condition than monotonicity. Consequently, our results represent a refinement and improvement of the known results.

Noor [90,91,93,94] has developed the technique of updating the solution to suggest and analyze a several projection-splitting methods for various classes of variational inequalities in conjunction with projection and Wiener-Hopf equations. Noor [99] has suggested and analyzed a class of self-adaptive projection methods by modifying the modified fixed-point equations involving a generalized residue vector associated with the general variational inequalities. The search direction in these methods is a combination of the generalized projection and modified extragradient direction, whereas the step-size depends upon the modified Wiener-Hopf equation. These methods are different from the existing one-step, two-step and three-step projection-splitting methods. In fact, these new methods coincide with the known splitting methods for special values of the step sizes and search line directions. It is shown that these modified methods converge for the pseudomonotone operators.

It is well known fact that to implement the projection-type methods, one has to evaluate the projection, which is itself a difficult problem. Secondly, the projection and Wiener-Hopf equations techniques can't be extended and generalized for some classes of variational inequalities involving the nonlinear (non)differentiable functions, see [92,94,108]. These facts motivated to use the auxiliary principle technique. This technique deals with finding the auxiliary variational inequality and proving that the solution of the auxiliary problem is the solution of the original problem by using the fixed-point approach. It turned out that this technique can be used to find the equivalent differentiable optimization problems, which enables us to construct gap (merit) functions. Glowinski et al. [38] used this technique to study the existence of a solution of mixed variational inequalities. Noor [92,93,108] has used to this technique to suggest some predictor-corrector methods for solving various classes of variational inequalities. It is well known that a substantial number of numerical methods can be obtained as special cases from this technique, see [15,29,85,88,92,108,120,161] and the references therein. We use this technique to suggest and analyze some explicit predictor-corrector methods for general variational inequalities. It is shown that the convergence of the predictor-corrector methods requires only the partially relaxed strongly monotonicity, which is a weaker condition than cocoercivity.

Proximal methods have been suggested for solving variational inequalities. These methods are in fact the implicit type methods, which arise in the context of discretization of the initial value problems. Martinet [64] considered these methods as a regularization technique for the convex optimization. Rockafellar [154] studied these methods for finding a zero of the maximal monotone operators. For the recent applications and developments, see [29,47,92,135,147] and the references therein. An other class of proximal methods has been considered by Alvarez and Attouch [6] for maximal monotone operators in the context of second order differential equations in time. These methods are called the inertial proximal methods. Noor [94,99] and Noor, Akhter and Noor [115] have introduced and considered these inertial methods for various classes of variational inequalities and have proved that the convergence criteria of the inertial proximal methods requires only the pseudomonotonicity. As a special cases of the inertial proximal methods, we obtain the proximal point methods. This clearly shows that our results represent a refinement of the previously known methods. Our approach is independent of the so-called Bregman function. In this paper, we give the basic idea of the inertial proximal methods. It is an open problem to compare the efficiency of the inertial methods with other methods and this is another direction for future research.

Related to the variational inequalities, we also consider the globally projected dynamical system using the various equivalent formulations. The concept of projected dynamical system in the context of variational inequalities was introduced by Dupuis and Nagurney [26] by using the fixed-point formulation of the variational inequalities. For the recent development and applications of the dynamical systems, see [25,26,32,33,69,70,95,96,106,128,162]. In this technique, we reformulate the variational inequality problem as an initial value problem. This equivalent formulation allows us to study the stability properties of the unique solution of the variational inequality problem. Noor [106] has introduced the Wiener-Hopf dynamical system for variational inequalities by using the equivalence between variational inequalities and the Wiener-Hopf equations. He has also proved the stability analysis of the Wiener-Hopf dynamical system for pseudomonotone operators thereby improving the previous known results. We use the equivalence between the variational inequalities and fixed-point problems to suggest some new dynamical systems associated with the variational inequalities and study some properties of the solution of the projected dynamical systems. Furthermore, we introduce the concept of the second order projected dynamical system for the general variational inequalities, the stability of which is still an open problem. We expect this concept will be useful in the study of differential equations and will have far reaching applications in biomathematics and regional sciences.

There are several projection-type iterative methods for solving variational inequalities. It is well known that the evaluation of the projection is itself a difficult problem except for some simple cases. Secondly the finite difference and similar numerical methods cannot be applied to find the approximate solutions of the obstacle, free and moving value problems due to the presence of the obstacle and other constraint conditions. However, it is known that if the obstacle is known then these obstacle and unilateral problems can be characterized by a system of differential equations in conjunction with the variational inequalities using the penalty function technique. This technique was used by Lewy and Stampacchia [55] to study the regularity of the solutions of the variational inequalities. Noor and Al-Said [118], Noor and Tirmizi [123] and Al-Said and Noor [4] used this technique to develop some numerical methods for solving these system of differential equations. The main advantage of this technique is its simple applicability for solving system of differential equations. We here give only the main idea of this technique for solving obstacle and unilateral problems. For more details, see [1-5, 118,124,125,127].

It is well known that many equilibrium problems arising in finance, economics, transportation and structural analysis can be studied via the variational inequalities. It is natural to study the behaviour of these problems due to change in the given data. Such type of study is known as the sensitivity analysis. Recently much attention has been given to develop a general sensitivity analysis framework for variational inequalities and related problems. The techniques suggested so far vary with the problem being studied, see [20,52,53,58,68,83,97,114,133,145,159,160] and the references therein. It is known that variational inequalities are equivalent to the fixed-point problems and the Wiener-Hopf equations. Dafermos [20] used the fixed-point formulation of variational inequalities to study the sensitivity analysis whereas Noor [83] used the Wiener-Hopf equations approach to study this problem. In this section, we again use the Wiener-Hopf equations technique to study the sensitivity analysis of the variational inequalities. This fixed-point formulation is obtained by suitable and appropriate rearrangement of the Wiener-Hopf equations. It is worth mentioning that this approach is easy to implement and provides an alternate approach to study the sensitivity analysis without assuming the differentiability of the given data.

We also consider the finite convergence criteria of the projection-type iterative methods, which play an important role in designing the algorithms for variational inequalities. Using the technique of Burke and More [14], we show that the sequence generated by the projection-type methods for solving the general variational inequalities terminates at a solution of the concerned problem or enter and remain in relative interior of the optimal face.

It is well-known that the concept of convexity plays an important part in the study of variational inequalities. This concept has been generalized in many directions using some novel and new techniques. A significant generalization of convex functions is preinvex(invex) functions. It has been shown in [81,87,105, 158] that the minimum of the preinvex(invex) functions on the invex sets can be characterized by a class of variational inequalities, known as variational-like inequalities or prevocational inequalities. Due to the nature of these problems, projection method and its variant forms cannot be used to suggest and analyze iterative methods for variational-like inequalities. This implies that the variational-like inequalities are not equivalent to the projection (resolvent) fixed-point problems. To overcome these drawbacks, we show that the auxiliary principle technique can be used to suggest and analyze some implicit and explicit iterative methods for solving variational-like inequalities. We also show that the variational-like inequalities are equivalent to the optimization problems, which can be used to study the associated optimal control problem. Such type of the problems have been not studied for variational-like inequalities and this is another direction for future research.

It is worth mentioning that the concept of projection operator has played a basic and significant part in the development of existence results and computational techniques for variational inequalities defined over the convex sets. Almost all the techniques and ideas are based on the properties of the projection operator over convex sets. If the set involved is not a convex set, then these properties of the projection operator may not hold as in the case of invex sets. To overcome these difficulties, one usually reformulates the variational inequalities into the equivalent variational problems over the uniformly prox-regular sets, which are convex sets, see [15,131] and the references therein. We here use the auxiliary principle technique to suggest and analyze some iterative schemes for solving noncongeal general variational inequalities. It will be interesting to extend this for other classes of variational inequalities involving nonlinear terms.

In recent years, much attention has been given to study the equilibrium problems as considered and studied by Blub and Fettle [12] and Noor and Fettle [111]. It is known that equilibrium problems include variational inequalities and complementarity problems as special cases. It is remarked that there are very few iterative methods for solving equilibrium problems, since the projection method and its variant forms including the Wiener-Hopf equations cannot be extended for these problems. This fact has motivated us to use the auxiliary principle technique to suggest and analyze some iterative type methods for solving equilibrium problems. We also discuss the convergence analysis of these iterative methods for pseudomonotone and partially relaxed strongly monotone functions. We also discuss the convergence analysis of these iterative methods for pseudomonotone and partially relaxed strongly monotone functions.

We introduce the concept of well-posedness for equilibrium problems, which was considered by Lucchetti and Patrone [61,62] for variational inequalities. We obtain some similar results. This technique also give an algorithms to compute the approximate solutions of the equilibrium problems. Despite of its importance, very little research has been carried out in this direction.

Theory of variational inequalities is quite broad, so we shall content ourselves here to give the flavour of the ideas and techniques involved. The techniques used to analysis the iterative methods and other results for variational inequalities are a beautiful blend of ideas of pure and applied mathematical sciences. In this paper, we have presented the main results regarding the development of various algorithms, their convergence analysis and the sensitivity analysis of the variational inequalities. Although this paper is expository in nature, our choice has been rather to consider a number of familiar and to us some interesting aspects of variational inequalities. We also include some new results which we and our coworkers have recently obtained. The language used is necessarily that of functional analysis and some knowledge of elementary Hilbert space theory is assumed. The framework chosen should be seen as a model setting for more general results for other classes of variational inequalities and variational inclusions. It is true that each of these areas of applications require special consideration of peculiarities of the physical problem at hand and the inequalities that model it. However, many of the concepts and techniques, we have discussed are fundamental to all of these applications. One of the main purposes of this expository paper is to demonstrate the close connection among various classes of algorithms for the solution of the variational inequalities and to point out that researchers in different field of variational inequalities and optimization have been considering parallel paths. General and unified frameworks are of important and significant scientific value, both as a means of summarizing existing techniques and to provide ideas and tools for explaining relationship and performing analysis. These unified frameworks also allow a cross-fertilization among the various diverse areas where both the theory and computational techniques have been applied. We would like to emphasize that the results obtained and discussed in this paper may motivate and bring a large number of novel, innovate and potential applications, extensions and interesting topics in these areas. We have given only a brief introduction of this fast growing field. The interested reader is advised to explore this field further and discover novel and fascinating applications of this theory in other areas of sciences.

2. Preliminaries and Basic Concepts

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ respectively. Let K be a closed convex set in H and $T : H \longrightarrow H$ be a nonlinear operator. For a given linear continuous functional $f \in H$, we now consider the problem of finding $u \in K$ such that

$$\langle Tu, v - u \rangle \geq \langle f, v - u \rangle, \quad \forall v \in K, \tag{2.1}$$

which is known as the variational inequality introduced and studied by Stampacchia [160] in 1964. For recent state-of-the-art in this field, see [1-160] and the references therein.

If $N(u) = \{w \in H : \langle w, v - u \rangle \leq 0$, for all $v \in K$ $\}$ is a normal cone to the convex set K at u, then the variational inequality (2.1) is equivalent to finding $u \in K$ such that

$$-T(u) + f \in N(u), \tag{2.2}$$

which are known as the *generalized equations.*

If P^t is the projection of $-(Tu - f)$ at $u \in K$, then it has been shown that the variational inequality problem (23.3) is equivalent to finding $u \in K$ such that

$$P^t[-Tu] := P^t(u) = 0, \tag{2.3}$$

which are known as the *tangent projection equations,* see Xiu et al [149]. This equivalence has been used to discuss the local convergence analysis of a wide class of iterative methods for solving variational inequalities (2.1).

If $K^* = \{u \in H : \langle u, v \rangle \geq 0$, for all $v \in K\}$ is a polar (dual) cone of a convex cone K in H, then problem (2.1) is equivalent to finding $u \in K$ such that

$$Tu - f \in K^* \quad \text{and} \quad \langle Tu - f, u \rangle = 0, \tag{2.4}$$

which is known as the generalized complementarity problem, see Karamardian [67] and Cottle et al. [22]. Such problems have been studied extensively in recent years.

If the operator T is a linear, symmetric and positive definite, then problem (2.1) is equivalent to finding the minimum of the function $I[v]$, where

$$I[v] = \langle Tv, v \rangle - 2\langle f, v \rangle, \quad \forall v \in K, \tag{2.5}$$

which is called the energy (virtual, potential) functional. Obviously the functional $I[v]$ is a quadratic function. Consequently the quadratics programming techniques can be used to develop some efficient methods for solving the variational inequalities.

If $\langle Tu, v\rangle = a(u, v), \quad \forall u, v \in K$, then problem (2.1) reduces to:
Find $u \in K$ such that

$$a(u, v-u) \geq \langle f, v-u\rangle, \quad \forall v \in K, \tag{2.6}$$

which is called the classical variational inequality, introduced and studied by Stampacchia[].

If $K = H$, then problem (2.6) collapses to:
Find $u \in H$, such that

$$a(u, v) = \langle f, v\rangle, \quad \forall v \in H, \tag{2.7}$$

which is known as the Lax-Milgram Lemma, see [86,128]. Lax-Milgram lemma implies that the linear continuous functional is represented by the bilinear continuous form $a(.,.) : H \times H \leftrightarrow H$.

If $a(u, v) = \langle u, v\rangle$, the Lax-Milgram reduces to:
Find $u \in H$ such that

$$\langle u, v\rangle = \langle f, v\rangle, \quad \forall v \in H, \tag{2.8}$$

which is known as the celebrated Reisz-Frechet representation theorem.

It is clear that problem (2.8) is equivalent to the finding the minimum of the quadratic functional $RF[v]$ of the form

$$RF[v] = \|v\|^2 - 2\langle f, v\rangle, \quad \forall v \in H, \tag{2.9}$$

which is a clearly strongly convex function. Thus, we conclude that the Reisz-Frechet representation has equivalent variational formulation. This equivalent variational formulation can be used to discuss the uniqueness of the solution.

If the bifunction $a(.,.)$ is bilinear, positive and symmetric and f is the function f is Frechet differentiable convex function, then $u \in H$is the minimum of the energy functional $ARF[v]$, defined as

$$RF[v] = a(v, v) - 2f(v), \quad \forall v \in H, \tag{2.10}$$

if and only if, $u \in H$ satisfies

$$a(u, v) = \langle f'(u), v\rangle, \quad \forall v \in H,$$

which can be viewed as a novel generalization of the Lax-Milgram. This result is due to Noor [86] and is the weak formulation of the mildly nonlinear boundary value problems. For the application of this result, see Noor and Whiteman [141]. Problem (2.10) is also known as the minimum of finding the difference of two convex functions, which is called DC problem and has been studied extensively recently, see Noor et al. [140].

We can rewrite the functional $I[v]$ defined by (2.10) as:

$$N[v] = a(v, v) - 2f(v), \quad \forall v \in K, \tag{2.11}$$

where the function f is a nonlinear continuous function.
If the function f is a differentiable convex function, then it has been shown that the minimum of the functional $N[v]$ defined by (2.11) is equivalent to finding $u \in K$ such that

$$a(u, v - u) \geq \langle f'(u), v - u \rangle, \quad \forall v \in K, \tag{2.12}$$

which is mainly introduced and studied by Noor [84, 85].
Motivated and inspired by the above idea, Noor [86] considered a more general variational inequality of which (2.12) is a special case. To be more precise, for given nonlinear operators T, A, consider the problem of finding $u \in K$ such that

$$a(u, v - u) \geq \langle A(u), v - u \rangle, \quad \forall v \in K, \tag{2.13}$$

which is known as the mildly nonlinear variational inequality. For the formulation, numerical methods and finite element analysis, see Noor [86, 87] and Noor et al [141].
If $\langle A(u), v - u \rangle = A(u, v - u)$, then problem (2.13) reduces to: Find $u \in K$ such that

$$a(u, v - u) \geq A(u, v - u), \quad \forall v \in K, \tag{2.14}$$

which is called the hemivariational inequality, introduced and investigated by Panagiotopoulos [144]. Hemivariational inequalities have important applications in structural analysis, see Demyanov et al [25].

Remark 2.1. *We would like to emphasize that the variational inequalities can be viewed as natural and novel generalization of the Riesz-Frechet representation theorem and the Lax-Milgram Lemma. For more details, see [128] and the references therein. For suitable and appropriate choice of the operators and spaces, one can obtain several classes of variational inequalities and related optimization problems. We have just highlighted some hidden aspects of the energy functionals associated with variational inequalities and Lax-Milgram Lemma. It is an interesting and fascinating problem to discover other aspects of the energy functionals.*

We now recall the following well known result [68], which is called the projection lemma(Best approximation lemma).

Lemma 2.1. *Let K be a nonempty, closed and convex set in the Hilbert space H. For a given $z \in H$, $u \in K$ satisfies the inequality*

$$\langle u - z, v - u \rangle \geq 0, \quad \forall \quad v \in K, \tag{2.15}$$

if and only if

$$u = P_K z,$$

where P_K is the projection of H onto the closed convex set K.

The projection operator P_K is nonexpansive, that is,

$$\|P_K(u) - P_K(v)\| \leq \|u - v\|, \quad \forall u, v \in H,$$

and satisfies the inequality

$$\|P_K z - u\|^2 \le \|z-u\|^2 - \|z - P_K z\|^2. \tag{2.16}$$

Also $P_K(u) = u, \quad \forall u \in K$.

Related to the variational inequalities, we now consider the problem of the Wiener-Hopf equations. To be more precise, let

$$Q_K = I - P_K,$$

where I is the identity operator and P_K is the projection of H onto K. For given nonlinear operator $T : H \to H$, consider the problem of finding $z \in H$ such that

$$\rho T P_K z + Q_K z = 0. \tag{2.17}$$

Equations of the type (2.17) are called the Wiener-Hopf equations(or normal equations), which were introduced and studied by Shi [155] and Robinson [153] in different settings independently. Using the projection operators technique, one can show that the variational inequalities are equivalent to the Wiener-Hopf equations. This equivalent alternative formulation has played a fundamental and important role in studying various aspects of variational inequalities. It has been shown that the Wiener-Hopf equations are more flexible and provide a unified framework to develop some efficient and powerful numerical techniques for solving variational inequalities and related optimization problems.

Applications: We now show that a wide class of obstacle, unilateral, free, moving and equilibrium problems arising in pure and applied sciences can be formulated in terms of (2.1). For simplicity and to illustrate the applications, we consider the second order obstacle boundary value problem of the type.

Example 2.1. *Find u such that*

$$\left.\begin{array}{ll} -u''(x) \ge f(x), & x \in \Omega = [a,b] \\ u(x) \ge \psi(x), & x \in \Omega = [a,b] \\ [-u''(x) - f(x)][u-\psi] = 0, & x \in \Omega[a,b] \\ u(a) = 0 \quad \textit{and} \quad u(b) = 0, & \end{array}\right\}, \tag{2.18}$$

where $\Omega = [a,b]$ is a domain, $\psi(x)$ and $f(x)$ are the given functions. The function ψ is known as the obstacle function. The region, where $u(x) = \psi(x), \quad$ for $x \in \Omega$ is called the contact region (set).

We not that problem (2.18) is a generalization of the second order boundary value problem

$$-\frac{d^2u(x)}{dx^2} = f(x) \qquad x \in \Omega \tag{2.19}$$

with boundary conditions

$$u(0) \quad = \quad u(b) \quad = 0.$$

Problems of type (2.18)and (2.19) arise in various branches of pure and applied sciences. For the formulations and applications of the obstacle, unilateral, free, moving and equilibrium problems, see [**?**, 24, 25, 50, 55, 56, 67, 68, 94, 138, 139].

To study the problem (2.18) in the general framework of the variational inequality, we define

$$K = \{u \in H_0^1(\Omega) : u(x) \geq \psi(x) \text{on } \Omega\},$$

which is closed convex set in $H_0^1(\Omega)$. For the definition and properties of the spaces $H_0^m(\Omega)$, see [52].

Using the technique of [52, 164], we can easily show that the energy functions associated with the problem (2.18) is

$$\begin{aligned} I[v] &= \int_a^b (-\frac{d^2v}{dx^2})vdx - 2\int_a^b f(x)vdx, \quad \text{for all } v \in K \\ &= \int_a^b (\frac{dv}{dx})(\frac{dv}{dx})dx - 2\int_a^b f(x)vdx \\ &= \langle Tv, v\rangle - 2\langle f, v\rangle, \end{aligned} \tag{2.20}$$

where

$$\langle Tu, v\rangle = \int_a^b \left(\frac{du}{dx}\right)\left(\frac{dv}{dx}\right) dx, \tag{2.21}$$

and

$$\langle f, v\rangle = \int_a^b f(x)vdx. \tag{2.22}$$

It is clear that the operator T defined by the relation (2.21) is symmetric, positive and linear. Thus the minimum of the functional $I[v]$, defined by (2.20) is equivalent to finding $u \in K$ such that the inequality (2.9) holds.

In fact, we conclude the problems equivalent to (2.18) are:

The Variational Problem.
Find $u \in H_0^1(\Omega)$, which gives the minimum value to the functional

$$I[v] = \langle Tv, v\rangle - 2\langle f, v\rangle, \quad \forall v \in K.$$

The Variational Inequality (Weak) Problem.
Find $u \in K$ such that

$$\langle Tu, v - u\rangle \geq \langle f, v - u\rangle, \quad \forall \quad v \in K.$$

Now, we recall some basics definitions and basic results.

Definition 2.1. $\forall u, v, z \in H$, *the operator* $T : H \longrightarrow H$ *is said to be:*

(i). strongly monotone, if there exist a constant $\alpha_1 > 0$ *such that*

$$\langle Tu - Tv, u - v\rangle \geq -\alpha_1\|u - v\|^2.$$

(ii). relaxed strongly monotone, if there exist a constant $\gamma > 0$ *such that*

$$\langle Tu - Tv, u - v\rangle \geq -\gamma\|u - v\|^2.$$

(iii). partially relaxed strongly monotone , if there exists a constant $\alpha > 0$ *such that*

$$\langle Tu - Tv, z - v\rangle \geq -\alpha\|u - z\|^2.$$

(iv). cocoercive, if there exists a constant $\mu > 0$ *such that*

$$\langle Tu - Tv, u - v\rangle \geq \mu\|Tu - Tv\|^2.$$

(v). monotone, if

$$\langle Tu - Tv, u - v\rangle \quad \geq 0.$$

(vi). strictly monotone, if

$$\langle Tu - Tv, u - v\rangle \quad > 0.$$

(vii). pseudomonotone, if

$$\langle Tu, v - u\rangle \geq 0 \quad \textit{implies} \quad \langle Tv, v - u\rangle \geq 0.$$

(viii). hemicontinuous, if
the mapping $t \in [0,1]$ *implies that* $\langle T(u + t(v - u)), v - u\rangle$ *is continuous.*

Remark 2.2. *Note that, if* $z = u$*, then partially relaxed strongly monotonicity reduces to monotonicity. This shows that the class of monotone mappings includes the class of partially relaxed strongly monotone mappings, but the converse is not true in general. It is well known [35] that monotonicity implies pseudomonotonicity, but the converse is not true. This shows that pseudomonotonicity is a weaker condition than monotonicity. It is known that cocoercivity implies partially relaxed strongly monotonicity. For the sake of completeness, we include its proof.*

Lemma 2.2. *If* T *is cocoercive with a constant* $\mu > 0$*, then the operator* T *is partially relaxed strongly monotone operator with constant* $\frac{1}{4\mu}$*.*

Proof. $\forall u, v, z \in H$, consider

$$\begin{aligned}\langle Tu - Tv, z - v\rangle &= \langle Tu - Tv, u - v\rangle + \langle Tu - Tv, z - u\rangle \\ &\geq \mu\|Tu - Tv\|^2 - \mu\|Tu - Tv\|^2 - \frac{1}{4\mu}\|z - u\|^2 \\ &\geq \frac{-1}{4\mu}\|z - u\|^2, \end{aligned} \tag{2.23}$$

which shows that T is partially relaxed strongly monotone with constant $\frac{1}{4\mu}$. □

Lemma 2.3. *Let the operator* T *be pseudomotone and hemicontinuous. Then* $u \in K$ *is a solution of (23.3), if and only if,* $u \in K$ *satisfies*

$$\langle Tv - f, v - u\rangle \geq 0, \quad \forall v \in K. \tag{2.24}$$

Proof. Let $u \in K$ be a solution of (23.3). Then

$$\langle Tu - f, v - u\rangle \geq 0, \quad \forall v \in K,$$

which implies, using the pseudomonotonicity of T,

$$\langle Tv - f, v - u\rangle \geq 0, \quad \forall v \in K,$$

the required (2.24).

Conversely let $u \in K$ be such that (2.24) hold. For $t \in [0,1], u, v \in K$,
$v_t = u + t(v - u) \in K$. Taking $v = v_t$ in (2.15), we have

$$0 \leq t\langle Tv_t - f, v - u\rangle.$$

Dividing the above inequality by t and letting $t \longrightarrow 0$, we have

$$\langle Tu - f, v - u\rangle \geq 0, \quad \forall v \in K,$$

the required (23.3). □

Lemma 2.3 is known as the Minty Lemma. Inequality (2.24) is also known as the *dual variational inequality.* From Lemma 2.3, it is clear that the solution sets of both problems (23.3) and (2.24) are equivalent. Lemma 2.3 plays an important part in the approximation of the variational inequalities.

Definition 2.2. *A function F is said to be strongly convex on the convex set K with modulus $\mu > 0$, if,*

$$F(u + t(v - u) \leq (1 - t)F(u) + tF(v) - t(1 - t)\mu\|v - u\|^2, \quad \forall u, v \in K, t \in [0, 1].$$

It is well known that if the differentiable convex function F is strongly convex function,then

$$(i). \quad F(v) - F(u) \geq \langle F'(u), v - u\rangle + \mu\|v - u\|^2, \quad \forall u, v \in K$$
$$(ii). \quad F'(u) - F'(v), u - v\rangle \geq \mu\|u - v\|^2, \quad \forall u, v \in K.$$

and conversely.
From (ii). it follows that $F'(u)$ is a strongly monotone operator.

3. Projection Technique.

In this section, we prove that the variational inequalities are equivalent to the fixed point problem. This equivalent alternative formulation is used to suggest and investigate several iterative methods for solving variational inequalities. For sake of simplicity and readers convenience, we take $f = 0$, in (2.1). To be more precise, for given nonlinear operator $T : H \longrightarrow H$, we now consider the problem of finding $u \in K$ such that

$$\langle Tu, v - u\rangle \geq 0, \quad \forall v \in K, \tag{3.1}$$

which is known as the variational inequality introduced and studied by Stampacchia [160] in 1964. For recent state-of-the-art in this field, see [1-160] and the references therein.

We use the projection technique to suggest and analyze some extragradient methods for solving the variational inequality (23.3). For this purpose, we need the following result, which can be proved by invoking Lemma 2.1.

Lemma 3.1. *The function $u \in K$ is a solution of (23.3), if and only if, $u \in K$ satisfies the relation*

$$u = P_K[u - \rho Tu], \tag{3.2}$$

where $\rho > 0$ is a constant.

Proof. Let $u \in K$ be a solution of (23.3). Then

$$\langle Tu, v - u\rangle \geq 0, \quad \forall v \in K,$$

which can be written as , for a constant $\rho > 0$,

$$\langle u - (u - \rho Tu), v - u \geq 0, \quad \forall u, v \in K,$$

which is equivalent to

$$u = P_K[u - \rho Tu],$$

which is the required result (23.3). □

Lemma 3.1 implies that problems (23.3) and (23.3)are equivalent. This alternative formulation is very important from the numerical analysis point of view. This alternative fixed-point formulation has been used to consider the existence of a solution of variational inequalities and to investigate various numerical methods for solving variational inequalities.

We use the fixed point formulation (23.3) to suggest a wide class of iterative methods.

Algorithm 3.1. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$u_{n+1} = P_K[u_n - \rho T u_n], \quad n = 0, 1, 2 \dots$$

For the convergence analysis of Algorithm 23.1, see [117, 137]. Xiu et al, [170] have proved that Algorithm 23.1 has the local convergence behaviour, which enables us to identify accurately the optimal constraint after finitely many iterations, see Section 11.

We now define the projection residue vector by the relation

$$R(u) = u - P_K[u - \rho T u]. \tag{3.3}$$

From Lemma 3.1, it is clear the $u \in K$ is a solution of (23.3), if and only if, $u \in K$ is a zero of the equation

$$R(u) = 0. \tag{3.4}$$

For a positive constant γ, we can rewrite equations (23.7) as

$$u + \rho T u = u + \rho T u - \gamma R(u).$$

This fixed-point formulation allows us to suggest and analyze the following iterative method.

Algorithm 3.2. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$u_{n+1} = u_n + \rho T u_n - \rho T u_{n+1} - \gamma R(u_n),$$

which is known as the implicit iterative method.
For $\gamma = 1$, the above iterative scheme can be written as

$$u_{n+1} = (I + \rho T)^{-1}\{P_K[I - \rho T] + \rho T\}u_n, \quad n = 0, 1, 2, \dots$$

which can be considered as an implicit operator-splitting method.

For the convergence analysis of Algorithm 23.2, we need the following result, which is due to He [63, 64]. To convey an idea of the technique and for the sake of completeness, we give its proof.

Lemma 3.2. *Let $\bar{u} \in H$ be a solution of (23.3). If the operator T is monotone, then*

$$\langle u - \bar{u} + \rho T u - \rho T \bar{u}, R(u) \rangle \geq \|R(u)\|^2, \quad \forall v \in K.$$

Proof. Let $\bar{u} \in H$ be a solution of the problem (23.3). Then

$$\langle T\bar{u}, v - \bar{u} \rangle \geq 0, \quad \forall v \in K. \tag{3.5}$$

Taking $v = P_K[u - \rho T u]$ in (23.8), we have

$$\langle T\bar{u}, P_K[u - \rho T u] - \bar{u} \rangle \geq 0. \tag{3.6}$$

Setting $z = u - \rho Tu, u = P_K[u - \rho Tu], v = \bar{u}$ in (2.15), we obtain

$$\langle R(u) + \rho Tu, \bar{u} - P_K[u - \rho Tu] \geq 0, \tag{3.7}$$

Adding (23.9), (23.10) and using the monotonicity of the operator T, we have

$$\langle u - \bar{u})\rho(Tu - T\bar{u}), R(u)\rangle \geq \|R(u)\|^2,$$

the required result. □

Lemma 3.3. *Let $\bar{u} \in H$ be a solution of the problem (23.3) and let u_{n+1} be the approximate solution obtained from Algorithm 23.2. Then*

$$\begin{aligned}\|u_{n+1} - \bar{u} + \rho(Tu_{n+1} - T\bar{u})\|^2 &\leq \|u_n - \bar{u} + \rho Tu_n - T\bar{u})\|^2 \\ &\quad -\gamma(2-\gamma)\|R(u_n)\|^2.\end{aligned}$$

Proof. Let $\bar{u} \in H$ be a solution of (23.3) and let u_{n+1} be an approximate solution obtained from Algorithm 23.2. Then

$$\begin{aligned}\|u_{n+1} - \bar{u} + \rho(Tu_n - T\bar{u})\|^2 &= \|u_n - \bar{u} + \rho(Tu_n - T\bar{u}) - \gamma R(u_n)\|^2 \\ &= \|u_n - \bar{u} + \rho(Tu_n - T\bar{u})\|^2 \\ &\quad +\gamma^2\|R(u_n)\|^2 - 2\gamma\langle u_n - \bar{u} \\ &\quad +\rho(Tu_n - T\bar{u}), R(u_n)\rangle \\ &\leq \|u_n - \bar{u} + \rho(Tu_n - T\bar{u})\|^2 \\ &\quad -\gamma(2-\gamma)\|R(u_n)\|^2,\end{aligned}$$

the required result. □

Theorem 3.1. *Let H be a finite dimensional space. Then the approximate solution u_{n+1} obtained from Algorithm 23.2 converges to a solution of $\bar{u} \in H$ of the variational inequality (23.3).*

For its proof, See Noor [117].

In order to implement Algorithm 23.2, one has to compute the solution implicitly, which is itself a difficult problem. In order to overcome this difficulty, we suggest another iterative method, the convergence of which also requires monotonicity of the operator.

For a positive step size γ, equation (23.7) can be written as

$$u = u - \gamma R(u). \tag{3.8}$$

This fixed-point formulation allows to suggest the following iterative method for solving the variational inequalities (23.3).

Algorithm 3.3. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$u_{n+1} = u_n - \gamma_n R(u_n), \quad n = 0, 1, 2 \ldots$$

Note that for $\gamma_n = 1$, Algorithm 3.3 coincides with Algorithm 23.1.

It is well known that the convergence analysis of Algorithm 23.1 requires that the operator T must be strongly monotone and Lipschitz continuous. These strict conditions rule out many important applications of Algorithm 23.1. It has been shown recently that one can prove that the convergence of Algorithm **??** requires only the partially relaxed strongly monotonicity. To overcome these drawbacks, one uses the technique of updating the solution. Using this technique, we can rewrite the equation (23.3) in the form

$$u = P_K[u - \rho T P_K[u - \rho T u]]. \tag{3.9}$$

We use this fixed-point formulation to suggest the following extragradient-type method for solving variational inequalities (23.3).

Algorithm 3.4. *For a given $u_0 \in K$, compute u_{n+1} by the iterative schemes:*
Predictor step.

$$v_n = P_K[u_n - \rho_n T u_n],$$

where ρ_n satisfies the relation

$$\rho_n \langle T u_n - T v_n, R(u_n)\rangle \leq \sigma \|R(u_n)\|^2, \quad \sigma \in (0,1).$$

Corrector step.

$$u_{n+1} = P_K[g(u_n) - \alpha_n T v_n],$$

where

$$\begin{aligned} \alpha_n &= \frac{(1-\sigma)\|R(u_n)\|^2}{\|T v_n\|^2} \\ T v_n &= T P_K[u_n - \rho_n T u_n]. \end{aligned}$$

Algorithm 3.4 is an improved version of the extragradient-type method of Korpelevich [69]. See also He and Liao [65] with different predictor search line and corrector step size.

Since K is convex set, $\forall \eta \in [0,1], u, P_K[u - \rho T u] \in K$, we have

$$w = (1-\eta)u + \eta P_K[u - \rho T u] = u - \eta R(u) \in K. \tag{3.10}$$

Using (23.12), we rewrite (23.3) in the form

$$u = P_K[u - \rho T(u - \eta R(u))]. \tag{3.11}$$

This fixed-point formulation is used to suggest and analyze the following modified extragradient method for variational inequalities (23.3).

Algorithm 3.5. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*
Predictor step.

$$w_n = u_n - \eta_n R(u_n), \tag{3.12}$$

where $\eta_n = a^{m_k}$, and m_k is the smallest nonnegative integer m such that

$$\rho_n \eta_n \langle T u_n - T(u_n - a^{m_k} R(u_n)), R(u_n)\rangle \leq \sigma \|R(u_n)\|^2, \quad \sigma \in (0,1). \tag{3.13}$$

Corrector step.

$$u_{n+1} = P_K[u_n - \alpha_n T(u_n - \eta_n R(u_n))], \quad n = 0, 1, 2 \ldots, \tag{3.14}$$

where

$$\alpha_n = \frac{(\eta_n - \sigma)\|R(u_n)\|^2}{\|T(u_n - \eta_n R(u_n))\|^2}. \tag{3.15}$$

For the convergence analysis of Algorithm 3.5, we need the following results.

Lemma 3.4. *Let $\bar{u} \in K$ be a solution of (23.3). If the operator T is pseudomonotone operator, then*

$$\langle u - \bar{u}, T(u - \eta R(u))\rangle \geq (\eta - \sigma)\|R(u)\|^2, \quad \forall u \in K, \tag{3.16}$$

Proof. Let $\bar{u} \in K$ be a solution of (23.3). Then

$$\langle T\bar{u}, v - \bar{u}\rangle \quad \geq \quad 0, \quad \forall v \in K,$$

implies

$$\langle Tv, v - \bar{u}\rangle \quad \geq \quad 0, \tag{3.17}$$

since T is a pseudomonotone operator.

Now taking $v = u - \eta R(u)$ in (23.17), we obtain

$$\langle T(u - \eta R(u)), u - \eta R(u) - \bar{u} \geq \quad 0,$$

from which we have

$$\begin{aligned} \langle u - \bar{u}, \rho T(u - \eta R(u))\rangle &\geq \eta\rho\langle R(u), T(u - \eta R(u))\rangle \\ &\geq -\eta\rho\langle R(u), Tu - T(u - \eta R(u))\rangle \\ &\quad +\rho\eta\langle Tu, R(u)\rangle \\ &\geq -\sigma\rho\|R(u)\|^2 + \rho\eta\langle Tu, R(u)\rangle. \end{aligned} \tag{3.18}$$

Taking $z = u - \rho Tu, = P_K[u - \rho Tu], v = u$ in (2.15), we obtain

$$\langle P_K[u - \rho Tu] - u + \rho Tu, u - P_K[u - \rho Tu]\rangle \geq 0,$$

from which, it follows that

$$\langle \rho Tu, R(u)\rangle \quad \geq \|R(u)\|^2. \tag{3.19}$$

Combining (23.18) and (23.19), we have

$$\langle u - \bar{u}, T(u - \eta R(u))\rangle \geq (\eta - \sigma)\|R(u)\|^2,$$

the required results. □

Lemma 3.5. *Let $\bar{u} \in K$ be a solution of (23.3) and let u_{n+1} be the approximate solution obtained from Algorithm 3.5. Then*

$$\|u_{n+1} - \bar{u}\|^2 \leq \|u_n - \bar{u}\|^2 - \frac{(\eta_n - \sigma)^2\|R(u_n)\|^4}{\|T(u_n - \eta_n R(u_n))\|^2}. \tag{3.20}$$

Proof. From (3.14) and (23.16), we have

$$\begin{aligned}
\|u_{n+1}-\overline{u}\|^2 &\leq \|u_n-\overline{u}-\alpha_n T(u_n-\eta_n R(u_n))\|^2 \\
&\leq \|u_n-\overline{u}\|^2 - 2\alpha_n\langle u-(\overline{u}, T(u_n-\eta_n R(u_n))\rangle \\
&\quad +\alpha_n^2\|T(u_n-\eta_n R(u_n))\|^2 \\
&\leq \|u_n-\overline{u}\|^2 - 2\alpha_n(\eta_n-\sigma)\|R(u_n)\|^2 \\
&\quad +\alpha_n^2\|T(u_n-\eta_n R(u_n))\|^2 \\
&\leq \|u_n-\overline{u}\|^2 - \frac{(\eta_n-\sigma)^2\|R(u_n)\|^4}{\|T(u_n-\eta_n R(u_n))\|^2},
\end{aligned}$$

the required result. □

Theorem 3.2. *Let u_{n+1} be the approximate solution obtained from Algorithm 3.5 and $\bar{u} \in K$ be the solution of (23.3). If H is a finite dimensional subspace, then $\lim_{n\to\infty} u_n = \bar{u}$.*

Proof. Let $u^* \in K$ be a solution of (23.3). Then, from (23.20), it follows that the sequence $\{u_n\}$ is bounded and

$$\sum_{n=0}^{\infty} \frac{(\eta_n-\sigma)^2||R(u_n)||^4}{\|T(u_n-\eta_n R(u_n))\|^2} \leq \|u_0-u^*\|^2,$$

which implies that either

$$\lim_{n\to\infty} R(u_n) = 0. \tag{3.21}$$

or

$$\lim_{n\to\infty} \eta_n = 0. \tag{3.22}$$

Assume that (23.21) holds. Let $\bar{u}$ be the cluster point of $\{u_n\}$ and the subsequence $\{u_{n_i}\}$ of the sequence $\{u_n\}$ converge to $\bar{u}$. Since R is continuous, it follows that

$$R(\bar{u}) = \lim_{i\to\infty} R(u_{n_i}) = 0,$$

which implies that $\bar{u}$ is a solution of (23.3) by invoking Lemma 3.1 and

$$||u_{n+1}-\bar{u}||^2 \leq \|u_n-\bar{u}\|^2. \tag{3.23}$$

Thus the sequence $\{u_n\}$ has exactly one cluster point and consequently

$$\lim_{n\to\infty} u_n = \bar{u}.$$

Assume that (23.22) holds, that is, $\lim_{n\to\infty} \eta_n = 0$. If (23.22) does not hold, then by a choice of η_n, we obtain

$$\sigma\|R(u_n)\|^2 \leq \rho_n\eta_n\langle Tu_n - T(u_n-\eta_n R(u_n), R(u_n)\rangle \tag{3.24}$$

Let $\bar{u}$ be a cluster point of $\{u_n\}$ and let $\{u_{n_i}\}$ be the corresponding subsequence of $\{u_n\}$ converging to $\bar{u}$. Taking the limit in (23.24), we have

$$\sigma\|R(\overline{u})\|^2 \leq 0,$$

which implies that $R(\bar{u}) = 0$, that is , $\bar{u} \in H$ is solution of (23.3) by invoking Lemma 3.1 and (23.24) holds. Repeating the above arguments, we conclude that $\lim_{n\to\infty} u_n = \bar{u}$, the required result. □

For a positive constant α_n, one can rewrite equation (23.3) as

$$u = P_K[u - \rho Tu - \alpha_n(u - u)], \tag{3.25}$$

which is the fixed-point problem. This equivalent fixed-point formulation allows us to suggest the following inertial iterative method.

Algorithm 3.6. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned} u_{n+1} &= P_K[u_n - \rho Tu_n - \alpha_n(u_n - u_{n-1})], \\ &= P_K[(1-\alpha_n)u_n + \alpha_n u_{n+1} - \rho Tu_{n+1}] \quad n = 0, 1, 2, \ldots \end{aligned}$$

which is an implicit type iterative method. To implement this, we use the predictor-corrector technique to rewrite Algorithm 3.6 as:

Algorithm 3.7. *For given $u_0, u_1 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\begin{aligned} y_n &= u_n - \alpha_n(u_n - u_{n-1}) \\ u_{n+1} &= P_K[y_n - \rho Ty_n], \end{aligned}$$

which is called the inertial iterative method.
Note that $\alpha_n = 0$, Algorithm 3.6 is equivalent to Algorithm 23.1. The process described above is reminiscent to a technique by which two-step methods can be derived as one step method. Compare this method for the heavy-ball method of Polyak [149]. Using the above technique, one can suggest a number of new and improved methods for the variational inequalities (23.3) and related problems.

Algorithm 3.8. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = P_K[u_n - \rho Tu_{n+1}], \quad n = 0, 1, 2, ... \tag{3.26}$$

which is known as the extragradient method, which was suggested and analyzed by Korperlevich [69]and has been studied extensively. Noor [109] has proved that the convergence of the extragradient for pseudomonotone operators, which can be viewed as a significant refinement of the Korperlevich's result [69].

Algorithm 3.8 can rewritten in the following form:

Algorithm 3.9. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= P_K[u_n - \rho Tu_n] \\ u_{n+1} &= P_K[u_n - \rho Ty_n], \quad n = 0, 1, 2, ... \end{aligned}$$

which is called the two-step iterative methods extragradient method of Korpelevich [69].
OR equivalently Algorithm 3.8 can rewritten in the following form:

Algorithm 3.10. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= (1-\alpha_n)u_n + \alpha_n u_{n-1} \\ u_{n+1} &= P_K[u_n - \rho Ty_n], \quad n = 0, 1, 2, ... \end{aligned}$$

which is known as the inertial type Koperlevich [69] method. To implement this method, one only use one projection. This method is being considered expensively in recent years.
Using the technique of the updating solution, Noor [102, 117] suggested and investigated the following implicit iterative method for solving the variational inequality (23.3):

Algorithm 3.11. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = P_K[u_{n+1} - \rho T u_{n+1}], \quad n = 0, 1, 2, ... \tag{3.27}$$

which is known as the modified double projection method and has been studied extensively and can be written in the equivalent form:

Algorithm 3.12. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= P_K[u_n - \rho T u_n] \\ u_{n+1} &= P_K[y_n - \rho T y_n], \quad n = 0, 1, 2, ... \end{aligned}$$

which is called the two-step iterative methods extragradient method of Noor [82, 117].
OR equivalently Algorithm 3.11 can rewritten in the following form:

Algorithm 3.13. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= (1 - \alpha_n) u_n + \alpha_n u_{n-1} \\ u_{n+1} &= P_K[y_n - \rho T y_n], \quad n = 0, 1, 2, ... \end{aligned}$$

which is known as the inertial type Noor iterative method and uses only one projection.

In recent years, Noor et al [134,137] have used the fixed point problem (23.3) in various different form to investigate a wide class of implicit and explicit iterative methods for solving variational inequalities.

We can rewrite the equation (23.3) as:

$$u = P_K[\frac{u + u}{2} - \rho T u]. \tag{3.28}$$

This fixed point formulation was used to suggest the following implicit method, which is due to Noor et al [134].

Algorithm 3.14. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = P_K[\frac{u_n + u_{n+1}}{2} - \rho T u_{n+1}], \quad n = 0, 1, 2, ... \tag{3.29}$$

For the implementation and numerical performance of Algorithm 23.1, Noor et al [134] used the predictor-corrector technique to suggest the following two-step iterative method for solving variational inequalities.

Algorithm 3.15. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= P_K[u_n - \rho T u_n] \\ u_{n+1} &= P_K[\frac{y_n + u_n}{2} - \rho T y_n], \quad \lambda \in [0, 1], \quad n = 0, 1, 2, ... \end{aligned}$$

which is an implicit method:

From equation (23.3), we have

$$u = P_K[u - \rho T(\frac{u + u}{2})]. \tag{3.30}$$

This fixed point formulation is used to suggest the implicit method for solving the variational inequalities as

Algorithm 3.16. *For a given* $u_0 \in H$, *compute* u_{n+1} *by the iterative scheme*

$$u_{n+1} = P_K[u_n - \rho T(\frac{u_n + u_{n+1}}{2})], \quad n = 0, 1, 2, ... \tag{3.31}$$

which is another implicit method, see Noor et al. [134]. To implement this implicit method, one can use the predictor-corrector technique to rewrite Algorithm 23.1 as equivalent two-step iterative method:

Algorithm 3.17. *For a given* $u_0 \in H$, *compute* u_{n+1} *by the iterative scheme*

$$\begin{aligned} y_n &= P_K[u_n - \rho T u_n], \\ u_{n+1} &= P_K[u_n - \rho T(\frac{u_n + y_n}{2})], \quad n = 0, 1, 2, ... \end{aligned}$$

which was suggested and studied by Noor et al [134] and is known as the mid-point implicit method for solving variational inequalities. For the convergence analysis and other aspects of Algorithm 3.16, see Noor et al [134].

It is obvious that Algorithm 3.14 and Algorithm 3.15 have been suggested using different variant of the fixed point formulations of the equation (23.3). It is natural to combine these fixed point formulation to suggest a hybrid implicit method for solving the variational inequalities and related optimization problems, which is the main motivation of this paper.

One can rewrite the (23.3) as

$$u = P_K[\frac{u + u}{2} - \rho T(\frac{u + u}{2})]. \tag{3.32}$$

This equivalent fixed point formulation enables to suggest the following method for solving the variational inequalities.

Algorithm 3.18. *For a given* $u_0 \in H$, *compute* u_{n+1} *by the iterative scheme*

$$u_{n+1} = P_K[\frac{u_n + u_{n+1}}{2} - \rho T(\frac{u_n + u_{n+1}}{2})], \quad n = 0, 1, 2, ... \tag{3.33}$$

which is an implicit method.

We would like to emphasize that Algorithm 3.17 is an implicit method. To implement the implicit method, one uses the predictor-corrector technique. We use Algorithm 23.1 as the predictor and Algorithm 3.17 as corrector. Thus, we obtain a new two-step method for solving variational inequalities.

Algorithm 3.19. *For a given* $u_0 \in H$, *compute* u_{n+1} *by the iterative scheme*

$$\begin{aligned} y_n &= P_K[u_n - \rho T u_n] \\ u_{n+1} &= P_K\left[\left(\frac{y_n + u_n}{2}\right) - \rho T\left(\frac{y_n + u_n}{2}\right)\right], \quad n = 0, 1, 2, ... \end{aligned}$$

which is two step method ad appears to be new one.

From the above discussion, it is clear that Algorithm 3.16 and Algorithm 3.18 are equivalent. It is enough to prove the convergence of Algorithm 3.16, which is the main motivation of our next result.

Theorem 3.3. *Let the operator T be strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$, respectively. Let $u \in K$ be solution of (23.3) and u_{n+1} be an approximate solution obtained from Algorithm 3.18. If there exists a constant $\rho > 0$, such that*

$$0 < \rho < \frac{2\alpha}{\beta^2}, \tag{3.34}$$

then the approximate solution u_{n+1} converge to the exact solution $u \in K$.

Proof. Let $u \in K$ be a solution of (23.3) and u_{n+1} be the approximate solution obtained from Algorithm 3.18. Then, from (3.32) and (3.33), we have

$$\begin{aligned} \|u_{n+1} - u\|^2 &= \|P_K[(\frac{u_n + u_{n+1}}{2}) - \rho T(\frac{u_n + u_{n+1}}{2})] \\ &\quad -P_K[\frac{u+u}{2}) - \rho T(\frac{u+u}{2})]\|^2 \\ &\leq \|\frac{u_{n+1} + u_n}{2} - \frac{u+u}{2}) \\ &\quad -\rho(T(\frac{u_{n+1} + u_n}{2}) - T(\frac{u+u}{2}))\|^2 \\ &\leq (1 - 2\rho\alpha + \rho^2\beta^2)\|\frac{u_n - u}{2} + \frac{u_{n+1} - u}{2}\|^2, \end{aligned} \tag{3.35}$$

where we have used the fact that the operator T is the strongly monotone with constant $\alpha > 0$ and Lipschitz continuous constant $\beta > 0$, respectively.
Thus, from (3.35), we have

$$\begin{aligned} \|u_{n+1} - u\| &\leq \sqrt{1 - 2\rho\alpha + \rho^2\beta^2}\{\|\frac{u_n - u}{2}\| + \|\frac{u_{n+1} - u}{2}\|\} \\ &= \frac{1}{2}\sqrt{1 - 2\rho\alpha + \rho^2\beta^2}\|u_n - u\| \\ &\quad +\frac{1}{2}\sqrt{1 - 2\rho\alpha + \rho^2\beta^2}\|u_{n+1} - u\|, \end{aligned} \tag{3.36}$$

which implies that

$$\begin{aligned} \|u_{n+1} - u\| &\leq \frac{\frac{1}{2}\sqrt{1 - 2\rho\alpha + \rho^2\beta^2}}{1 - \frac{1}{2}\sqrt{1 - 2\rho\alpha + \rho^2\beta^2}}\|u_n - u\| \\ &= \theta\|u_n - u\|, \end{aligned} \tag{3.37}$$

where

$$\theta = \frac{\frac{1}{2}\sqrt{1 - 2\rho\alpha + \rho^2\beta^2}}{1 - \frac{1}{2}\sqrt{1 - 2\rho\alpha + \rho^2\beta^2}}.$$

From (3.34), it follows that $\theta < 1$. This shows that the approximate solution u_{n+1} obtained from Algorithm 3.18 converges to the exact solution $u \in K$ satisfying the variational inequality (23.3). □

From equation (23.3), for a constant ξ, we have

$$u = P_K[u - \xi(u - u) - \rho T(u\xi(u - u))].$$

This fixed point equivalent formulation is used to suggest iterative method for solving the variational inequalities.

Algorithm 3.20. *For given $u_0, u_1 \in H$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = P_K[u_n - \xi(u_n - u_{n-1}) - \rho T(u_n - \xi(u_n - u_{n-1}))], \quad n = 0, 1, 2, ...$$

Algorithm 3.20 is known as the inertial projection iterative method. For different and suitable choice of the parameter ξ, one can obtain various known and new known inertial projection type methods for solving variational inequalities and related optimization problems, see Noor [117].
Algorithm 3.20 can be written in the following two step method:

Algorithm 3.21. *For a given $u_0, u_1 \in H$, compute u_{n+1} by the iterative schemes*

$$\begin{aligned} y_n &= u_n - \xi(u_n - u_{n-1}) \\ u_{n+1} &= P_K[y_n - \rho T y_n], \quad n = 0, 1, 2, ..., \end{aligned}$$

which is the subject of recent investigation and have been extended for other classes of variational inequalities. It is worth mentioning that to implement the inertial-type methods, one has to choose two initial values, which is the main draw back of these inertial methods.

In a similar way, we can suggest the following four-step inertial method for solving the quasi variational inequalities (2.16).

Algorithm 3.22. *For given $u_0, u_1 \in H$, compute u_{n+1} by the recurrence relation*

$$\begin{aligned} \omega_n &= u_n - \theta_n \left(u_n - u_{n-1}\right), \\ x_n &= (1-\gamma_n)u_n + \gamma_n P_K\left[\omega_n - \rho T \omega_n\right]\}, \\ y_n &= (1-\beta_n)u_n + \beta_n P_K\left[x_n - \rho T x_n\right]\}, \\ u_{n+1} &= (1-\alpha_n)u_n + \alpha_n P_K\left[y_n - \rho T y_n\right]\}, \end{aligned}$$

where $\alpha_n, \beta_n, \gamma_n, \theta_n \in [0, 1], \quad \forall n \geq 1$.

Using the essentially the technique of Shehu et al [58], Jabeen et al [29] and Noor et al. [51], one can investigate the convergence analysis of the four-step inertial Algorithm 3.22. For appropriate suitable choice of the parameters θ, γ and α, one can obtain one-step inertial Mann iteration, two-step inertial two-step inertial Ishikawa iteration and three-step inertial Noor iterations for solving the variational inequalities. The implementation and comparison of these inertial methods with other technique is an interesting problem for future research.

We now explain the Algorithm 3.18 to illustrate the efficiency of Algorithm 3.18.

Algorithm 3.18
Step 0. Let $\rho_0 > 0, \delta := 0.95 < 1, \epsilon > 0, k = 0$ and $u^0 \in K$.
Step 1. If $\|\|r(u^k, \rho_k)\|_\infty \leq \epsilon$, then stop. Otherwise, go to Step 2.
Step 2.
$y^k = P_K[u^k - \rho_k T(u^k)], \qquad \varepsilon^k = \rho_k(T(\tilde{u}^k) - T(u^k)),$
$r = \frac{\|\varepsilon^k\|}{\|u^k - \tilde{u}^k\|}.$
While $(r > \delta)$
$\rho_k = \frac{0.8}{r} * \rho_k, \qquad y^k = P_K[u^k - \rho_k T(u^k)],$
$\varepsilon^k = \rho_k(T(\tilde{u}^k) - T(u^k)), \qquad r = \frac{\|\varepsilon^k\|}{\|u^k - \tilde{u}^k\|}.$
end While
Step 3.

$u^{k+1} = P_K\left[\frac{u^k+y^k}{2} - \rho T(\frac{u^k+y^k}{2})\right].$

Step 4. $\rho_{k+1} = \begin{cases} \frac{\rho_k * 0.7}{r} & \text{if} \quad r \leq 0.5; \\ \rho_k & \text{otherwise.} \end{cases}$

Step 5. k:=k+1; go to Step 1.

We now consider some examples to illustrate the implementation and efficiency of the proposed method.

3.1. **Numerical Experiments I.** In order to verify the theoretical assertions, we consider the variational inequality (23.3) of finding $u \in K$, such that

$$\langle Tu, v - u \rangle \geq 0, \quad \forall v \in K, \tag{3.38}$$

where

$$Tu = D(u) + Mu + q, \tag{3.39}$$

$D(u)$ and $Mu + q$ are the nonlinear part and the linear part of Tu, respectively.

We form the linear part in the test problems similarly as in Harker and Pang [61]. The matrix $M = A^T A + B$, where A is an $n \times n$ matrix whose entries are randomly generated in the interval $(-5, +5)$ and a skew-symmetric matrix B is generated in the same way. The vector q is generated from a uniform distribution in the interval $(-500, 500)$. In $D(u)$, the nonlinear part of Tu, the components are chosen to be $D_j(u) = d_j * \arctan(u_j)$, where d_j is a random variable in $(0, 1)$.

In all tests we take $\delta = 0.95$ and $\gamma = 1.98$. All iterations start with $u^0 = (1, ..., 1)^T$ and $\rho_0 = 1$, and stopped whenever $\|r(u^k, 1)\|_\infty \leq 10^{-7}$. All codes are written in Matlab. The iteration numbers and the computational time for Algorithm 23.2, the methods in [65] and in [10] with different dimensions are given in the Table 3.1.

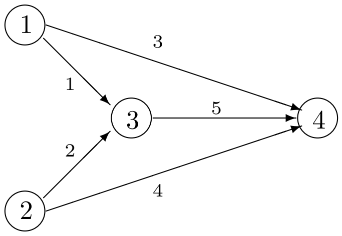
Table 4.1: Numerical results for problem ((23.3))

Dimension of the problem	Algorithm 23.15		The method in [65]		The method in [10]	
	No. It.	CPU(Sec.)	No. It.	CPU(Sec.)	No. It.	CPU(Sec.)
n=100	122	0.009	261	0.06	164	0.04
n=200	194	0.022	402	0.59	250	0.53
n=300	178	0.047	442	1.94	282	1.30
n=500	221	90.107	496	5.91	312	3.29
n=700	181	0.169	479	16.99	310	10.68

3.2. **Numerical Experiments II.** In this subsection, we apply the proposed method to the traffic equilibrium problems and present corresponding numerical results.

Consider a network $[N, L]$ of nodes N and directed links L, which consists of a finite sequence of connecting links with a certain orientation. Let a, b, etc., denote the links, and let p, q, etc., denote the paths. We let ω denote an origin/destination (O/D) pair of nodes of the network and P_ω denotes the set of all paths connecting O/D pair ω. Note that the path-arc incidence matrix and the path-O/D pair incidence matrix, denoted by A and B, respectively, are determined by the given network and

O/D pairs. To see how to convert a traffic equilibrium problem into a variational inequality, we take into account a simple example depicted in Fig.1.



O/D Pairs	Path No. & the link on the path
ω_1 : ① → ④	$p_1 \equiv \{3\}$, $p_2 \equiv \{1, 5\}$
ω_2 : ② → ④	$p_3 \equiv \{4\}$, $p_4 \equiv \{2, 5\}$

FIGURE 1. An illustrative example of given directed network and the O/D pairs

For the given example in Fig. 1, the path-arc incidence matrix A and the path-O/D pair incidence matrix B have the following forms:

$$\begin{array}{cc} \text{No. link} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ A = & \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}, \end{array} \qquad \begin{array}{cc} \text{No. O/D pair} & \begin{array}{cc} \omega_1 & \omega_2 \end{array} \\ B = & \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}. \end{array}$$

Let u_p represent the traffic flow on path p and f_a denote the link load on link a, then the arc-flow vector f is given by

$$f = A^T u.$$

Let d_ω denote the traffic amount between O/D pair ω, which must satisfy

$$d_\omega = \sum_{p \in P_\omega} u_p.$$

Thus, the O/D pair-traffic amount vector d is given by

$$d = B^T u.$$

Let $t(f) = \{t_a, a \in L\}$ be the vector of link travel costs, which is a function of the link flow. A user travelling on path p incurs a (path) travel cost θ_p. For given link travel cost vector t, the path travel cost vector θ is given by

$$\theta = At(f) \qquad \text{and thus} \qquad \theta(u) = At(A^T u).$$

Associated with every O/D pair ω, there is a travel disutility $\lambda_\omega(d)$. Since both the path costs and the travel disutilities are functions of the flow pattern u, the traffic network equilibrium problem is to seek the path flow pattern u^* such that

$$u^* \geq 0 \qquad \langle F(u^*), u - u^* \rangle \geq 0, \qquad \forall u \geq 0 \tag{3.40}$$

where

$$F_p(u) = \theta_p(u) - \lambda_\omega(d(u)), \qquad \forall\, \omega, \;\; p \in P_\omega$$

and thus

$$F(u) = At(A^T u) - B\lambda(B^T u).$$

We apply the proposed method to the example taken from [83] (Example 7.5 in [83]), which consisted of 25 nodes, 37 links and 6 O/D pairs.

The network is depicted in Figure 2. For this example, there are together 55 paths for the 6 given O/D pairs and hence the dimension of the variable u is 55. Therefore, the path-arc incidence matrix A is a 55×37 matrix and the path-O/D pair incidence matrix B is a 55×6 matrix.

In all test implementations we take $u^0 = (1, ..., 1)^T$ as starting point, $\rho_0 = 1$ and the stop criterion is

$$\frac{\| \min\{u, T(u)\}\|_\infty}{\| \min\{u^0, T(u^0)\}\|_\infty} \le \varepsilon. \tag{3.41}$$

The numbers of iteration and the CPU time for different ε are reported in Table 3.2.

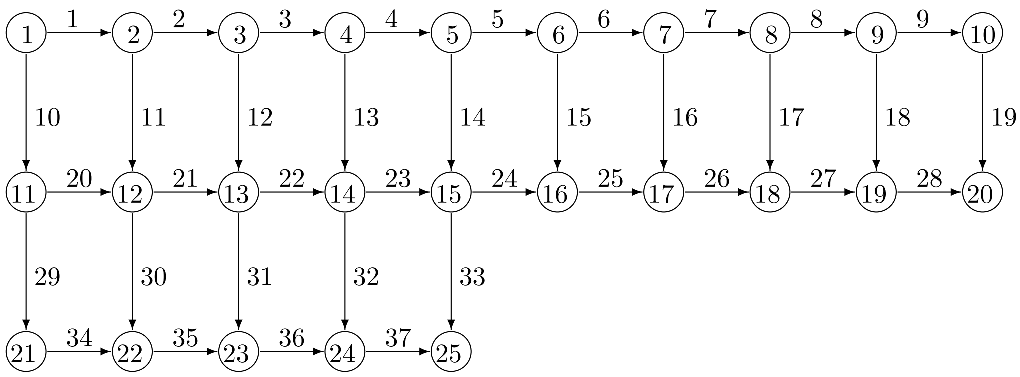


FIGURE 2. A directed network with 25 nodes and 37 links

Table 3.2: Numerical results for different ε

Different	Algorithm 3.18		The method in [65]		The method in [10]	
ε	No. It.	CPU(Sec.)	No. It.	CPU(Sec.)	No. It.	CPU(Sec.)
10^{-5}	85	0.010	316	0.11	199	0.08
10^{-6}	109	0.015	412	0.14	255	0.10
10^{-7}	133	0.017	506	0.18	308	0.12
10^{-8}	157	0.019	602	0.21	363	0.14
10^{-9}	184	0.024	697	0.31	424	0.17

Remark 3.1. *In this section, we have used the equivalence between the variational inequalities and fixed point formulation to suggest some new iterative methods for solving the variational inequalities. Convergence analysis of the proposed method is investigated under suitable conditions. These new implicit methods include extragradient method and modified double projection methods as special cases.*

Some examples are given to illustrate the efficiency which shows that the proposed methods are robust and perform better than the known methods. Comparison of the proposed methods with other methods need further efforts. Using the ideas and techniques of this paper, one can suggest and investigate several new implicit methods for solving various classes of variational inequalities and related problems.

4. Wiener-Hopf equations Technique

In this section, we suggest another class of modified extragradient-type method for solving variational inequalities (23.3) using the Wiener-Hopf equations technique. For this purpose, we need the following result.

Lemma 4.1. *The variational inequality (23.3) has a unique solution $u \in K$, if and only if, the Wiener-Hopf equation (2.17) has a unique solution $z \in H$, where*

$$u = P_K z \tag{4.1}$$

$$z = u - \rho T u. \tag{4.2}$$

Proof. Let $u \in K$ be a solution of (23.3). Then, from Lemma 3.1, we have

$$u = P_K[u - \rho T u],$$

where $\rho > 0$ is a constant.

Let $z = u - \rho T u$. Then

$$u = P_K[z],$$

which implies that

$$z = P_K z - \rho T P_K z,$$

that is,

$$T P_K z + \rho^{-1} Q_K = 0,$$

the required Wiener-Hopf equation (2.17).

Conversely, let $z \in H$ be a solution of (2.17). Then

$$\rho T P_K z = -Q_K z = P_K z - z. \tag{4.3}$$

Now, from (4.3) and Lemma 2.1, we have

$$0 \leq \langle P_K z - z, v - P_K z \rangle = \langle \rho T P_K z, v - P_K z \rangle \quad \forall \quad v \in K.$$

Thus $u = P_K z$ is a solution of (23.3) and from (4.3), we have

$$z = u - \rho T u,$$

the required result. □

Lemma 4.1 implies that the variational inequality (23.3) is equivalent to the Wiener-Hopf equation (2.17). This equivalence has been used to suggest and analyzed a number of iterative methods for solving variational inequalities and complementarity problems.

I. The Wiener-Hopf equations (2.17) can be written as

$$Q_K z = -\rho T P_K,$$

which implies, using (4.1), that

$$z = P_K z - \rho T P_K z = u - \rho T u.$$

This formulation enables us to suggest the following iterative method.

Algorithm 4.1. *For a given $z_0 \in H$, compute z_{n+1} by the iterative scheme*

$$\begin{aligned} u_n &= P_K z_n, \\ z_{n+1} &= u_n - \rho T u_n, \quad n = 0, 1, 2, \ldots \end{aligned}$$

II. By an appropriate and suitable rearrangement of the terms, the Wiener-Hopf equations (2.17) can be written in the form as:

$$\begin{aligned} z &= P_K z - \rho T P_K z + (1 - \rho^{-1}) Q_K z \\ &= u - \rho T u + (1 - \rho^{-1}) Q_K z, \end{aligned}$$

which is another equivalent fixed-point formulation. Using this equivalent fixed-point formulation, we suggest the following iterative method.

Algorithm 4.2. *For a given $z_0 \in H$, compute z_{n+1} by the iterative scheme*

$$\begin{aligned} u_n &= P_K z_n, \\ z_{n+1} &= u_n - \rho T u_n + (1 - \rho^{-1}) Q_K z_n, \quad n = 0, 1, 2, \ldots \end{aligned}$$

III. If T is linear and T^{-1} exists, then the Wiener-Hopf equation (2.17) can be written as

$$z = (I - \rho^{-1} T^{-1}) Q_K z.$$

This fixed-point formulation allows to suggest the following iterative method.

Algorithm 4.3. *For a given $z_0 \in H$, compute z_{n+1} by the iterative scheme*

$$z_{n+1} = (I - \rho^{-1} T^{-1}) Q_K z_n \quad n = 0, 1, 2, \ldots$$

We would like to point out that a special case of Algorithm 4.1 has been used to compute the numerical solutions of the obstacle and unilateral problems by Pitonyak et al. [147]. The results so far obtained are very encouraging. There is still no comparison among various iterative schemes for solving the variational inequalities (23.3). This is an open problem and offers an interesting and fascinating area for further research.

It is well known that the convergence analysis of Algorithms 4.1-4.3 requires that the operator T is both strongly monotone and Lipschitz continuous. These strict conditions rule out many applications of these algorithms. This motivated to modify and develop other iterative methods. From lemma 4.1, we see that both the problems (23.3) and (2.17) are equivalent. Using (4.1) and (4.2), we can rewrite the Wiener-Hopf equations(2.17) in the form

$$\begin{aligned} u - P_K[u - \rho T u] &- \rho T u + \rho P_K[u - \rho T u] \\ &= R(u) - \rho T u + \rho T P_K[g(u) - \rho T u] = 0. \end{aligned} \tag{4.4}$$

Invoking Lemma 2.1 and 3.1, one can easily show that $u \in K$ is solution of (23.3), if and only if, $u \in K$ is a zero of the equation (4.4).

For a positive step α, equation (4.4) can be written as

$$u = u - \alpha d_1(u), \tag{4.5}$$

where

$$d_1(u) = R(u) - \rho Tu + \rho TP_K[u - \rho Tu]. \tag{4.6}$$

This fixed-point formulation has been used to develop some very efficient iterative projection methods for solving various classes of variational inequalities and complementarity problems. He and Liao [65] and Sun [161,162] used $d_1(u)$, defined by (4.6), as the step size in considering the improvement of the extragradient method.

We here use the fixed-point formulation (4.5) to suggest the following modified projection-type method for variational inequalities (23.3).

Algorithm 4.4. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$u_{n+1} = P_K[u_n - \alpha_n d_1(u_n)], \quad n = 0, 1, 2 \ldots,$$

where ρ_n, (prediction), satisfies

$$\rho_n \langle Tu_n - TP_K[u_n - \rho_n Tu_n], R(u_n)\rangle \le \sigma \|R(u_n)\|^2, \quad \sigma \in (0,1),$$

and

$$\begin{aligned} d_1(u_n) &= R(u_n) - \rho_n Tu_n + \rho_n TP_K[u_n - \rho_n Tu_n] \\ \alpha_n &= \frac{(1-\sigma)\|R(u_n)\|^2}{\|d_1(u_n)\|^2}, \end{aligned}$$

is the corrector step size.

Algorithm 4.4 can be viewed as an improvement of the modified projection-type methods of He [63,64], Solodov and Tseng [159] and Noor [102,117] for solving variational inequalities (23.3) with different (predictor, corrector) step sizes.

Now we suggest an improved version of an extragradient-type method, which involves the Wiener-Hopf equation as a step size.

Algorithm 4.5. *For a give $u_0 \in H$, compute u_{n+1} by the iterative schemes*
Predictor step.

$$v_n = P_K[u_n - \rho_n Tu_n], \quad n = 0, 1, 2, \ldots,$$

where ρ_n satisfies

$$\rho_n \langle Tu_n - TP_K[u_n - \rho_n Tu_n], R(u_n)\rangle \le \sigma \|R(u_n)\|^2, \quad \sigma \in (0,1).$$

Corrector step

$$u_{n+1} = P_K[u_n - \rho_n Tv_n], , \quad n = 0, 1, 2 \ldots,$$

where

$$\begin{aligned} d_1(u_n) &= R(u_n) - \rho_n Tu_n + \rho_n TP_K[u_n - \rho Tu_n], \\ \alpha_n &= \frac{(1-\sigma)\|R(u_n)\|^2}{\|d_1(u_n)\|^2,} \end{aligned}$$

is the corrector step size.

We now modify the Wiener-Hopf equation (2.17) to propose some modified projection methods for solving the variational inequalities (23.3). Using (23.12), we can rewrite (4.5) in the form

$$u = u - \alpha d(u), \tag{4.7}$$

where

$$d(u) = \eta R(u) - \eta\rho Tu + \rho T(u - \eta R(u)), \tag{4.8}$$

where α and η are positive constants.

Noor and Rassias [136] used the fixed-point formulation (4.7) to suggest and analyze the following modified projection method for solving the variational inequalities (23.3).

Algorithm 4.6. *For a given $u_0 \in H$, compute u_{n+1} by the iterative schemes*
Predictor step.

$$w_n = u_n - \eta_R(u_n),$$

where $\eta_n = a^{m_n}$ and m_n is the nonnegative integer such that

$$\langle T(u_n) - \rho_n T(u_n - \eta_n R(u_n)), R(u_n)\rangle \leq \sigma \|R(u_n)\|^2, \quad \sigma \in (0,1).$$

Corrector step.

$$\begin{aligned} u_{n+1} &= P_K[u_n - \alpha_n d(u_n)], \quad n = 0,1,2\ldots, \\ d(u_n) &= \eta_n R(u_n) - \eta_n\rho_n Tu_n + \rho_n T(u_n - \eta_n R(u_n)) \\ \alpha_n &= \frac{(\eta-\sigma)\|R(u_n)\|^2}{\|d(u_n)\|^2.} \end{aligned}$$

We now suggest a new unified extragradient methods which combines the main features of Algorithm 4.5 and 4.6 and this is the main motivation of our next algorithms.

Algorithm 4.7. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes:*
Predictor step.

$$w_n = u_n - \eta_n R(u_n), \quad n = 0,1,2\ldots, \tag{4.9}$$

where η_n satisfies

$$\rho_n\langle Tu_n - \eta_n T(u_n - \eta R(u_n)), R(u_n)\rangle \leq \sigma\|R(u_n)\|^2, \quad \sigma \in (0,1). \tag{4.10}$$

Corrector step.

$$u_{n+1} = P_K[u_n - \alpha_n D(u_n)], \quad n = 0,1,2\ldots \tag{4.11}$$

where

$$D(u_n) = R(u_n) - \rho_n Tu_n + \eta_n\rho_n T(u_n - \eta_n R(u_n)) \tag{4.12}$$

$$\alpha_n = \frac{\langle R(u_n), D(u_n)\rangle}{\|D(u_n)\|^2}. \tag{4.13}$$

For the sake of simplicity and without loss of generality, we take $\rho_n = 1$, we denote u_n by u, η_n by η, α_n by α, and

$$M(u) = Tg(u - \eta R(u)) \tag{4.14}$$

$$D(u) = R(u) - Tu + \eta M(u) \tag{4.15}$$

$$u(\alpha) = P_K[g(u) - \alpha M(u)]. \tag{4.16}$$

From (4.10) and (4.12), one can easily obtain

$$\begin{aligned}\langle R(u), D(u)\rangle &= \|R(u)\|^2 - \langle R(u), Tu - \eta M(u)\rangle \\ &\geq (1-\sigma)\|R(u)\|^2. \end{aligned} \tag{4.17}$$

and

$$\begin{aligned}\langle R(u), D(u)\rangle &= \|R(u)\|^2 - \langle R(u), Tu - \eta M(u)\rangle \\ &= \frac{1}{2}\|R(u)\|^2 - \langle R(u), Tu - \eta M(u)\rangle + \frac{1}{2}\|Tu - \eta M(u)\|^2 \\ &\geq \frac{1}{2}\|D(u)\|^2. \end{aligned} \tag{4.18}$$

We now study the convergence of Algorithm 4.7 and show that its convergence requires only pseudomonotonicity, which is a weaker condition than monotonicity. These results are mainly due to Noor [117].

Theorem 4.1. *Let $\bar{u} \in K$ be solution of (23.3). If the operator $T : H \longrightarrow H$ is a pseudomonotone operator, then*

$$\|\bar{u} - u(\alpha)\|^2 \leq \|\bar{u} - u\|^2 - \frac{(1-\sigma)}{2}\|R(u)\|^2, \quad \textit{for all } v \in K. \tag{4.19}$$

Proof. Let $\bar{u} \in K$ be a solution of (23.3). Then, as in Lemma 3.2, we obtain

$$\langle u - \bar{u}, M(u)\rangle \geq \eta\langle R(u), M(u)\rangle = \langle R(u), \eta M(u)\rangle. \tag{4.20}$$

Setting $z = u - Tu, v = u(\alpha), u = P_K[u - Tu]$ in (2.16), we have

$$\langle P_K[u - Tu] - u + Tu, u(\alpha) - P_K[u - Tu]\rangle \geq 0,$$

which implies that

$$\langle u - u(\alpha), R(u) - Tu\rangle \geq \langle R(u), R(u) - Tu\rangle. \tag{4.21}$$

From (2.16) and (4.21), we have

$$\begin{aligned}\|u(\alpha) - u\|^2 &\leq \|u - \alpha M(u) - \bar{u}\|^2 - \|u - \alpha M(u) - u(\alpha)\|^2 \\ &= \|u - \bar{u}\|^2 - \|u - u(\alpha)\|^2 + 2\alpha\langle u - u(\alpha), M(u)\rangle \\ &\quad -2\alpha\langle u - \bar{u}, M(u)\rangle \\ &\leq \|u\|^2 - \|u - u(\alpha)\|^2 - 2\alpha\langle R(u), \eta M(u)\rangle \\ &\quad -2\alpha\langle g(u) - u(\alpha), M(u)\rangle, \end{aligned}$$

from which it follows

$$\begin{aligned}\|u - \bar{u}\|^2 - \|u - u(\alpha)\|^2 &\geq 2\alpha\langle R(u), \eta M(u)\rangle + \|u - u(\alpha)\|^2 \\ &\quad -2\alpha\langle u - u(\alpha), M(u)\rangle \\ &= 2\alpha\langle R(u), \eta M(u)\rangle + \|u - u(\alpha) - \alpha D(u)\|^2 \\ &\quad -\alpha^2\|D(u)\|^2 + 2\alpha\langle u - u(\alpha), D(u) - \eta M(u)\rangle \\ &\geq 2\alpha\langle R(u), \eta M(u)\rangle + \|u - u(\alpha) - \alpha D(u)\|^2 \\ &\quad -\alpha^2\|D(u)\|^2 + 2\alpha\langle R(u), R(u) - Tu\rangle, \quad \text{using (4.19).} \\ &= 2\alpha\langle R(u), R(u) - Tu + \eta M(u)\rangle \\ &\quad -\alpha^2\|D(u)\|^2, \end{aligned} \tag{4.22}$$

which is a quadratic in α and has a maximum at

$$\alpha^* = \frac{\langle R(u), R(u) - Tu + \eta M(u)\rangle}{\|D(u)\|^2} = \frac{\langle R(u), D(u)\rangle}{]|D(u)\|^2} = h(u). \tag{4.23}$$

From (4.17),(4.18),(4.22) and (4.23), we have

$$\begin{aligned}\|u-\bar{u}\|^2 - \|u-u(\alpha)\|^2 &\geq \alpha^*\langle R(u), D(u)\rangle \\ &= h(u)\langle R(u), D(u)\rangle = \frac{1}{2}h(u)\|D(u)\|^2 \\ &= \frac{1}{2}\langle R(u), D(u)\rangle \geq \frac{(1-\sigma)}{2}\|R(u)\|^2,\end{aligned}$$

that is,

$$\|\bar{u} - u(\alpha)\|^2 \leq \|u-\bar{u}\|^2 - \frac{(1-\sigma)}{2}\|R(u)\|^2,$$

the required result. □

Theorem 4.2. *Let u_{n+1} be the approximate solution obtained from Algorithm 4.7 and let $\bar{u} \in K$ be a solution of (2.1). Then*

$$\lim_{n\to\infty}(u_n) = \bar{u}.$$

Proof. Its proofs is similar to that of Theorem 3.1. □

We now use the Wiener-Hopf equations (2.16) to suggest some new iterative methods for solving the variational inequalities (23.3).
From (4.1) and (4.2),

$$\begin{aligned} z &= P_K z - \rho T P_K z \\ &= P_K[u-\rho Tu] - \rho T P_K[u-\rho Tu].\end{aligned}$$

Thus, we have

$$u = \rho Tu + \big[P_K[u-\rho Tu] - \rho T P_K[u-\rho Tu].$$

Consequently, for a constant $\alpha_n > 0$, we have

$$\begin{aligned} u &= (1-\alpha_n)u + \alpha_n\{P_K[P_K[u-\rho Tu] + \rho Tu - \rho T P_K[u-\rho Tu]]\} \\ &= (1-\alpha_n)u + \alpha_n\{P_K[y - \rho Ty + \rho Tu]\}, \end{aligned} \tag{4.24}$$

where

$$y = P_K[u-\rho Tu]. \tag{4.25}$$

Using (4.24) and (4.25), we can suggest the following new predictor-corrector method for solving variational inequalities.

Algorithm 4.8. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$y_n = P_K[u_n - \rho Tu_n] \tag{4.26}$$

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n\Big\{P_K[y_n - \rho Ty_n + \rho Tu_n]\Big\}. \tag{4.27}$$

Algorithm 4.8 can be rewritten in the following equivalent form:

Algorithm 4.9. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} u_{n+1} &= (1-\alpha_n)u_n \\ &+ \alpha_n\{P_K[P_K[u_n-\rho Tu_n]-\rho TP_K[u_n-\rho Tu_n]+\rho Tu_n]\}, \end{aligned} \quad (4.28)$$

which is an explicit iterative method and appears to be a new one.
If $\alpha_n = 1$, then Algorithm 4.9 reduces to

Algorithm 4.10. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$y_n = P_K[u_n - \rho Tu_n] \quad (4.29)$$

$$u_{n+1} = P_K[y_n - \rho Ty_n + \rho Tu_n]\}, \quad (4.30)$$

which appears to be a new one.

We now discuss some applications of the Wiener-Hopf equations (2.17) in the complementarity theory.

I. We show that the Wiener-Hopf equations (2.17) are equivalent to the general complementarity problem (2.4) with $f = 0$, by using the change of variables method. We note that the problem (2.4) can be rewritten in the following form:

$$w = u \in K, \quad v = Tu \in K^*, \quad \langle Tu, u\rangle = 0. \quad (4.31)$$

which is useful in developing a fixed point formulation.
We recall the following known concepts.

$\forall w \in H$, we define the absolute value of w as

$$|w| = w^+ + w^-, \quad (4.32)$$

where

$$w^+ = \sup(0, w), \quad \text{and} \quad w^- = \text{-inf}(0, w).$$

It is well known that for any arbitrary element $w \in H$, we have

$$w = w^+ - w^- \quad \text{and} \quad \langle w^+, w^-\rangle = 0. \quad (4.33)$$

From (4.33) and (4.32), we have

$$|w| + w = 2w^+ \quad \text{and} \quad |w| - w = 2w^-. \quad (4.34)$$

Following the idea of Noor [92], we consider the following change of variables

$$w = u = \frac{|z| + z}{2} = z^+ = P_K(z) \quad (4.35)$$

$$v = \frac{|z| - z}{2\rho} = \rho^{-1}z^-. \quad (4.36)$$

From the above equations, we have

$$u = P_K z \quad \text{and} \quad z = z^+ - z^- = u - \rho TP_K z,$$

which is equivalent to the general Wiener-Hopf equation (2.17) by using Lemma 4.1. This formulation has been used to suggest some iterative type methods for complementarity problems.

II. It is well known that for $z \in H$, we have

$$z = P_K z + P_{-K^*} z = P_K z + P_{K^*}(-z). \tag{4.37}$$

From the Wiener-Hopf equations (2.17), we have

$$\rho T P_K z = P_K - z = P_{K^*}(-z).$$

Form the above discussion, it follows that the complementarity problem (2.4) is equivalent to finding $z \in K$, such that

$$P_K z \in K, \quad P_{K^*}(-z) \in K^* \quad \text{and} \quad \langle P_K(z), P_{K^*}(-z)\rangle = 0.$$

which is equivalent to finding $z \in K$ such that

$$\langle \rho T P_K z, v - P_K(z)\rangle \geq 0, \quad \text{for all} \quad v \in K.$$

if and only if

$$P_K z = P_K[P_K z - \rho T P_K z] \tag{4.38}$$

by invoking Lemma 3.1.

Using (4.1) and (4.2), the equation (4.38) can be written as:

$$u = P_K[P_K[u - \rho T u] - \rho T P_K[u - \rho T u]],$$

which is another equivalent fixed-point formulation of the generalized complementarity problem (2.4). This fixed-point formulation is used to suggest some modified projection methods for complementarity problems. Thus we conclude that with suitable and appropriate rearrangement of the terms of the Wiener-Hopf equations, one can suggest several new and previously known methods for solving variational inequalities and related optimization problems. This clearly shows that the Wiener-Hopf equations provides us very general, unifying and flexible techniques. We hope that the interested reader will be able to use this technique to discover new and innovative methods for variational inequalities and related complementarity problems.

We now discuss some variant forms of Algorithm 4.7 for solving variational inequalities (23.3), which was suggested and considered by Noor et al. [142]. We also include some computational experiments of these special cases. See [142] for further details.

Algorithm 4.11. *For a given $u_0 \in K$, compute*

$$z_n := P_K[u_n - T(u_n)].$$

If $\|R(u_n))\| = 0$, stop; otherwise compute

$$y_n := (1 - \eta_n)u_n + \eta_n z_n,$$

where $\eta_n = \gamma_{m_n}$ with m_n being the smallest nonnegative integer satisfying

$$\langle T(u_n) - T(u_n - \gamma_m R(u_n)), R(u_n)\rangle \leq \sigma \|R(u_n)\|^2.$$

Compute

$$u_{n+1} := P_K[u_n + \alpha_n d_n], \quad n = 0, 1, 2, \ldots$$

where

$$\begin{aligned} d_n &= -(\eta_n R(u_n) - \eta_n T(u_n) + T(y_n)) \\ \alpha_n &= \frac{\eta_n \langle R(u_n), R(u_n) - T(u_n) + T(y_n)\rangle}{\|d_n\|}. \end{aligned}$$

If we choose $\overline{\alpha}_n$ as step size in Algorithm 4.11, then we obtain another convergent algorithm. Obviously, $\overline{\alpha}_n$ guarantees that the distance between the new iterative point and the solution set has a larger decrease, so we call α_n the basic step and $\overline{\alpha}_n$ the optimal step. However, in practice, if K does not possess any special structure, it is much expensive to compute $\overline{\alpha}_n$. That is, we need to find a simple way to compute the projection $P_K[u_n + \overline{\alpha}_n d_n]$. Following the proof of Lemma **??** in [172], we can show that $u_n(\overline{\alpha}_n) = P_{K\cap H_n}[u_n + \alpha_n d_n]$, where $H_n = \{u \in R^n \mid \eta_n\langle R(u_n), R(u_n) - T(u_n) + T(y_n)\rangle + \langle u_n - u, d_n\rangle = 0\}$.

Thus, we can obtain our improved double-projection method for solving variational inequalities.

Algorithm 4.12. *For a given $u_0 \in K$, compute*

$$z_n := P_K[u_n - T(u_n)]$$

If $\|R(u_n)\| = 0$, stop; otherwise compute

$$y_n := (1 - \eta_n)u_n + \eta_n z_n,$$

where $\eta_n = \gamma^{m_n}$ with m_n being the smallest nonnegative integer satisfying

$$\langle T(u_n) - T(u_n - \gamma_m R(u_n)), R(u_n)\rangle \leq \sigma\|R(u_n)\|^2.$$

Compute

$$u_{n+1} = P_{H_n\cap K}[u_n + \alpha_n d_n], \quad n = 0, 1, 2, \ldots$$

where

$$\begin{aligned} d_n &= -(\eta_n R(u_n) - \eta_n T(u_n) + T(y_n)) \\ \alpha_n &= \frac{\eta_n \langle R(u_n), R(u_n) - T(u_n) + T(y_n)\rangle}{\|d_n\|}. \end{aligned}$$

Notice that at each iteration in Algorithm 4.12, the latter projection region is different from the former. More precisely, the latter projection region is an intersection of the domain set K and a hyperplane, so it does not increase computation cost, if K is a polyhedral.

We now give some numerical experiments for Algorithm 4.11 and Algorithm 4.12 and some comparison with other double-projection methods. Throughout the computational experiments, the parameters used are set as $\sigma = 0.5, \gamma = 0.8$, and we use $\|R(u_n)\| \leq 10^{-7}$ as stopping criteria. All computational results were undertaken on a PC-II by MATLAB. We use symbol e to denote the vector whose components are all ones.

Example 4.1. *Consider the mapping $T : R^n \to R^n$ defined by*

$$F(x_1, x_2, x_3, x_4) = \begin{pmatrix} -x_2 + x_3 + x_4 \\ x_1 - (4.5x_3 + 2.7x_4)/(x_2 + 1) \\ 5 - x_1 - (0.5x_3 + 0.3x_4)/(x_3 + 1) \\ 3 - x_1 \end{pmatrix}.$$

with the domain set

$$K = \{x \in R^n_+ \mid e^\top x = 1\}, \qquad \textit{see cite121.}$$

Example 4.2. *This example was tested by Sun [161]. Let*

$$T(x) = Mx + q$$

, where

$$M = \begin{pmatrix} 4 & -1 & 0 & \cdots & 0 \\ -1 & 4 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & -1 \\ 0 & 0 & 0 & \cdots & 4 \end{pmatrix}, \ q = \begin{pmatrix} -1 \\ -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix}$$

with the domain set

$$K = \{x \in R^n_+ \mid x_i \leq 1, i = 1, 2 \cdots, n\}.$$

It is easy to see that T is strongly monotone on R^n.

Example 4.3. *Define*

$$T(x) = Mx + q$$

, where

$$M = diag(1/n, 2/n, \cdots, 1), \ q = (-1, -1, \cdots, -1)^\top.$$

with the domain set

$$K = \{x \in R^n_+ \mid x_i \leq 1, i = 1, 2 \cdots, n\}, \quad \textit{see [142].}$$

Again T is strongly monotone on K. The corresponding strongly monotonicity modulus depends on the dimension n and approaches zeros when n tends to infinity. Obviously, $x = e$ is its unique solution.

We choose the starting point $u_0 = e$ for Example 4.1 and choose $u = (0, \cdots, 0)^\top$ as starting point for Example 4.2 and 4.3 for different dimensions n. For double-projection methods [117, 137], there always exist two step size rules just as in Algorithm 4.11 and Algorithm 4.12. In the following, we give numerical comparison for these methods using two different steps. The numerical results for double- projection methods using the basic step for Examples 4.1, 4.2, 4.3 are listed in Table 1, and the numerical results for double projection methods using the optimal step for Examples 4.1, 4.2, 4.3 are listed in Table 2 (the symbol "\" denotes the number of iterations exceeds 1000 times).

Table 4.1. Numbers experience for Algorithm 4.11

Problem	Alg. [65]	Alg. [168]	Alg. 4.11
5.1(n=4)	\	\	\
5.2($n = 10$)	47	57	47
5.2($n = 20$)	50	60	50
5.2($n = 50$)	52	62	52
5.2($n = 100$)	53	64	53
5.3($n = 10$)	\	\	\
5.3($n = 20$)	\	\	\
5.3($n = 50$)	\	\	\
5.3($n = 100$)	\	\	\

Table 4.2. Numbers experience for Algorithm 4.12

Problem	Alg. [137]	Alg.[147]	Alg. 4.12
5.1(n=4)	65	49	96
5.2($n = 10$)	44	54	44
5.2($n = 20$)	47	57	47
5.2($n = 50$)	49	59	49
5.2($n = 100$)	50	60	50
5.3($n = 10$)	32	73	35
5.3($n = 20$)	33	79	37
5.3($n = 50$)	33	85	40
5.3($n = 100$)	40	90	43

Obviously, optimal step $\overline{\alpha}_n$ is better than the basic step α_n for any direction. Compared with other double projection methods, Algorithm 4.9 also shows a better behavior. From Table 4.1 and Table 4.2, it is clear that our new methods are as efficient as the methods of Iusem and Svaiter [66] and Solodov and Svaiter [158]. This shows that our Algorithm 4.11 and Algorithm 4.12 can be considered as practical alternative to the extragradient and other modified projection methods. The comparison of new methods developed in this paper with the recent methods of He and Liao [65] and Noor [117] is an interesting problem for future research.

As mentioned earlier, one can modify the projection fixed-point equation (23.3) in several way to suggest and analyze a wide class of iterative methods for solving variational inequalities. Continuing this way, we now introduce a new residue vector for the variational inequalities (23.3). For a positive constant $\gamma > 0$, we define the residue vector

$$R_2(u) = u - P_K[u - \gamma R(u)], \tag{4.39}$$

where $R(u)$ is defined by (23.4).

Note that for $\gamma = 1, R_2(u) \equiv R(u)$. It is clear from Lemma 3.1 that $u \in H$ is a solution of the variational inequality (23.3) if and only if $u \in H$ is a zero of the equation (4.32).

Using this fixed-point formulation, one can easily show that the variational inequality (23.3) is equivalent to finding $u \in H$ such that

$$\eta R_2(u) - \eta\rho Tu + \rho T(u - \eta R_2(u)) = 0, \tag{4.40}$$

where $\eta > 0$ is a constant. Equations of the type (4.40) are known as the modified general Wiener-Hopf equations. Note that for $\eta = 1\gamma$, we obtain the original form of the Wiener-Hopf equations (2.17). This alternative equivalent formulation is used to suggest and analyze the following iterative method for solving (23.3).

Algorithm 4.13. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme:*
Predictor step.

$$w_n = u_n - \eta_n R_2(u_n),$$

where $\eta_n = b^{m_n}$ and m_n is the nonnegative integer such that

$$\rho_n\eta_n\langle T(u_n) - Tw_n, R_2(u_n)\rangle \leq \sigma\||R_2(u_n)\|^2 \quad \sigma \in (0, 1).$$

Corrector step.

$$\begin{aligned} u_{n+1} &= P_K[u_n - \alpha_n N(u_n)], \quad n = 0,1,2,\ldots, \\ N(u_n) &= \eta_n R_2(u_n) - \eta_n \rho_n T u_n + \rho_n T(u_n - \eta_n R_2(u_n)) \\ \alpha_n &= \frac{(\eta_n - \sigma)\|R_2(u_n)\|^2}{\|N(u_n)\|^2,} \end{aligned}$$

where α_n is called the corrector step size which depends upon the modified general Wiener-Hopf equations. For different choice of the positive constants γ and η, we can obtain a variety of new and known algorithms for solving variational inequalities and complementarity problems. This clearly shows that Algorithm 4.13 is flexible and unifying one. Using the techniques developed earlier, one can easily show that Algorithm 4.13 converges for pseudomonotone operators.

5. Complementarity problems

In this section, we consider the nonlinear complementarity problem (NCP). For given nonlinear mapping $F : H \to H$, find $x \in K$ such that

$$x \geq 0, \quad F(x) \geq 0, \quad \langle F(x), x\rangle = 0. \tag{5.1}$$

If F is linear mapping, then problem (5.1) is known as the linear complementarity problem, which was first introduced in game theory. Cottle [39] studied the complementarity problem in nonlinear programs with positively bounded Jacobians. These problems have applications in management sciences and engineering sciences and formulated in finite dimensional spaces. It is pointed out that the variational inequalities are usually studied in the infinite dimensional spaces. Karamardian [67] introduced the general complementarity problem (2.17). He proved that, if the underlying convex set K is a convex cone, then the variational inequality problem (23.3) and complementarity problem (5.1). are equivalent. This important results played a very important and significant role in the developments of numerical methods for solving these both problems. Here our focus is on the development of some numerical methods for solving complementarity problem, which can be used to solve the variational inequalities. Based on the logarithmic-quadratic proximal (LQP) method [10], Bnouhacem and Noor [31] proposed a new prediction-correction method for nonlinear complementarity problem (5.1). They obtained the predictor through a simplified inexact logarithmic-quadratic proximal method under a relaxed inexact criterion. The corrector is obtained by the improved extragradient method. Under certain conditions, the global convergence of the proposed method is proved. Preliminary numerical results indicate the efficiency of the proposed method. This section is mainly due to Bnouhacem and Noor [31].

Throughout this section, we assume that F is continuous differentiable and pseudomonotone with respect to R^n_+ and the solution set of (5.1), denoted by Ω^*, is nonempty.
It is well known that problem (5.1) can be alternatively formulated as finding the zero of an appropriately maximal monotone operator T, namely, find $x^* \in R^n_+$ such that $0 \in T(x^*)$, where $N_{R^n_+}(.)$ is the normal cone operator to R^n_+ defined by

$$N_{R^n_+}(x) := \begin{cases} \{y : y^T(v - x) \leq 0, \quad \forall v \in R^n_+\}, & \text{if } \ x \in R^n_+; \\ \emptyset, & \text{otherwise.} \end{cases}$$

A well known method to find the zero of a maximal monotone operator T is the proximal point algorithm, which starting with any vector $x^0 \in R^n_+$ and $\beta_k \geq \beta > 0$, iteratively updates x^{k+1} conforming the following problem:

$$0 \in \beta_k T(x) + \nabla_x q(x, x^k), \tag{5.2}$$

where

$$q(x, x^k) = \frac{1}{2}\|x - x^k\|^2 \tag{5.3}$$

is a quadratic function of x. Motivation for studying the algorithms of problem (5.2) could be found in several studies, in place of usual quadratic term, where many researchers have used some nonlinear functions $r(x, x^k)$. For instance, we quoted reference [49] for the iterative schemes of the form (5.2) using the Bregman-based functional instead of (5.3).
The inexact version of the proximal point algorithm (PPA) [154] generates iteratively sequence $\{x^k\} \subset R^n_+$ satisfying

$$e^k \in (x^{k+1} - x^k) + \beta_k T(x^{k+1}), \tag{5.4}$$

where $e^k \in \mathcal{R}^n$ is the error term. The exact form of PPA corresponds to the case $e^k = 0, \forall k$. In order to ensure convergence, many researcher [49,154] have developed various kinds of additional conditions on the sequence $\{e^k\}$.
For example, Eckstein [10] supposed that

$$\sum_{k=1}^{\infty} \|e^k\| < +\infty \tag{5.5}$$

and

$$\sum_{k=1}^{\infty} \langle e^k, x^k \rangle < +\infty. \tag{5.6}$$

Recently, Auslender et al. [10] have proposed a new type of proximal interior method through replacing the quadratic function (5.3) by $d_\phi(x, x^k)$ which could be defined as

$$d_\phi(x, y) = \sum_{j=1}^{n} y_j^2 \phi(y_j^{-1} x_j).$$

The fundamental difference here is that the term d_ϕ is used to force the iterates $\{x^{k+1}\}$ to stay in the interior of the nonegative orthant R^n_{++}.
Among the possible choices of ϕ, there exists a particular one which enjoys several attractive properties for developing efficient algorithms to solve the problem (5.1).

Let $\nu > \mu > 0$ be given fixed parameters, and define

$$\phi(t) = \begin{cases} \frac{\nu}{2}(t-1)^2 + \mu(t - \log t - 1) & \text{if} \quad t > 0 \\ +\infty & \text{otherwise.} \end{cases}$$

Auslender et al. [10, 11]. used a very special logarithmic-quadratic proximal (LQP) method (with $\nu = 2, \mu = 1$) for solving variational inequalities over polyhedra.

Let $\mu \in (0, 1)$ be a constant, in this paper we consider the function ϕ used in [30] which is proposed by Auslender in [10, 11] defined by

$$\phi(t) = \begin{cases} \frac{1}{2}(t-1)^2 + \mu(t \log t - t + 1) & \text{if} \quad t > 0 \\ +\infty & \text{otherwise.} \end{cases}$$

Then the problem (5.2) becomes for given $x^k \in R^n_{++}$ and $\beta_k \geq \beta > 0$, the new iterate x^{k+1} is solution of the following set-valued equation:

$$0 \in \beta_k T(x) + \nabla_x Q(x, x^k), \tag{5.7}$$

where

$$Q(x, x^k) = \begin{cases} \frac{1}{2}\|x - x^k\|^2 + \mu \sum\limits_{j=1}^{n} (x_j^k)^2 \left(\frac{x_j}{x_j^k} \log \frac{x_j}{x_j^k} - \frac{x_j}{x_j^k} + 1 \right), & \text{if} \quad x \in R^n_{++}; \\ +\infty, & \text{otherwise}. \end{cases} \tag{5.8}$$

We now state the main result proven in [10].

Proposition 5.1. *For each $\beta_k > 0$, $x^k \in R^n_{++}$, there exists a unique $x^{k+1} \in R^n_{++}$ satisfying (5.7).*

Proof. See Section 3 in [10]. □

It is easy to see that

$$\nabla_x Q(x, x^k) = x - x^k + \mu X_k \log \frac{x}{x^k},$$

where $X_k = diag(x_1^k, ..., x_n^k)$ and $\log \frac{x}{x^k} = (\log \frac{x_1}{x_1^k}, ..., \log \frac{x_n}{x_n^k})^T$.
Then the problem (5.7)-(5.8) is equivalent to the following systems of nonlinear equations

$$\beta_k F(x) + x - x^k + \mu X_k \log \frac{x}{x^k} = 0. \tag{5.9}$$

It is more practical to find approximate solutions of (5.9) rather than the exact solutions due the fact that in general it exclude some practical applications. Driven by the fact of eliminating this drawback, very recently, Bnouhachem [30] presented Logarithmic-Quadratic Proximal prediction-correction method for solving nonlinear complementarity problem. The key features of this method are the predictor is obtained by solving (5.9) approximately, with more relaxed conditions than [10,11] and the new iterate is obtained by convex combination of the previous points. Now, we suggest and analyze a new LQP method for solving nonlinear complementarity problems (5.1) by using a new step size α_k which provides a significant refinement and improvement of the method in [11]. We list some important results which will be required in our following analysis.

First, we denote $P_{R^n_+}(.)$ as the projection under the Euclidean norm, i.e.,

$$P_{R^n_+}(z) = \min\{\|z - x\| \mid x \in R^n_+\}.$$

From the above definition, it follows that

$$(v - P_{R^n_+}(v))^T (u - P_{R^n_+}(v)) \leq 0, \qquad \forall u \in R^n_+, \forall v \in R^n. \tag{5.10}$$

From (5.10), it is easy to verify that

$$\|P_{R^n_+}(v) - P_{R^n_+}(u)\| \leq \|v - u\|, \qquad \forall u, v \in R^n, \tag{5.11}$$

and

$$\|P_{R^n_+}(v) - u\|^2 \leq \|v - u\|^2 - \|v - P_{R^n_+}(v)\|^2, \qquad \forall v \in R^n, u \in R^n_+. \tag{5.12}$$

Definition 2.1 $\forall u, v \in R^n$, the operator $F : R^n \to R^n$ is said to be pseudomonotone, if

$$(v - u)^T F(u) \geq 0 \Rightarrow (v - u)^T F(v) \geq 0.$$

The following lemma is a special case of [[10], Lemma3.4]. For the sake of completeness and to convey an idea of the technique involved, we include its proof.

Lemma 5.1. *For given $x^k > 0$ and $q \in R^n$, let x be the positive solution of the following equation:*

$$q + x - x^k + \mu X_k \log \frac{x}{x^k} = 0, \tag{5.13}$$

where $X_k = diag(x_1^k, ..., x_n^k)$ and $\log \frac{x}{x^k} = (\log \frac{x_1}{x_1^k}, ..., \log \frac{x_n}{x_n^k})$,
then for any $y \geq 0$ we have

$$(y - x)^T q \geq \tfrac{1+\mu}{2}\left(\|x - y\|^2 - \|x^k - y\|^2\right) + \tfrac{1-\mu}{2}\|x^k - x\|^2. \tag{5.14}$$

Proof. For each $t > 0$ we have $1 - \frac{1}{t} \leq \log t \leq t - 1$, then we obtain after multiplication by $y_j x_j^k \geq 0$ for each $j = 1, ..., n$,

$$y_j x_j^k \log \frac{x_j}{x_k^j} \leq y_j x_j^k \left(\frac{x_j}{x_j^k} - 1\right) = y_j(x_j - x_j^k)$$

and after multiplication by $x_j x_j^k \geq 0$ for each $j = 1, ..., n$,

$$-x_j x_j^k \log \frac{x_j}{x_j^k} \leq x_j x_j^k \left(\frac{x_j^k}{x_j} - 1\right) = x_j^k(x_j^k - x_j),$$

adding the two inequalities, then obtained

$$(y_j - x_j)(x_j - x_j^k + \mu x_j^k \log \frac{x_j}{x_j^k}) \leq \mu(y_j - x_j^k)(x_j - x_j^k) + (x_j - x_j^k)(y_j - x_j)$$

Using the identities

$$(y_j - x_j^k)(x_j - x_j^k) = \frac{1}{2}\left((x_j - x_j^k)^2 - (x_j - y_j)^2 + (y_j - x_j^k)^2\right)$$

$$(x_j - x_j^k)(y_j - x_j) = \frac{1}{2}\left((y_j - x_j^k)^2 - (y_j - x_j)^2 - (x_j - x_j^k)^2\right)$$

and recalling (5.13) thus obtained

$$(x_j - y_j)(-q_j) \geq \frac{1+\mu}{2}\left((x_j - y_j)^2 - (x_j^k - y_j)^2\right) + \frac{1-\mu}{2}(x_j^k - x_j)^2.$$

Summing over $j = 1, ..., n$, encountered (5.14). □

Lemma 5.2. *Let $x^* \in \Omega^*$, $0 < a_1 < 1, 0 < \mu < 1$, $\tilde{x}^k \in R_+^n$, $\beta_k > 0$, and $x^{k+1}(\alpha)$ is defined by $x^{k+1}(\alpha_k) = a_1 x^k + (1 - a_1) P_{R_+^n}\left[x^k - \frac{\alpha_k \beta_k}{1+\mu} F(\tilde{x}^k)\right]$, then we have*

$$\|x^{k+1}(\alpha_k) - x^*\|^2 \leq \|x^k - x^*\|^2 - (1-a_1)\{\|x^k - x_*^k(\alpha_k)\|^2 + \frac{2\alpha_k\beta_k}{1+\mu}\langle x_*^k(\alpha_k) - x^k, F(\tilde{x}^k)\rangle + \frac{2\alpha_k\beta_k}{1+\mu}\langle x^k - x^*, F(\tilde{x}^k)\rangle\},$$

where $x_*^k(\alpha_k) = P_{R_+^n}\left[x^k - \frac{\alpha_k\beta_k}{1+\mu}F(\tilde{x}^k)\right]$.

Proof. Since $x^* \in \Omega^* \subset R_+^n$ and $x_*^k(\alpha_k) = P_{R_+^n}\left[x^k - \frac{\alpha_k\beta_k}{1+\mu}F(\tilde{x}^k)\right]$, it follows from (5.12) that

$$\|x_*^k(\alpha_k) - x^*\|^2 \leq \|x^k - \frac{\alpha_k\beta_k}{1+\mu}F(\tilde{x}^k) - x^*\|^2 - \|x^k - \frac{\alpha_k\beta_k}{1+\mu}F(\tilde{x}^k) - x_*^k(\alpha_k)\|^2. \tag{5.15}$$

and

$$\begin{aligned}\|x^{k+1}(\alpha_k) - x^*\|^2 &= \|a_1(x^k - x^*) + (1 - a_1)(x_*^k(\alpha_k) - x^*)\|^2 \\ &= a_1^2\|x^k - x^*\|^2 + (1 - a_1)^2\|x_*^k(\alpha_k) - x^*\|^2 + 2a_1(1 - a_1)\langle x^k - x^*, x_*^k(\alpha_k) - x^*\rangle.\end{aligned}$$

Using the following identity

$$2\langle a + b, b\rangle = \|a + b\|^2 - \|a\|^2 + \|b\|^2$$

for $a = x^k - x_*^k(\alpha_k), b = x_*^k(\alpha_k) - x^*$ and (5.15), and using $0 < a_1 < 1$, we obtain

$$\begin{aligned}
\|x^{k+1}(\alpha_k) - x^*\|^2 &= a_1^2\|x^k - x^*\|^2 + (1-a_1)^2\|x_*^k(\alpha_k) - x^*\|^2 + a_1(1-a_1)\{\|x^k - x^*\|^2 \\
&\quad -\|x^k - x_*^k(\alpha_k)\|^2 + \|x_*^k(\alpha_k) - x^*\|^2\} \\
&= a_1\|x^k - x^*\|^2 + (1-a_1)\|x_*^k(\alpha_k) - x^*\|^2 - a_1(1-a_1)\|x^k - x_*^k(\alpha_k)\|^2 \\
&\leq a_1\|x^k - x^*\|^2 + (1-a_1)\{\|x^k - \frac{\alpha_k\beta_k}{1+\mu}F(\tilde{x}^k) - x^*\|^2 - \|x^k - \frac{\alpha_k\beta_k}{1+\mu}F(\tilde{x}^k) - x_*^k(\alpha_k)\|^2\} \\
&\quad -a_1(1-a_1)\|x^k - x_*^k(\alpha_k)\|^2. \\
&\leq \|x^k - x^*\|^2 - (1-a_1)\{\|x^k - x_*^k(\alpha_k)\|^2 + \frac{2\alpha_k\beta_k}{1+\mu}\langle x_*^k(\alpha_k) - x^k, F(\tilde{x}^k)\rangle \\
&\quad + \frac{2\alpha_k\beta_k}{1+\mu}\langle x^k - x^*, F(\tilde{x}^k)\rangle\}
\end{aligned}$$

□

We now suggest the iterative method for solving complementarity problems.

At the k-th iteration, LQP method finds the exact solution for the following system of equations:

$$\beta_k F(x) + x - x^k + \mu X_k \log\frac{x}{x^k} = 0. \tag{5.16}$$

We now present an LQP method-based prediction-correction method for solving (5.1). For given $x^k > 0$ and $\beta_k > 0$, each iteration of the proposed method consists of two steps, the first step offers a predictor $\tilde{x}^k$ and the second step produces the new iterate x^{k+1}.

Prediction step: Find an approximate positive solution $\tilde{x}^k$ of (5.16), called predictor, such that

$$0 \approx \beta_k F(\tilde{x}^k) + \tilde{x}^k - x^k + \mu X_k \log\frac{\tilde{x}^k}{x^k} = \xi^k \tag{5.17}$$

and ξ^k which satisfies

$$\|\xi^k\| \leq \eta\|x^k - \tilde{x}^k\|, \qquad 0 < \eta < 1. \tag{5.18}$$

Latter, we show how to choose the new iterates for the proposed method.

Remark 5.1. *(5.18) implies that*

$$|(x^k - \tilde{x}^k)^T\xi^k| \leq \eta\|x^k - \tilde{x}^k\|^2, \qquad 0 < \eta < 1. \tag{5.19}$$

Remark 5.2. *In general the prediction step is implementable. Sometimes we can get the approximate solution of (5.17) directly via choosing a suitable small* $\beta_k > 0$*. For example in the special case* $\xi^k = \beta_k(F(\tilde{x}^k) - F(x^k))$*, if* F *is Lipschitz continuous in* R_+^n *with Lipschitz constant* $L > 0$*, i.e.,*

$$\|F(x^k) - F(\tilde{x}^k)\| \leq L\|x^k - \tilde{x}^k\|.$$

If we choose β_k *satisfying* $0 < \beta_k \leq \frac{\eta}{L}$*, then the above inequality (5.18) is satisfied.*

Remark 5.3. *Note that in the special case* $\xi^k = \beta_k(F(\tilde{x}^k) - F(x^k))$*, then (5.17) is equivalent to the following system of nonlinear equations*

$$\beta_k F(x^k) + \tilde{x}^k - x^k + \mu X_k \log\frac{\tilde{x}^k}{x^k} = 0, \tag{5.20}$$

hence

$$\tilde{x}_j^k + \mu x_j^k \log\tilde{x}_j^k + (\beta_k F_j(x^k) - x_j^k - \mu x_j^k \log x_j^k) = 0, \qquad j = 1, ..., n. \tag{5.21}$$

The recursion of classical Newton method for the above problem is

$$\tilde{x_j}^k := {x_j}^k - \frac{\beta_k}{1+\mu} F_j(x^k).$$

The solution $\tilde{x}^k$ *of (5.21) is positive, to avoid the non-positive value* $\tilde{x_j}^k$ *in the iteration process, we take*

$$\tilde{x_j}^k := \max\{{x_j}^k - \frac{\beta_k}{1+\mu} F_j(x^k), 0\}, \qquad j = 1, ..., n.$$

Remark 5.4. *Auslender et al. [10] proposed the following conditions*

$$\sum_{k=1}^{\infty} \|\xi^k\| < +\infty \quad \text{and} \quad \sum_{k=1}^{\infty} \langle \xi^k, x^k \rangle < +\infty \tag{5.22}$$

to ensure convergence.

The following result is very useful in analysis of the proposed method.

Lemma 5.3. *For given* $x^k > 0$ *and* $\beta_k > 0$, *let* $\tilde{x}^k$ *to be obtained by the prediction step (5.17), then for each* $x \geq 0$, *we have*

$$(x - \tilde{x}^k)^T (\beta_k F(\tilde{x}^k) - \xi^k) \geq (x^k - \tilde{x}^k)^T \{(1+\mu)x - (\mu x^k + \tilde{x}^k)\}. \tag{5.23}$$

Proof. By setting $q = \beta_k F(\tilde{x}^k) - \xi^k$ in (5.13) and $y = x$ in (5.14), it follows from (5.17) that

$$\begin{aligned} (x - \tilde{x}^k)^T (\beta_k F(\tilde{x}^k) - \xi^k) &\geq \frac{1+\mu}{2}(\|\tilde{x}^k - x\|^2 - \|x^k - x\|^2) + \frac{1-\mu}{2}\|x^k - \tilde{x}^k\|^2 \\ &= (1+\mu)x^T x^k - (1+\mu)x^T \tilde{x}^k - (1-\mu)(\tilde{x}^k)^T x^k - \mu\|x^k\|^2 + \|\tilde{x}^k\|^2 \\ &= (1+\mu)x^T(x^k - \tilde{x}^k) - (x^k - \tilde{x}^k)^T(\mu x^k + \tilde{x}^k) \\ &= (x^k - \tilde{x}^k)^T\{(1+\mu)x - (\mu x^k + \tilde{x}^k)\}. \end{aligned}$$

□

Now, we show how to choose the new iterate.
Setting $x = x^k$ in (5.23) and using (5.19), we get

$$\begin{aligned} (x^k - \tilde{x}^k)^T (\beta_k F(\tilde{x}^k)) &\geq \|x^k - \tilde{x}^k\|^2 + (x^k - \tilde{x}^k)^T \xi^k \\ &\geq (1-\eta)\|x^k - \tilde{x}^k\|^2. \end{aligned} \tag{5.24}$$

Since $\tilde{x}^k \in R^n_+$ and x^* is solution of(5.1), using pseudomonotonicity of F we have

$$(\tilde{x}^k - x^*)^T F(x^*) \geq 0 \Rightarrow (\tilde{x}^k - x^*)^T F(\tilde{x}^k) \geq 0$$

and consequently

$$(x^k - x^*)^T \left(\frac{\beta_k}{1+\mu} F(\tilde{x}^k)\right) \geq (x^k - \tilde{x}^k)^T \left(\frac{\beta_k}{1+\mu} F(\tilde{x}^k)\right). \tag{5.25}$$

From (5.24) and (5.36), we get

$$(x^k - x^*)^T \left(\frac{\beta_k}{1+\mu} F(\tilde{x}^k)\right) \geq \left(\frac{1-\eta}{1+\mu}\right) \|x^k - \tilde{x}^k\|^2.$$

Then $-\frac{\beta_k}{1+\mu} F(\tilde{x}^k)$ is a descent direction of the distance function at x^k, so along $-\frac{\beta_k}{1+\mu} F(\tilde{x}^k)$, one can find a new iterate which is closer to the solution set. Due to this fact, we construct to new iterate as:

Correction step :

For $\alpha_k > 0$ and $0 < a_1 < 1$, the new iterate $x^{k+1}(\alpha)$ is defined by

$$x^{k+1}(\alpha) = a_1 x^k + (1 - a_1) P_{R^n_+}\Big[x^k - \frac{\alpha_k \beta_k}{1+\mu} F(\tilde{x}^k)\Big]. \tag{5.26}$$

How to choose a suitable step length $\alpha_k > 0$ to force convergence. For this purpose, we need the following theorem.

Theorem 5.1. *Let* $x_*^k(\alpha_k) := P_{R_+^n}\left[x^k - \frac{\alpha_k\beta_k}{1+\mu}F(\tilde{x}^k)\right]$ *and* $\Theta(\alpha_k) := \|x^k - x^*\|^2 - \|x^{k+1}(\alpha_k) - x^*\|^2$, *then we have*

$$\Theta(\alpha_k) \geq (1-a_1)\Psi(\alpha_k) \geq (1-a_1)\Phi(\alpha_k) \quad (5.27)$$

where

$$\Psi(\alpha_k) = \|x^k - x_*^k(\alpha_k)\|^2 + \frac{2\alpha_k\beta_k}{1+\mu}\langle x_*^k(\alpha_k) - \tilde{x}^k, F(\tilde{x}^k)\rangle, \quad (5.28)$$

$$e\Phi(\alpha_k) = 2\alpha_k\varphi_k - \alpha_k^2\|d^k\|^2, \quad (5.29)$$

$$\varphi_k = \frac{1}{1+\mu}\|x^k - \tilde{x}^k\|^2 + \frac{1}{1+\mu}\langle x^k - \tilde{x}^k, \xi^k\rangle, \quad (5.30)$$

and

$$d^k = (x^k - \tilde{x}^k) + \frac{1}{1+\mu}\xi^k. \quad (5.31)$$

Proof. By setting $q = \beta_k F(\tilde{x}^k) - \xi^k$ in (5.13) and $y = x_*^k(\alpha_k)$ in (5.14), it follows that

$$\begin{aligned}\langle x_*^k(\alpha_k) - \tilde{x}^k, \frac{1}{1+\mu}(\xi^k - \beta_k F(\tilde{x}^k))\rangle &\leq \frac{1}{2}\left(\|x^k - x_*^k(\alpha_k)\|^2 - \|\tilde{x}^k - x_*^k(\alpha_k)\|^2\right) \\ &\quad -\frac{1-\mu}{2(1+\mu)}\|x^k - \tilde{x}^k\|^2. \end{aligned} \quad (5.32)$$

Recall

$$\langle x_*^k(\alpha_k) - \tilde{x}^k, x^k - \tilde{x}^k\rangle = \frac{1}{2}\left(\|\tilde{x}^k - x_*^k(\alpha_k)\|^2 - \|x^k - x_*^k(\alpha_k)\|^2\right) + \frac{1}{2}\|x^k - \tilde{x}^k\|^2. \quad (5.33)$$

Adding (5.32) and (5.33) we then obtain

$$\langle x_*^k(\alpha_k) - \tilde{x}^k, x^k - \tilde{x}^k + \frac{1}{1+\mu}(\xi^k - \beta_k F(\tilde{x}^k))\rangle \leq \frac{\mu}{1+\mu}\|x^k - \tilde{x}^k\|^2,$$

which implies

$$2\alpha_k\langle x_*^k(\alpha_k) - \tilde{x}^k, x^k - \tilde{x}^k + \frac{1}{1+\mu}(\xi^k - \beta_k F(\tilde{x}^k))\rangle - \frac{2\alpha_k\mu}{1+\mu}\|x^k - \tilde{x}^k\|^2 \leq 0. \quad (5.34)$$

Using the definition of $\Theta(\alpha_k)$, we get

$$\begin{aligned}\Theta(\alpha_k) &\geq (1-a_1)\{\|x^k - x_*^k(\alpha_k)\|^2 + \frac{2\alpha_k\beta_k}{1+\mu}\langle x_*^k(\alpha_k) - x^k, F(\tilde{x}^k)\rangle \\ &\quad + \frac{2\alpha_k\beta_k}{1+\mu}\langle x^k - x^*, F(\tilde{x}^k)\rangle\}. \end{aligned} \quad (5.35)$$

Since $\tilde{x}^k \in R_+^n$ and x^* is a solution of the problem (5.1) using the pseudomonotonicity of F we have

$$\langle \tilde{x}^k - x^*, F(x^*)\rangle = \langle \tilde{x}^k, F(x^*)\rangle \geq 0 \Rightarrow \langle \tilde{x}^k - x^*, F(\tilde{x}^k\rangle \geq 0$$

and consequently

$$\langle x^k - x^*, F(\tilde{x}^k)\rangle \geq \langle x^k - \tilde{x}^k, F(\tilde{x}^k)\rangle. \quad (5.36)$$

Applying (5.36) to the last term in the right side of (5.35), we obtain

$$\begin{aligned}\Theta(\alpha_k) &\geq (1-a_1)\{\|x^k - x_*^k(\alpha_k)\|^2 + \frac{2\alpha_k\beta_k}{1+\mu}\langle x_*^k(\alpha_k) - \tilde{x}^k, F(\tilde{x}^k)\rangle\} \\ &= (1-a_1)\Psi(\alpha_k). \end{aligned} \quad (5.37)$$

Adding $\Psi(\alpha_k)$ and (5.34) and using the notation of d^k in (5.31), we get

$$\begin{aligned}\Psi(\alpha_k) &\geq \|x_*^k(\alpha_k) - x^k\|^2 + 2\alpha_k\langle x_*^k(\alpha_k) - x^k, d^k\rangle + 2\alpha_k\langle x^k - \tilde{x}^k, d^k\rangle - \frac{2\alpha_k\mu}{1+\mu}\|x^k - \tilde{x}^k\|^2 \\ &= \|x_*^k(\alpha_k) - x^k + \alpha_k d^k\|^2 - \alpha_k^2\|d^k\|^2 + 2\alpha_k\langle x^k - \tilde{x}^k, d^k\rangle - \frac{2\alpha_k\mu}{1+\mu}\|x^k - \tilde{x}^k\|^2 \\ &\geq 2\alpha_k\langle x^k - \tilde{x}^k, x^k - \tilde{x}^k + \frac{1}{1+\mu}\xi^k\rangle - \frac{2\alpha_k\mu}{1+\mu}\|x^k - \tilde{x}^k\|^2 - \alpha_k^2\|d^k\|^2 \\ &= 2\alpha_k\left(\frac{1}{1+\mu}\|x^k - \tilde{x}^k\|^2 + \frac{1}{1+\mu}\langle x^k - \tilde{x}^k, \xi^k\rangle\right) - \alpha_k^2\|d^k\|^2 \\ &= 2\alpha_k\varphi_k - \alpha_k^2\|d^k\|^2 \\ &= \Phi(\alpha_k).\end{aligned}$$

□

Proposition 5.2. *Assume that F is continuously differentiable, then we have*

(i) $\Psi'(\alpha_k) = \frac{2\beta_k}{1+\mu}\langle x_*^k(\alpha_k) - \tilde{x}^k, F(\tilde{x}^k)\rangle$.

(ii) $\Psi'(\alpha_k)$ *is decreasing function with respect to $\alpha_k > 0$, i.e., $\Psi(\alpha_k)$ is concave.*

Proof. For given $\beta_k, x^k, \tilde{x}^k$, let

$$h(\alpha_k, y) = \|y - [x^k - \frac{\alpha_k\beta_k}{1+\mu}F(\tilde{x}^k)]\|^2 - \frac{\alpha_k^2\beta_k^2}{(1+\mu)^2}\|F(\tilde{x}^k)\|^2 - \frac{2\alpha_k\beta_k}{1+\mu}\langle\tilde{x}^k - x^k, F(\tilde{x}^k)\rangle \tag{5.38}$$

It easy to see that the solution of the following problem

$$\min_y\{h(\alpha_k, y) | y \in R_+^n\}$$

is $y^* = P_{R_+^n}[x^k - \frac{\alpha_k\beta_k}{1+\mu}F(\tilde{x}^k)]$. Substituting y^* into (5.38) and simplifying it, we have

$$\Psi(\alpha_k) = h(\alpha_k, y)|_{y = P_{R_+^n}[x^k - \frac{\alpha_k\beta_k}{1+\mu}F(\tilde{x}^k)]} = \min_y\{h(\alpha_k, y) | y \in R_+^n\}$$

It follows that $\Psi(\alpha_k)$ is differentiable and its derivative is given by

$$\begin{aligned}\Psi'(\alpha_k) &= \frac{\partial h(\alpha_k, y)}{\partial\alpha_k}|_{y = P_{R_+^n}[x^k - \frac{\alpha_k\beta_k}{1+\mu}F(\tilde{x}^k)]} \\ &= \frac{2\beta_k}{1+\mu}\langle x_*^k(\alpha_k) - x^k + \frac{\alpha_k\beta_k}{1+\mu}F(\tilde{x}^k), F(\tilde{x}^k)\rangle - \frac{2\alpha_k\beta_k^2}{(1+\mu)^2}\|F(\tilde{x}^k)\|^2 - \frac{2\beta_k}{1+\mu}\langle\tilde{x}^k - x^k, F(\tilde{x}^k)\rangle \\ &= \frac{2\beta_k}{1+\mu}\langle x_*^k(\alpha_k) - \tilde{x}^k, F(\tilde{x}^k)\rangle.\end{aligned}$$

and the first conclusion is proved. We now establish the proof of the second assertion. Let $\bar{\alpha_k} > \alpha_k > 0$, we will prove that

$$\Psi'(\bar{\alpha_k}) \leq \Psi'(\alpha_k)$$

i.e.,

$$\langle x_*^k(\bar{\alpha_k}) - x_*^k(\alpha_k), F(\tilde{x}^k)\rangle \leq 0. \tag{5.39}$$

Setting $z := x^k - \frac{\bar{\alpha_k}\beta_k}{1+\mu}F(\tilde{x}^k), v := x_*^k(\alpha_k)$ and $z := x^k - \frac{\alpha_k\beta_k}{1+\mu}F(\tilde{x}^k), v := x_*^k(\bar{\alpha_k})$ in (5.10), respectively, we have

$$\langle x^k - \frac{\bar{\alpha_k}\beta_k}{1+\mu}F(\tilde{x}^k) - x_*^k(\bar{\alpha_k}), x_*^k(\alpha_k) - x_*^k(\bar{\alpha_k})\rangle \leq 0 \tag{5.40}$$

and

$$\langle x^k - \frac{\alpha_k\beta_k}{1+\mu}F(\tilde{x}^k) - x_*^k(\alpha_k), x_*^k(\bar{\alpha_k}) - x_*^k(\alpha_k)\rangle \leq 0. \tag{5.41}$$

Adding (5.40) and (5.41), we obtain

$$\langle x_*^k(\bar{\alpha_k}) - x_*^k(\alpha_k), x_*^k(\bar{\alpha_k}) - x_*^k(\alpha_k) + \frac{(\bar{\alpha_k} - \alpha_k)\beta_k}{1+\mu} F(\tilde{x}^k)\rangle \le 0,$$

that is,

$$\|x_*^k(\bar{\alpha_k}) - x_*^k(\alpha_k)\|^2 + \frac{(\bar{\alpha_k} - \alpha_k)\beta_k}{1+\mu} \langle x_*^k(\bar{\alpha_k}) - x_*^k(\alpha_k), F(\tilde{x}^k)\rangle \le 0.$$

It follows that

$$\langle x_*^k(\bar{\alpha_k}) - x_*^k(\alpha_k), F(\tilde{x}^k)\rangle \le -\frac{1+\mu}{(\bar{\alpha_k} - \alpha_k)\beta_k} \|x_*^k(\bar{\alpha_k}) - x_*^k(\alpha_k)\|^2 \le 0.$$

Then, we obtain the inequality (5.39) and complete the proof. □

Now for the same kth approximate solution x^k, let

$$\alpha_{k_2}^* = \arg\max_{\alpha}\{\Phi(\alpha) | \alpha > 0\} \tag{5.42}$$

and

$$\alpha_{k_1}^* = \arg\max_{\alpha}\{\Psi(\alpha) | 0 < \alpha \le m_1 \alpha_{k_2}^*\}, \tag{5.43}$$

where $m_1 \ge 1$.
Note that $\Phi(\alpha)$ is a quadratic function of α and it reaches its maximum at

$$\alpha_{k_2}^* = \frac{\varphi_k}{\|d^k\|^2} \qquad \text{which is used in [30]} \tag{5.44}$$

and

$$\Phi(\alpha_{k_2}^*) = \alpha_{k_2}^* \varphi_k. \tag{5.45}$$

Based on the theorem 5.1 and proposition 5.2, the following convergence results can be proved easily.

Proposition 5.3. *Let $\alpha_{k_1}^*$ and $\alpha_{k_2}^*$ be defined by (5.43) and (5.42) respectively, F be pseudomonotone and continuously differentiable, then we have*

(i) $\|x^k - x^*\|^2 - \|x^{k+1}(\alpha_{k_1}^*) - x^*\|^2 \ge (1-a_1)\Psi(\alpha_{k_1}^*)$
(ii) $\|x^k - x^*\|^2 - \|x^{k+1}(\alpha_{k_2}^*) - x^*\|^2 \ge (1-a_1)\Phi(\alpha_{k_2}^*)$
(iii) $\Psi(\alpha_{k_1}^*) \ge \Phi(\alpha_{k_2}^*)$
Furthermore, if $\Psi'(\alpha_{k_1}^*) = 0$, we have
(iv) $\|x^k - x^*\|^2 - \|x^{k+1}(\alpha_{k_1}^*) - x^*\|^2 \ge (1-a_1)\|x^k - x^{k+1}(\alpha_{k_1}^*)\|^2$.

Remark 5.5. *Proposition 5.3 shows theoretically that the proposed method is expected to make more progress than the method in [30] at each iteration.*

For the convergence analysis of the proposed method, we need the following results.

Theorem 5.2. *[30] For given $x^k \in R_+^n$ and $\beta_k > 0$, let $\tilde{x}^k$ and ξ^k satisfied to the condition (5.18), then we have the following*

$$\alpha_{k_2}^* \ge \frac{1-\eta}{2(1+\mu)} \tag{5.46}$$

and

$$\Phi(\alpha_{k_2}^*) \ge \frac{(1-\eta)^2}{2(1+\mu)^2}\|x^k - \tilde{x}^k\|^2. \tag{5.47}$$

Proof. If $\langle x^k - \tilde{x}^k, \xi^k \rangle \leq 0$, since $\mu > 0$ it follows from (5.18) and (5.31) that

$$\begin{aligned} \|d^k\|^2 &\leq \|x^k - \tilde{x}^k\|^2 + \frac{1}{(1+\mu)^2}\|\xi^k\|^2 \\ &\leq \|x^k - \tilde{x}^k\|^2 + \|\xi^k\|^2 \\ &\leq 2\|x^k - \tilde{x}^k\|^2, \end{aligned} \tag{5.48}$$

from (9.32) and (5.48), we obtain

$$\alpha^*_{k_2} = \frac{\varphi_k}{\|d^k\|^2} \geq \frac{1-\eta}{2(1+\mu)}.$$

Otherwise, if $\langle x^k - \tilde{x}^k, \xi^k \rangle \geq 0$, it follows from $0 < \mu < 1$ and (5.18) that

$$\begin{aligned} \varphi_k &= \frac{1}{1+\mu}\|x^k - \tilde{x}^k\|^2 + \frac{1}{1+\mu}\langle x^k - \tilde{x}^k, \xi^k \rangle \\ &\geq \frac{1}{1+\mu}\left(\frac{1}{2}\|x^k - \tilde{x}^k\|^2 + \frac{1}{1+\mu}\langle x^k - \tilde{x}^k, \xi^k \rangle + \frac{1}{2}\|x^k - \tilde{x}^k\|^2\right) \\ &\geq \frac{1}{1+\mu}\left(\frac{1}{2}\|x^k - \tilde{x}^k\|^2 + \frac{1}{(1+\mu)}\langle x^k - \tilde{x}^k, \xi^k \rangle + \frac{1}{2(1+\mu)^2}\|\xi^k\|^2\right) \\ &= \frac{1}{2(1+\mu)}\|d^k\|^2 \end{aligned}$$

and thus

$$\alpha^*_{k_2} \geq \frac{1}{2(1+\mu)} \geq \frac{1-\eta}{2(1+\mu)}.$$

It follows from (5.19) and (5.30) that

$$\varphi_k = \frac{1}{1+\mu}\|x^k - \tilde{x}^k\|^2 + \frac{1}{1+\mu}(x^k - \tilde{x}^k)^T\xi^k \geq \left(\frac{1-\eta}{1+\mu}\right)\|x^k - \tilde{x}^k\|^2. \tag{5.49}$$

Using (5.49), (9.32) and (5.46) directly we obtained (5.47). □

For fast convergence, we take a relaxation factor $\gamma \in [1, 2)$ and set the step size $\alpha^*_{k_2} = \gamma\alpha^*_{k_2}$. Through simple manipulations we obtain

$$\begin{aligned} \Phi(\gamma\alpha^*_{k_2}) &= 2\gamma\alpha^*_{k_2}\varphi_k - (\gamma^2\alpha^*_{k_2})(\alpha^*_{k_2}\|d^k\|^2) \\ &= (2\gamma\alpha^*_{k_2} - \gamma^2\alpha^*_{k_2})\varphi_k \\ &= \gamma(2-\gamma)\Phi(\alpha^*_{k_2}) \end{aligned} \tag{5.50}$$

It follows from Theorem 5.1, Proposition 5.3, Theorem 5.2 and (5.50) that there is a constant

$$c := \frac{\gamma(2-\gamma)(1-a_1)(1-\eta)^2}{2(1+\mu)^2} > 0$$

such that

$$\|x^{k+1} - x^*\|^2 \leq \|x^k - x^*\|^2 - c\|x^k - \tilde{x}^k\|^2 \qquad \forall x^* \in \Omega^*. \tag{5.51}$$

Theorem 5.3. *[?] If $\inf_{k=0}^{\infty}\beta_k = \beta > 0$, then the sequence $\{x^k\}$ generated by the proposed method converges to some x^∞ which is a solution of the problem (5.1).*
Proof. It follows from (5.51) that $\{x^k\}$ is a bounded sequence and

$$\lim_{k\to\infty}\|x^k - \tilde{x}^k\| = 0. \tag{5.52}$$

Consequently, $\{\tilde{x}^k\}$ is also bounded. Since $\lim_{k\to\infty}\|x^k-\tilde{x}^k\|=0$, $\|\xi^k\|\le\eta\|x^k-\tilde{x}^k\|$ and $\beta_k\ge\beta>0$, it follows from (5.23) that

$$\lim_{k\to\infty}(x-\tilde{x}^k)^T F(\tilde{x}^k)\ge 0, \qquad \forall x\in R^n_+.$$

Because $\{\tilde{x}^k\}$ is bounded, it has at least one cluster point. Let x^∞ be a cluster point of $\{\tilde{x}^k\}$ and the subsequence $\{\tilde{x}^{k_j}\}$ converges to x^∞. It follows that

$$\lim_{j\to\infty}(x-\tilde{x}^{k_j})^T F(\tilde{x}^{k_j})\ge 0, \qquad \forall x\in R^n_+.$$

and consequently

$$(x-x^\infty)^T F(x^\infty)\ge 0, \qquad \forall x\in R^n_+.$$

Since R^n_+ is closed set, $\tilde{x}^{k_j}\ge 0$ and the subsequence $\{\tilde{x}^{k_j}\}$ converges to x^∞, then $x^\infty\ge 0$ and x^∞ is a solution of the problem (5.1). Note that the inequality (5.51) is true for all solution points of NCP and hence we have

$$\|x^{k+1}-x^\infty\|^2\le\|x^k-x^\infty\|^2, \qquad \forall k\ge 0. \tag{5.53}$$

Since $\tilde{x}^{k_j}\to x^\infty(j\to\infty)$ and $x^k-\tilde{x}^k\to 0$ $(k\to\infty)$, for any $\epsilon>0$, there exists an $l>0$ such that

$$\|\tilde{x}^{k_l}-x^\infty\|<\frac{\epsilon}{2} \quad\text{and}\quad \|x^{k_l}-\tilde{x}^{k_l}\|<\frac{\epsilon}{2}. \tag{5.54}$$

Therefore, for any $k\ge k_l$, it follows from (5.53) and (5.54) that

$$\|x^k-x^\infty\|\le\|x^{k_l}-x^\infty\|\le\|x^{k_l}-\tilde{x}^{k_l}\|+\|\tilde{x}^{k_l}-x^\infty\|<\epsilon.$$

This implies that the sequence $\{x^k\}$ converges to x^∞ which is a solution of NCP. □

The proposition 5.2 motivates us that we can improve the LQP method by choosing a more proper step size α based on finding $\alpha^*_{k_1}$ instead of $\alpha^*_{k_2}$ which is used in [32] and extending the step size by solving the following subproblem:

$$\alpha_k=\max_\alpha\{\alpha^*_{k_1}\le\alpha\le m_2\alpha^*_{k_1}\mid\Psi(\alpha)\ge\sigma\Psi(\alpha^*_{k_1})\}, \tag{5.55}$$

where $\sigma\in(0,1)$ and $m_2\ge 2$.

We now describe the new algorithm as follows.

Algorithm 5.1. *Step 0. Let* $\beta_0>0$, $\varepsilon>0$, $0<\mu<1$, $0<\sigma<1$, $0<a_1<1$, $0<\eta<1$, $m_1\ge 1$, $m_2\ge 2$, $x^0\in R^n_{++}$ *and set* $k:=0$.

Step 1. If $\|\min(x,F(x))\|_\infty\le\epsilon$, *then stop. Otherwise, go to Step 2.*

Step 2. $\tilde{x}^k=P_{R^n_+}[x^k-\frac{\beta_k}{1+\mu}F(x^k)]$, $\quad\xi^k:=\beta_k(F(\tilde{x}^k)-F(x^k))$,

$r:=\|\xi^k\|/\|x^k-\tilde{x}^k\|$.

while $(r>\eta)$

$\beta_k:=\beta_k*0.7*\min(1,1/r)$, $\quad\tilde{x}^k=P_{R^n_+}[x^k-\frac{\beta_k}{1+\mu}F(x^k)]$,

$\xi^k:=\beta_k(F(\tilde{x}^k)-F(x^k))$, $\quad r:=\|\xi^k\|/\|x^k-\tilde{x}^k\|$.

while

Step 3. Searching step size α^*_k :

Let $\bar{\alpha}_k=arg\max_\alpha\{\Phi(\alpha)\mid\alpha>0\}$, *where* $\Phi(\alpha)$ *is defined by (5.29).*

Solve the following optimization problem

$\alpha^*_k=arg\max_\alpha\{\Psi(\alpha)\mid 0<\alpha\le m_1\bar{\alpha}_k\}$, *where* $\Psi(\alpha)$ *is defined by (5.28).*

Step 4. Extending the step size:

$$\alpha_k = \max_{\alpha}\{\alpha_k^* \le \alpha \le m_2\alpha_k^* \mid \Psi(\alpha) \ge \sigma\Psi(\alpha_k^*)\}$$

and

$$x^{k+1} = a_1x^k + (1-a_1)P_{R_+^n}[x^k - \tfrac{\alpha_k\beta_k}{1+\mu}F(\tilde{x}^k)],$$

Step 5. $\beta_{k+1} = \begin{cases} \frac{\beta_k * 0.9}{r}, & if \quad r \le 0.3; \\ \beta_k, & otherwise. \end{cases}$

Step 6. k:=k+1; go to Step 1.

In this section, we consider the nonlinear complementarity problems to illustrate the efficiency of the proposed algorithm.

$$x \ge 0, \qquad F(x) \ge 0, \qquad \langle F(x), x\rangle) = 0, \tag{5.56}$$

where

$$F(x) = D(x) + Mx + q,$$

$D(x)$ and $Mx + q$ are the nonlinear part and linear part of $F(x)$ respectively.

The matrix $M = A^TA + B$ is computed as follows. A is $n \times n$ matrix whose entries are randomly generated in the interval (-5,+5) and the skew-symmetric matrix B is generated in the same way. The components of D(x) are $D_j(x) = d_j * arctan(x_j)$ and d_j is chosen randomly in $(0, 1)$.

In all tests we take the logarithmic proximal parameter $\mu = 0.01, a_1 = 0.01$ and all iterations start with $x^0 = (1, ..., 1)^T$ and $\beta_0 = 1$. The stopping criterion was set to be

$$\frac{\| \min\{x, F(x)\}\|_\infty}{\| \min\{x^0, F(x^0)\}\|_\infty} \le 10^{-7}.$$

All codes are written in Matlab and run on a P4-2.00G note book computer. We test the problems (5.56) with different dimensions and $q \in (-500,\ 500)$ in Table 1, and $q \in (-500,\ 0)$ in Table 2. We compared the proposed method with that of Bnouhachem's method [30]. In all tests we take $\sigma = 0.05, m_1 = 3, m_2 = 4, \eta = 0.9, \gamma = 1.8$. The test results for problems (5.56) are reported in tables **??**-2. k is the number of iterations and l denotes the number of evaluations of mapping F.

TABLE 1. The numerical results for problem (5.56) with $q \in (-500,\ 500)$

	The proposed method		The method in [30]	
n	k	l	k	l
200	179	398	233	508
300	201	446	258	562
400	205	456	264	575
500	228	509	293	640
700	216	482	276	603
1000	208	465	264	577

Table 2. The numerical results for problem (5.56) with $q \in (-500,\ 0)$

	The proposed method		The method in [30]	
n	k	l	k	l
200	358	791	437	950
300	335	740	429	932
400	457	1008	538	1168
500	468	1035	557	1212
700	412	912	491	1069
1000	458	1013	521	1134

The numerical results show that the new method is attractive in practice. Moreover, it demonstrates computationally that the new method is more effective in the sense that the new method needs fewer iteration and less evaluation numbers of F, which clearly illustrate its efficiency and thus justifies the theoretical assertions.

6. Splitting Methods

In this Section, we use the technique of updating the solution to suggest a class of three-step forward-backward projection-splitting methods for solving the variational inequalities (23.3).

Using this technique, we again rewrite the equation (23.3) in the form:

$$\begin{aligned} u &= P_K[P_K[u-\rho Tu]-\rho TP_K[u-\rho Tu]] \\ &= P_K[I-\rho T]P_K[I-\rho T]u \\ &= (I+\rho T)^{-1}\{P_K[I-\rho T]P_K[I-\rho T]+\rho T\}u \end{aligned} \tag{6.1}$$

or

$$y = P_K[u-\rho Tu] \tag{6.2}$$

$$u = P_K[y-\rho Ty] \tag{6.3}$$

Using this fixed-point formulation, one can suggest and analyze the following iterative methods.

Algorithm 6.1. *For a given $u_0 \in H$, calculate the approximate solution u_{n+1} by the iterative schemes*

$$u_{n+1} = P_K[P_K[u_n-\rho Tu_n]-\rho TP_K[u_n-\rho Tu_n]]$$

or

$$\begin{aligned} y_n &= P_K[u_n-\rho Tu_n] \\ u_{n+1} &= P_K[y_n-\rho Ty_n], \quad n=0,1,2,\ldots \end{aligned}$$

or

$$\begin{aligned} u_{n+1} &= P_K[I-\rho T]P_K[I-\rho T]u_n, \\ &= (I+\rho T)^{-1}\{P_K[I-\rho T]P_K[I-\rho T]+\rho T\}u_n, \quad n=0,1,2,\ldots \end{aligned}$$

which are known as the two-step forward-backward splitting methods and are different from the forward-backward splitting methods of Tseng [166]. Using the technique of Tseng [166], one can derive a number of iterative methods for solving mathematical programming problems.

For a positive constant α, one can rewrite the equation (23.3) as:

$$\begin{aligned} u &= P_K[u - \alpha\{\eta(u - P_K[u - \rho Tu]) + \rho T(u - \eta R(u))\}] \\ &= P_K[u - \alpha\{\eta R(u) + \rho T(u - \eta R(u))\}] \\ &= P_K[u - \alpha d_2(u)] \qquad (6.4) \end{aligned}$$

where

$$d_2(u) = \eta R(u) + \rho T(u - \eta R(u)).$$

Note that for $\alpha = 1$ and $\eta = 1$ equation (6.4) is equivalent to equations (6.1). This equivalent formulation is flexible and is used to suggest and analyze the following iterative method for solving the variational inequalities (23.3).

Algorithm 6.2. *For a given $u_0 \in H$, compute u_{n+1} by the following iterative schemes*

Predictor step

$$g(w_n) = (1 - \eta_n)g(u_n) + \eta_n P_K[g(u_n) - \rho_n Tu_n] = g(u_n) - \eta_n R(u_n),$$

where η_n satisfies

$$\eta_n \rho_n \langle Tu_n - Tw_n, R(u_n)\rangle \leq \sigma \|R(u_n)\|^2, \quad \sigma \in (0, 1).$$

Corrector step.

$$g(u_{n+1}) = P_K[g(u_n) - \alpha_n d_2(u_n)], \quad n = 0, 1, 2, \dots$$

where

$$\begin{aligned} d_2(u_n) &= \eta_n R(u_n) + \rho_n Tw_n \\ \alpha_n &= \frac{\eta_n \langle R(u_n), R(u_n) - \rho_n Tu_n + \rho Tw_n\rangle}{\|d_2(u_n)\|^2}, \end{aligned}$$

which is called the self-adaptive projection method. Some variant forms of Algorithm 6.2 have been studied by Han and Lo [60], Wang, et al [167, 168], and Noor [117]. It is interesting to note that for $\eta_n = 1$ and $\alpha_n = 1$, Algorithm 6.2 is exactly Algorithm 6.1, which was suggested by Noor [102, 117]. This shows that projection-type methods studied in [60, 165] are generalizations of the extragradient methods of Noor [102, 117]. One can study the convergence analysis of Algorithm 6.2 using the technique of Noor [117] and Han and Lo [60].

In a similar way, one can rewrite equation (23.3) as:

$$\begin{aligned} u &= P_K[P_K[P_K[u - \rho Tu] - \rho TP_K[u - \rho Tu]] \\ &\quad - \rho TP_K[P_K[u - \rho Tu] - \rho TP_K[u - \rho Tu]]] \\ &= P_K[P_K[y - \rho Ty] - \rho TP_K[y - \rho Ty]] \\ &= P_K[w - \rho Tw] \qquad (6.5) \end{aligned}$$

where

$$y = P_K[u - \rho Tu] \qquad (6.6)$$

$$w = P_K[y - \rho Ty] \qquad (6.7)$$

We use the fixed-point formulation (6.5) to suggest the following three-step forward-backward splitting method:

Algorithm 6.3. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\begin{aligned} y_n &= P_K[u_n - \rho T u_n] \\ w_n &= P_K[y_n - \rho T y_n] \\ u_{n+1} &= P_K[u_n - \rho T u_n], \quad n = 0, 1, 2, \ldots \end{aligned}$$

Algorithm 6.3 is also known as the predictor-corrector method which has been studied by Noor [102, 117] for problems of the type (23.3).

We now write the equation (6.5) in the form:

$$\begin{aligned} u &= P_K[I - \rho T]P_K[I - \rho T]P_K[I - \rho T]u \\ &= (I + \rho T)^{-1}\{P_K[I - \rho T]P_K[I - \rho T]P_K[I - \rho T] + \rho T\}u. \end{aligned}$$

This fixed-point formulation can be used to suggest and analyze the following iterative method.

Algorithm 6.4. *For a given $u_0 \in H$, compute the approximate solution by the iterative schemes:*

$$\begin{aligned} u_{n+1} &= P_K[I - \rho T]P_K[I - \rho T]P_K[I - \rho T]u_n \\ &= (I + \rho T)^{-1}\{P_K[I - \rho T]P_K[I - \rho T]P_K[I - \rho T] \\ &\quad + \rho T\}u_n, \quad n = 0, 1, 2, \ldots \end{aligned}$$

Algorithm 6.4 is a three-step forward-backward projection splitting methods, which is different from the splitting method of Glowinski and Le Tallec [58] and can be viewed as a generalization of the modified forward-backward splitting method of Tseng [166]. Using the technique of Tseng [166], one can develop various splitting-type methods for solving optimization and mathematical programming problems. For the convergence analysis and applications of three-step splitting methods in variational inequalities and mathematical programming problems, see [58, 102, 117, 137, 138, 142, 159, 166–168, 171, 181].

We now consider a self-adaptive projection-splitting method using the fixed-point formulation (6.1). For this purpose, we define the modified residue vector $R_1(u)$ by

$$\begin{aligned} R_1(u) &= u - w = u - P_K[y - \rho T y] \\ &= u - P_K[P_K[u - \rho T u] - \rho T P_K[u - \rho T u]] \end{aligned}$$

From Lemma 3.1, it follows that $u \in H$ is a solution of (23.3), if and only if, $u \in H$ is a zero of the equation

$$R_1(u) = 0.$$

Since K is a convex set, for all $\eta \in [0, 1]$, $\quad u, P_K[y - \rho T y] \in K$,

$$x = (1 - \eta)u + \eta P_K[y - \rho T y] = u - R_1(u) \in K.$$

Based on the above discussions and observations, we can rewrite equation (23.3) in the form:

$$u = P_K[u - \alpha d_3(u)],$$

where

$$\begin{aligned} d_3(u) &= \eta R_1(u) + \rho T x \\ &= \eta R_1(u) + \rho T(u - \eta R_1(u)) \end{aligned}$$

and α is a positive constant. This fixed-point has been used to suggest and analyze the following iterative method.

Algorithm 6.5. *For a given $u_0 \in H$, compute u_{n+1} by the iterative schemes:*
Predictor step.

$$\begin{aligned} y_n &= P_K[u_n - \rho_n T u_n] \\ w_n &= P_K[y_n - \rho_n T y_n] \\ x_n &= u_n - \eta_n R_1(u_n), \end{aligned}$$

where η_n satisfies

$$\eta_n \rho_n \langle T u_n - T(u_n - \eta_n R_1(u_n)), R_1(u_n)\rangle \leq \sigma \|R_1(u_n)\|^2, \quad \sigma \in (0,1).$$

Corrector step.

$$u_{n+1} = P_K[u_n - \alpha_n d_3(u_n)], \quad n = 0, 1, 2, \dots$$

$$\begin{aligned} d_3(u_n) &= \eta_n R_1(u_n) + \rho_n T x_n \\ \alpha_n &= \frac{\eta_n \langle R_1(u_n), D_1(u_n)\rangle}{\|d_3(u_n)\|^2} \\ D_1(u_n) &= R_1(u_n) - \rho_n T u_n + \rho T x_n. \end{aligned}$$

Here α_n is called the corrector step size which depends upon the modified Wiener-Hopf equation. For $\alpha_n = 1$ and $\eta_n = 1$, Algorithm 5.5 coincides with the projection-splitting Algorithm 6.3 and Algorithm 6.4. Note that Algorithm 6.5 is quite different from the Algorithm 6.2 and other methods. For the convergence analysis of Algorithm 6.5, see Noor [117], where it has been shown that the convergence of Algorithm 6.5 requires only pseudomonotonicity. Using essentially the technique of updating the solution, one can develop several one-step, two-step, three-step and four-step forward-backward projection splitting methods for variational inequalities.

Noor [102,117] used the technique of updating the solution to suggest and analyze several two-step and three-step iterative methods for solving the various classes of variational inequalities. To convey an idea of this technique, we include the necessary details. To do so, we can rewrite the relation (2.5) in the following form.

$$\begin{aligned} w &= P_K[u - \rho T u] \\ y &= P_K[w - \rho T w] \\ u &= P_K[u - \rho T y], \end{aligned}$$

which is another fixed point formulation of the variational inequality (23.3). This fixed point formulation can be used to suggest and analyze the following three-step iterative method for solving the variational inequalities (23.3).

Algorithm 6.6. *For a given $u_0 \in K$, find the approximate solution u_{n+1} by the iterative schemes*

$$\begin{aligned} u_{n+1} &= P_K[u_n - \rho T y_n] \\ y_n &= P_K[w_n - \rho T w_n] \\ w_n &= P_K[u_n - \rho T u_n], \quad n = 0, 1, 2, \dots . \end{aligned}$$

We would like to remark that Algorithm 6.6 is new one and is remarkably different from the ones, which were suggested and analyzed for solving the variational inequalities. It is an open problem to consider the convergence of Algorithm 6.6. Comparison of Algorithm 6.6 with other two-step and three-step methods needed further research efforts.

From Lemma 3.1, it is clear that u is solution of the variational inequality (23.3), if and only if, u is a zero point of the function

$$r(u, \rho) = u - P_K[u - \rho T(u)].$$

Lemma 6.1. *[10].* $\forall u \in H$ *and* $\rho' \geq \rho > 0$, *it holds that*

$$\|r(u, \rho')\| \geq \|r(u, \rho)\| \tag{6.8}$$

and

$$\frac{\|r(u, \rho')\|}{\rho'} \leq \frac{\|r(u, \rho)\|}{\rho}. \tag{6.9}$$

Throughout this paper, we make following assumptions.

Assumptions:

- H is a finite dimension space.
- T is continuous and pseudomonotone operator on H i.e.,

$$\langle T(u), u' - u\rangle \geq 0 \Rightarrow \langle T(u'), u' - u\rangle \geq 0 \qquad \forall u', u \in H.$$

- The solution set of problem (23.3) denoted by Ω^* is nonempty.

In this section, inspired and motivated by Algorithm 6.6, we suggest and analyze a self-adaptive method for solving the variational inequalities (23.3).

For given $u^k \in H$ and $\rho_k > 0$, each iteration of the proposed method consists of three steps, the first step offers $\tilde{u}^k$, the second step makes $\bar{u}^k$ and the third step produces the new iterate u^{k+1}.

Algorithm 6.7. Step 1. *Given $u^0 \in H$, $\epsilon > 0$, $\rho_0 = 1$, $\nu > 1$, $\mu \in (0, \sqrt{2})$, $\gamma \in (0, 2)$, $\tau \in (0, 1)$, $\eta_1 \in (0, \tau)$, $\eta_2 \in (\tau, \nu)$ and let $k = 0$.*

Step 2. *If $\|r(u^k, 1)\| \leq \epsilon$, then stop. Otherwise, go to Step 3.*

Step 3. 1) *For a given $u^k \in H$, calculate the two predictors*

$$\tilde{u}^k = P_K[u^k - \rho_k T(u^k)], \tag{6.10a}$$

$$\bar{u}^k = P_K[\tilde{u}^k - \rho_k T(\tilde{u}^k)]. \tag{6.10b}$$

2) *If $\|r(\tilde{u}^k, 1)\| \leq \epsilon$, then stop . Otherwise, continue.*

3) *If ρ_k satisfies both*

$$r_1 := \frac{\|\rho_k[\langle \tilde{u}^k - \bar{u}^k, T(u^k) - T(\tilde{u}^k)\rangle - \langle u^k - \bar{u}^k, T(\tilde{u}^k) - T(\bar{u}^k)\rangle]\|}{\|\tilde{u}^k - \bar{u}^k\|^2} \leq \mu^2 \tag{6.11}$$

and

$$r_2 := \frac{\|\rho_k(T(\tilde{u}^k) - T(\bar{u}^k))\|}{\|\tilde{u}^k - \bar{u}^k\|} \leq \nu, \tag{6.12}$$

then go to Step 4; otherwise, continue.

4) *Perform an Armijo-like line search via reducing ρ_k*

$$\rho_k := \rho_k * \frac{0.8}{\max(r_1, 1)} \tag{6.13}$$

and go to Step 3.

Step 4. *Take the new iteration u^{k+1}, by setting*

$$u^{k+1}(\alpha_k) = P_K[u^k - \alpha_k d(\tilde{u}^k, \bar{u}^k))], \tag{6.14}$$

where

$$\alpha_k = \frac{\langle u^k - \bar{u}^k, d(\tilde{u}^k, \bar{u}^k)\rangle}{\|d(\tilde{u}^k, \bar{u}^k)\|^2} \tag{6.15}$$

and

$$d(\tilde{u}^k, \bar{u}^k) := (\tilde{u}^k - \bar{u}^k) - \rho_k(T(\tilde{u}^k) - T(\bar{u}^k)). \tag{6.16}$$

Step 5. *Adaptive rule of choosing a suitable ρ_{k+1} as the start prediction step size for the next iteration*

1) *Prepare a proper ρ_{k+1},*

$$\rho_{k+1} := \begin{cases} \rho_k * \tau/r_2 & \text{if } r_2 \leq \eta_1, \\ \rho_k * \tau/r_2 & \text{if } r_2 \geq \eta_2, \\ \rho_k & \text{otherwise.} \end{cases} \tag{6.17}$$

2) *Return to Step 2, with k replaced by $k+1$.*

We show that Algorithm 6.7 is well-defined. To see this, we need to show that the Armijo-like line search procedure is well defined.

Lemma 6.2. *In the kth iteration, if $\|r(u^k, 1)\| \geq \epsilon$, then the Armijo-like line search procedure with criteria (6.11) and (6.12) is finite.*

Proof. Assume for contradiction that ρ_k does not satisfy criterion (6.11) or (6.12) in finite Armijo-like line search procedure. Consequently, $\rho_k \to 0(see(6.13))$. Without losing generality, we can assume $\rho_k < 1$. Let us consider two possible cases.

Case 1. Criterion (6.11) fails to be satisfied. It follows that

$$\mu^2\|\tilde{u}^k - \bar{u}^k\|^2 < \|\rho_k[\langle \tilde{u}^k - \bar{u}^k, T(u^k) - T(\tilde{u}^k)\rangle - \langle u^k - \bar{u}^k, T(\tilde{u}^k) - T(\bar{u}^k)\rangle]\|.$$

This implies that either

$$\frac{1}{2}\mu^2\|\tilde{u}^k - \bar{u}^k\|^2 < \|\rho_k\langle \tilde{u}^k - \bar{u}^k, T(u^k) - T(\tilde{u}^k)\rangle\| \tag{6.18}$$

or

$$\frac{1}{2}\mu^2\|\tilde{u}^k - \bar{u}^k\|^2 < \|\rho_k\langle u^k - \bar{u}^k, T(\tilde{u}^k) - T(\bar{u}^k)\rangle\| \tag{6.19}$$

holds.

If (6.18) holds, by using the Cauchy-Schwarz inequality and dividing both sides of (6.18) by ρ_k, we have that

$$\frac{\mu^2\|\tilde{u}^k - \bar{u}^k\|}{2\rho_k} < \|T(u^k) - T(\tilde{u}^k)\|. \tag{6.20}$$

Note that

$$\|\tilde{u}^k - \bar{u}^k\| = \|\tilde{u}^k - J_\varphi[\tilde{u}^k - \rho_k T(\tilde{u}^k)]\| = \|r(\tilde{u}^k, \rho_k)\|, \tag{6.21}$$

substituting above equality into (6.20) and using inequality (6.9), we find that

$$\frac{1}{2}\mu^2\|r(\tilde{u}^k, 1)\| \le \frac{\mu^2\|r(\tilde{u}^k, \rho_k)\|}{2\rho_k} < \|T(\tilde{u}^k) - T(\bar{u}^k)\|. \tag{6.22}$$

It is easy to see that $\tilde{u}^k \to u^k, \bar{u}^k \to u^k$ (since $\rho_k \to 0$). Consequently, $T(\tilde{u}^k) \to T(u^k), T(\bar{u}^k) \to T(u^k)$ and $r(\tilde{u}^k, 1) \to r(u^k, 1)$ due to continuity of $T(u)$ and $r(u, 1)$, respectively. When we take $\rho_k \to 0$ in (6.22), we get $\|r(u^k, 1)\| \le 0$. But this contradicts the assertion that $\epsilon \le \|r(u^k, 1)\|$.

Let us turn to deal with (6.19). Since u^k is bounded, then we have $\|T(u^k)\| \le M$. Note that

$$\|u^k - \tilde{u}^k\| = \|u^k - P_K[u^k - \rho_k T(u^k)]\| \le \|\rho_k T(u^k)\| \le \rho_k M,$$

then we have

$$\|u^k - \bar{u}^k\| \le \|u^k - \tilde{u}^k\| + \|\tilde{u}^k - \bar{u}^k\| \le \|\rho_k T(u^k)\| + \|\tilde{u}^k - \bar{u}^k\| \le \rho_k M + \|\tilde{u}^k - \bar{u}^k\|. \tag{6.23}$$

In (6.19), using Cauchy-Schwarz inequality and (6.23), we get immediately,

$$\frac{1}{2}\mu^2\|\tilde{u}^k - \bar{u}^k\|^2 < \rho_k\|u^k - \bar{u}^k\|\|T(\tilde{u}^k) - T(\bar{u}^k)\| \le \rho_k(\rho_k M + \|\tilde{u}^k - \bar{u}^k\|)\|T(\tilde{u}^k) - T(\bar{u}^k)\|. \tag{6.24}$$

Dividing both sides of (6.24) by ρ_k^2, using the equality (6.21) and inequality (6.9) again, we obtain

$$\frac{1}{2}\mu^2\|r(\tilde{u}^k, 1)\|^2 \le \frac{\mu^2\|r(\tilde{u}^k, \rho_k)\|^2}{2\rho_k^2} < (M + \frac{\|r(\tilde{u}^k, \rho_k)\|}{\rho_k})\|T(\tilde{u}^k) - T(\bar{u}^k)\|. \tag{6.25}$$

By taking $\rho_k \to 0$ in above inequality, we obtain $\|r(u^k, 1)\| \le 0$. Therefore, $\|r(u^k, 1)\| = 0$, contradicting that u^k is not a solution.

Case 2. Condition (6.12) is violated. Then we must have

$$\nu\|\tilde{u}^k - \bar{u}^k\| < \|\rho_k(T(\tilde{u}^k) - T(\bar{u}^k))\|. \tag{6.26}$$

The proof is quite similar to the Case 1. Dividing both sides of (6.26) by ρ_k and taking $\rho_k \to 0$, we get the contradiction.From the above observations, we assert that our proposed algorithm is well-defined. □

Lemma 6.3. *Let $u^* \in K$ be a solution of the variational inequality (2.1). For given $u^k \in H$, let $\tilde{u}^k, \bar{u}^k$ be the predictors produced by (6.10a) and (6.10b), then we have*

$$\langle u^k - \bar{u}^k, d(\tilde{u}^k, \bar{u}^k)\rangle \ge (2 - \mu^2)\|\tilde{u}^k - \bar{u}^k\|^2. \tag{6.27}$$

Proof. Note that $\tilde{u}^k = P_K[u^k - \rho_k T(u^k)]$, $\bar{u}^k = P_K[\tilde{u}^k - \rho_k T(\tilde{u}^k)]$, we can apply (2.4) with $v = u^k - \rho_k T(u^k)$, $w = \tilde{u}^k - \rho_k T(\tilde{u}^k)$ to obtain

$$\langle u^k - \rho_k T(u^k) - (\tilde{u}^k - \rho_k T(\tilde{u}^k)), \tilde{u}^k - \bar{u}^k \rangle \geq \|\tilde{u}^k - \bar{u}^k\|^2.$$

By some manipulations, we have

$$\langle u^k - \tilde{u}^k, \tilde{u}^k - \bar{u}^k \rangle \geq \|\tilde{u}^k - \bar{u}^k\|^2 + \rho_k \langle \tilde{u}^k - \bar{u}^k, T(u^k) - T(\tilde{u}^k)\rangle.$$

Then, we obtain

$$\begin{aligned}\langle u^k - \tilde{u}^k, d(\tilde{u}^k, \bar{u}^k)\rangle &= \langle u^k - \tilde{u}^k, \tilde{u}^k - \bar{u}^k\rangle - \rho_k \langle u^k - \tilde{u}^k, T(\tilde{u}^k) - T(\bar{u}^k)\rangle \\ &\geq \|\tilde{u}^k - \bar{u}^k\|^2 + \rho_k \langle \tilde{u}^k - \bar{u}^k, T(u^k) - T(\tilde{u}^k)\rangle \\ &\quad -\rho_k \langle u^k - \tilde{u}^k, T(\tilde{u}^k) - T(\bar{u}^k)\rangle. \end{aligned} \tag{6.28}$$

Using (6.28), (6.11) and the definition of $d(\tilde{u}^k, \bar{u}^k)$, we get

$$\begin{aligned}\langle u^k - \bar{u}^k, d(\tilde{u}^k, \bar{u}^k)\rangle &= \langle u^k - \tilde{u}^k, d(\tilde{u}^k, \bar{u}^k)\rangle + \langle \tilde{u}^k - \bar{u}^k, d(\tilde{u}^k, \bar{u}^k)\rangle \\ &\geq \|\tilde{u}^k - \bar{u}^k\|^2 + \rho_k \langle \tilde{u}^k - \bar{u}^k, T(u^k) - T(\tilde{u}^k)\rangle - \rho_k \langle u^k - \tilde{u}^k, T(\tilde{u}^k) - T(\bar{u}^k)\rangle \\ &\quad +\|\tilde{u}^k - \bar{u}^k\|^2 - \rho_k \langle \tilde{u}^k - \bar{u}^k, T(\tilde{u}^k) - T(\bar{u}^k)\rangle \\ &\geq (2 - \mu^2)\|\tilde{u}^k - \bar{u}^k\|^2. \end{aligned}$$

Hence, (6.27) holds and the proof is completed. □

We consider the convergence criteria of the proposed method. The following theorem plays a crucial role in the convergence of the proposed

Theorem 6.1. *Let u^* be a solution of the variational inequality (23.3) and let $u^{k+1} = u^{k+1}(\gamma\alpha)$ be the sequence obtained from algorithm 3.1. Then u^k is bounded and*

$$\|u^{k+1} - u^*\|^2 \leq \|u^k - u^*\|^2 - \frac{\gamma(2-\gamma)(2-\mu^2)^2}{(1+\nu)^2}\|\tilde{u}^k - \bar{u}^k\|^2. \tag{6.29}$$

Proof. For any $u^* \in \Omega^*$ solution of the variational inequality (23.3), we have

$$\langle \rho_k T(u^*), \bar{u}^k - u^*\rangle \geq 0.$$

Using the pseudomonotonicity of T, we obtain

$$\langle \rho_k T(\bar{u}^k), \bar{u}^k - u^*\rangle \geq 0 \tag{6.30}$$

Substituting $w = \tilde{u}^k - \rho_k T(\tilde{u}^k)$ and $v = u^*$ into (**??**), we get

$$\langle \tilde{u}^k - \rho_k T(\tilde{u}^k) - \bar{u}^k, \bar{u}^k - u^*\rangle + \rho_k \varphi(u^*) - \rho_k \varphi(\bar{u}^k) \geq 0. \tag{6.31}$$

Adding (9.25) and (9.26), and using the definition of $d(\tilde{u}^k, \bar{u}^k)$, we have

$$\langle d(\tilde{u}^k, \bar{u}^k), \bar{u}^k - u^*\rangle \geq 0. \tag{6.32}$$

Since $u^* \in K$ be a solution of the variational inequality (23.3), then

$$\begin{aligned}\|u^{k+1} - u^*\|^2 &\leq \|u^k - u^* - \gamma\alpha_k d(\tilde{u}^k, \bar{u}^k)\|^2 \\ &= \|u^k - u^*\|^2 - 2\gamma\alpha_k \langle u^k - u^*, d(\tilde{u}^k, \bar{u}^k)\rangle + \gamma^2\alpha_k^2 \|d(\tilde{u}^k, \bar{u}^k)\|^2 \end{aligned} \tag{6.33}$$

Adding (6.32) (multiplied by $2\gamma\alpha_k$) to (9.31) and using (6.15)

$$\begin{aligned}\|u^{k+1}-u^*\|^2 &\le \|u^k-u^*\|^2 - 2\gamma\alpha_k\langle u^k-\bar{u}^k, d(\tilde{u}^k,\bar{u}^k)\rangle + \gamma^2\alpha_k^2\|d(\tilde{u}^k,\bar{u}^k)\|^2 \\ &= \|u^k-u^*\|^2 - \gamma(2-\gamma)\alpha_k\langle u^k-\bar{u}^k, d(\tilde{u}^k,\bar{u}^k)\rangle \\ &\le \|u^k-u^*\|^2 - \gamma(2-\gamma)\alpha_k(2-\mu^2)\|\tilde{u}^k-\bar{u}^k\|^2 \end{aligned} \tag{6.34}$$

where the last inequality follows from (6.27)

Recalling the definition of $d(\tilde{u}^k,\bar{u}^k)$ (see (6.16)) and applying Criterion (6.12), it is easy to see that

$$\|d(\tilde{u}^k,\bar{u}^k)\|^2 \le (\|\tilde{u}^k-\bar{u}^k\| + \|\rho_k(T(\tilde{u}^k)-T(\bar{u}^k))\|)^2 \le (1+\nu)^2\|\tilde{u}^k-\bar{u}^k\|^2. \tag{6.35}$$

Moreover, by using (6.27) together with (6.35), we get

$$\alpha_k = \frac{\langle u^k-\bar{u}^k, d(\tilde{u}^k,\bar{u}^k)\rangle}{\|d(\tilde{u}^k,\bar{u}^k)\|^2} \ge \frac{2-\mu^2}{(1+\nu)^2} > 0, \qquad \mu\in(0,\sqrt{2}). \tag{6.36}$$

Substituting (6.36) in (6.34), we get the assertion of this theorem. Since $\gamma\in[1,2)$ and $\mu\in(0,\sqrt{2})$ we have

$$\|u^{k+1}-u^*\| \le \|u^k-u^*\| \le \ldots \le \|u^0-u^*\|.$$

Then the sequence u^k is bounded. □

We now present the convergence result of the proposed method.

Theorem 6.2. *If* $\inf_{k=0}^{\infty}\rho_k := \rho > 0$*, then any cluster point of the sequence* $\{\tilde{u}^k\}$ *generated by the proposed method is a solution of problem (23.3).*

Proof. It follows from (6.29) that

$$\lim_{k\to\infty}\|\tilde{u}^k-\bar{u}^k\| = 0.$$

Since the sequence u^k is bounded, $\{\tilde{u}^k\}$ is also bounded, it has at least a cluster point. Let u^∞ be a cluster point of $\{\tilde{u}^k\}$ and the subsequence $\{\tilde{u}^{k_j}\}$ converges to u^∞. Using the continuity of $r(u,\rho)$ and inequality (6.8), it follows that

$$\|r(u^\infty,\rho)\| = \lim_{k_j\to\infty}\|r(\tilde{u}^{k_j},\rho)\| \le \lim_{k_j\to\infty}\|r(\tilde{u}^{k_j},\rho_{k_j})\| = \lim_{k_j\to\infty}\|\tilde{u}^{k_j}-\bar{u}^{k_j}\| = 0.$$

This means that u^∞ is a solution of problem (2.1). □

To illustrate the implementation of the proposed Algorithm 6.6, we consider the nonlinear complementarity problems:

$$u\ge 0, \qquad T(u)\ge 0, \qquad \langle T(u),u\rangle = 0, \tag{6.37}$$

where

$$T(u) = D(u) + Mu + q,$$

$D(u)$ and $Mu+q$ are the nonlinear part and linear part of $T(u)$ respectively. We form the linear part in the test problems as

$$M = \begin{pmatrix} 1 & 2 & 2 & \cdots & 2 \\ 0 & 1 & 2 & \cdots & 2 \\ 0 & 0 & 1 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}_{n\times n} \qquad \text{and} \qquad q = \begin{pmatrix} -1 \\ -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix}_{n\times 1}.$$

Table 6.1 Numerical results for problem (6.37)

Dimension of the problem	Method [10]			Method [11]			New method		
	k	l	CPU(Sec.)	k	l	CPU(Sec.)	k	l	CPU(Sec.)
n=100	281	572	0.07	228	465	0.04	144	467	0.05
n=200	315	637	0.19	242	496	0.17	156	487	0.2
n=500	508	1090	4.77	531	1109	4.97	228	789	3.59
n=600	569	1243	7.59	520	1160	7.30	256	1023	6.18
n=800	657	1427	14.57	568	1236	13.11	297	1194	9.83
n=900	788	1621	21.22	635	1321	17.40	375	1178	15.58
n=1000	735	1567	21.57	574	1285	17.02	303	1000	13.14

In $D(u)$, the nonlinear part of $T(u)$, the components are chosen to be $D_j(u) = d_j * \arctan(u_j)$, where d_j is a random variable in $(0, 1)$.

In all tests we take $\rho_0 = 1, \tau = 0.7, \eta_1 = 0.2, \eta_2 = 0.95, \mu = 0.95, \nu = 1.95, \gamma = 1.9$. We employ $\|r(u, 1)\| \leq 10^{-7}$ as the stopping criterion and choose $u^0 = 0$ as the initial iterative points. All codes were written in Matlab, we compare the proposed method with those in [10] and [11]. The test results for problems (6.37) with different dimensions are reported in Table 5.1. k is the number of iterations and l denotes the number of evaluations of mapping T.

In all tests we take $\rho_0 = 1, \tau = 0.7, \eta_1 = 0.2, \eta_2 = 0.95, \mu = 0.95, \nu = 1.95, \gamma = 1.9$. We employ $\|r(u, 1)\| \leq 10^{-7}$ as the stopping criterion and choose $u^0 = 0$ as the initial iterative points. All codes were written in Matlab, we compare the proposed method with those in [10] and [11]. The test results for problems (6.37) with different dimensions are reported in Table 6.1. k is the number of iterations and l denotes the number of evaluations of mapping T. Table 5.1 shows that the proposed method is very efficient algorithm even for large-scale classical nonlinear complementarity problems. Moreover, it demonstrates computationally that the new method is more effective than the methods presented in [10] and [11] in the sense that the new method needs fewer iteration and less evaluation numbers of T, which clearly illustrate its efficiency..

7. Auxiliary Principle Technique

In the previous sections, we have considered and analyzed several projection-type methods for solving variational inequalities. It is well known that to implement such type of the methods, one has to evaluate the projection, which is itself a difficult problems. Secondly, one can't extend the technique of projection for solving some classes of variational inequalities. These facts motivated to consider other methods. One of these techniques is known as the auxiliary principle. This technique is basically due to Lions and Stampacchia [56]. See also Noor [71,73]. Glowinski, Lions and Tremolieres [38] used this technique to study the existence of a solution of mixed variational inequalities. Noor [103,104] has used this technique to develop some predictor-corrector methods for solving variational inequalities. It can be shown [15,29,85,92,94,104,164] that various classes of methods including projection, Wiener-Hopf, decomposition and descent can be obtained from this technique as special cases.

For a given $u \in K$ satisfying (23.3), consider the problem of finding a unique $w \in K$ such that

$$\langle \rho Tu + w - u, v - w \rangle \geq 0, \quad \forall v \in K, \tag{7.1}$$

where $\rho > 0$ is a constant.

Note that, if $w = u$, then w is clearly a solution of the variational inequality (23.3). This simple observation enables us to suggest and analyze the following predictor-corrector method.

Algorithm 7.1. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\langle \mu T u_n + y_n - u_n, v - y_n \rangle \geq 0, \quad \text{for all} \quad v \in K, \tag{7.2}$$

$$\langle \beta T y_n + w_n - y_n, v - w_n \rangle \geq 0, \quad \text{for all} \quad v \in K, \tag{7.3}$$

$$\langle \rho T w_n + u_{n+1} - w_n, v - u_{n+1} \rangle \geq 0, \quad \text{for all} \quad v \in K, \tag{7.4}$$

where $\rho > 0, \beta > 0$ and $\mu > 0$ are constants

Algorithm 7.1 can be considered as a three-step predictor-corrector method, which was suggested and studied by Noor [104]. If $\mu = 0$, then Algorithm 7.1 reduces to:

Algorithm 7.2. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes;*

$$\langle \beta T u_n + w_n - u_n, v - w_n \rangle \geq 0, \quad \text{for all} \quad v \in K,$$
$$\langle \rho T w_n + u_{n+1} - w_n, v - u_{n+1} \rangle \geq 0, \quad \text{for all} \quad v \in K$$

which is known as the two-step predictor-corrector method, see [104].

If $\mu = 0, \beta = 0$, then Algorithm 7.1 becomes:

Algorithm 7.3. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$\langle \rho T u_n + u_{n+1} - u_n, v - u_{n+1} \rangle \geq 0, \quad \text{for all} \quad v \in K.$$

Using the projection technique, Algorithm 7.1 can be written as

Algorithm 7.4. *For a given $u_0 \in H$, compute u_{n+1} by the iterative schemes*

$$\begin{aligned} y_n &= P_K[u_n - \mu T u_n] \\ w_n &= P_K[y_n - \beta T y_n] \\ u_{n+1} &= P_K[w_n - \rho T w_n], \quad n = 0, 1, 2, \ldots \end{aligned}$$

or

$$u_{n+1} = P_K[I - \mu T] P_K[I - \beta T] P_K[I - \rho T] u_n, \quad n = 0, 1, 2, \ldots$$

or

$$\begin{aligned} u_{n+1} &= (I + \rho T)^{-1} \{ P_K[I - \rho T] P_K[I - \rho T] P_K[I - \rho T] \\ &\quad + \rho T \} u_n, \quad n = 0, 1, 2, \ldots, \end{aligned}$$

which is three-step forward-backward method and coincides with Algorithm 6.3 and Algorithm 6.4 for $\rho = \beta = \mu$. Algorithm 7.1 is compatible with three-step splitting method of Glowinski and Le Tallec [39] and also can be considered as a generalization of a two-step forward-backward splitting method of Tseng [146]. In a similar way, one can show that Algorithm 7.2 and Algorithm 7.3 are exactly Algorithm 6.1 and Algorithm 23.1.

For the analysis of Algorithm 7.1, we need the following concepts.

Lemma 7.1. *For all $u, v \in H$, we have*

$$\begin{aligned} 2\langle u, v\rangle &= \|u+v\|^2 - \|u\|^2 - \|v\|^2 \qquad (7.5)\\ \{\frac{-1}{4}\}\|v\|^2 &\leq \langle u, v\rangle + \|u\|^2. \end{aligned}$$

We now study the convergence criteria of Algorithm 7.1. This result is due to Noor [104]. We include their proofs for the sake of completeness and to convey the idea of technique involved.

Theorem 7.1. *Let $\bar{u} \in K$ be a solution of (23.3) and $T : H \longrightarrow H$ be a partially relaxed strongly monotone operator. If u_{n+1} is the approximate solution obtained from Algorithm 7.1, then*

$$\|u_{n+1} - \bar{u}\|^2 \leq \|w_n - \bar{u}\|^2 - (1 - 2\alpha\rho)\|u_{n+1} - w_n\|^2 \qquad (7.6)$$

$$\|w_n - \bar{u}\|^2 \leq \|u_n - \bar{u}\|^2 - (1 - 2\alpha\beta)\|w_n - y_n\|^2 \qquad (7.7)$$

$$\|y_n - \bar{u}\|^2 \leq \|u_n - \bar{u}\|^2 - (1 - 2\alpha\mu)\|y_n - u_n\|^2 \qquad (7.8)$$

Proof. Let $\bar{u} \in K$ be a solution of (23.3). Then

$$\rho\langle T\bar{u}, v - \bar{u}\rangle \geq 0, \quad \text{for all} \quad v \in K \qquad (7.9)$$

$$\beta\langle T\bar{u}, v - \bar{u}\rangle \geq 0, \quad \text{for all} \quad v \in K \qquad (7.10)$$

$$\mu\langle T\bar{u}, v - \bar{u}\rangle \geq 0, \quad \text{for all} \quad v \in K. \qquad (7.11)$$

Taking $v = \bar{u}$ in (7.2), $v = u_{n+1}$ in (7.9) and adding the resultant, we have

$$\begin{aligned} \langle u_{n+1} - w_n, \bar{u} - u_{n+1}\rangle\rangle &\geq \rho\langle Tw_n - T\bar{u}, u_{n+1} - \bar{u}\rangle \\ &\geq -\rho\alpha\|u_{n+1} - w_n\|^2, \qquad (7.12) \end{aligned}$$

since T is partially relaxed strongly monotone with constant α.

Taking $v = \bar{u} - u_{n+1}$ and $u = u_{n+1} - w_n$ in (7.5), we have

$$\begin{aligned} 2\langle u_{n+1} - w_n, \bar{u} - u_{n+1} &= \|w_n - \bar{u}\|^2 - \|\bar{u} - u_{n+1}\|^2 \\ &\quad -\|u_{n+1} - w_n\|^2. \qquad (7.13) \end{aligned}$$

From (7.12) and (7.13), we have

$$\|u_{n+1}g - \bar{u}\|^2 \leq \|w_n - \bar{u}\|^2 - (1 - 2\rho\alpha)\|u_{n+1} - w_n\|^2,$$

the required (7.6).

Now taking $v = \bar{u}$ in (7.3), $v = y_n$ in (7.10), adding the resultant and using the partially relaxed strongly monotonicity of T, we have

$$\langle w_n - y_n, \bar{u} - w_n \geq -\beta\rho\|y_n - w_n\|^2,$$

which implies , Using Lemma 7.1,

$$\|w_n - \bar{u}\|^2 \leq \|y_n - \bar{u}\|^2 - (1 - 2\alpha\beta)\|y_n - w_n\|^2,$$

the required (7.7).

In a similar way, by taking $v = \bar{u}$ in (7.2), $v = y_n$ in (7.11), adding the resultant, using the partially relaxed strongly monotonicity and invoking Lemma 7.1, we obtain

$$\|y_n - \bar{u}\|^2 \leq \|u_n - \bar{u}\|^2 - (1 - 2\alpha\mu)\|y_n - u_n)\|^2,$$

the required (7.8). □

Theorem 7.2. *Let $\bar{u} \in K$ be a solution of (23.3) and u_{n+1} be the approximate solution obtained from Algorithm 7.1. If H is a finite dimensional space and $0 < \rho < 1/2\alpha, 0 < \beta < 1/2\alpha, 0 < \mu < 1/2\alpha$, then $\lim_{n\to\infty} u_n = \bar{u}$.*

Proof. Let $\bar{u} \in K$ be a solution of (23.3). Then, from (7.6), (7.7) and (7.8), it follows that the sequences $\{\|u_n - \bar{u}\|\}$, $\{\|y_n - \bar{u}\|\}$ and $\{\|w_n - \bar{u}\|\}$ are nonincreasing and consequently the sequences $\{w_n\}, \{y_n\}$ and $\{u_n\}$ are bounded and

$$\begin{aligned}
\sum_{n=0}^{\infty}(1 - 2\alpha\rho)\|u_{n+1} - u_n\|^2 &\leq \|w_0 - \bar{u}\|^2 \\
\sum_{n=0}^{\infty}(1 - 2\alpha\beta)\|w_n - u_n\|^2 &\leq \|y_0 - \bar{u}\|^2 \\
\sum_{n=0}^{\infty}(1 - 2\alpha\mu)\|y_n - u_n\|^2 &\leq \|u_0 - \bar{u}\|^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
\lim_{n\to\infty} \|u_{n+1} - w_n\| &= 0 \\
\lim_{n\to\infty} \|w_n - y_n\| &= 0 \\
\lim_{n\to\infty} \|y_n - u_n\| &= 0.
\end{aligned}$$

Thus

$$\begin{aligned}
\lim_{n\longrightarrow\infty} \|u_{n+1} - u_n\| &= \lim_{n\longrightarrow\infty} \|u_{n+1} - w_n\| + \lim_{n\longrightarrow\infty} \|w_n - y_n\| \\
&\quad + \lim_{n\longrightarrow\infty} \|y_n - u_n\| = 0. \qquad (7.14)
\end{aligned}$$

Let $\hat{u}$ be a cluster point of $\{u_n\}$; there exists a subsequence $\{u_{n_i}\}$ such that $\{u_{n_i}\}$ converges to $\hat{u}$. Replacing w_n and y_n by u_{n_i} in (7.2), (7.3) and (7.4); and taking the limits and using (7.14), we have

$$\langle T\hat{u}, v - \hat{u}\rangle \geq 0, \quad \forall v \in K.$$

This shows that $\hat{u} \in H$ solves the variational inequality problem (23.3) and

$$\|u_{n+1} - \hat{u})\|^2 \leq \|u_n - \hat{u}\|^2,$$

which implies that the sequence $\{u_n\}$ has a unique cluster point and $\lim_{n\to\infty} u_n = \hat{u}$, is the solution of (23.3), the required result. □

Remark 7.1. *It is worth mentioning that the one-step scheme, that is, Algorithm 23.1 convergence under the assumptions of Theorem 7.1 and Theorem 7.2. This clearly improves the convergence results for the one-step scheme. Our method can be considered as a new approach to consider the convergence analysis of three-step schemes. In the implementation of this scheme, one does not have to evaluate the projection, which is itself a problem. Our method of convergence is very simple as compared with other methods. Following the technique of Tseng [146], one can obtained new parallel and decomposition algorithms for solving a number of problems arising in optimization and mathematical programming.*

Remark 7.2. *We note that the auxiliary problem (6.1) is equivalent to finding the minimum of the functional $I[w]$ on the convex set K, where*

$$\begin{aligned} I[w] &= \frac{1}{2}\langle w-u, w-u\rangle + \langle \rho Tu, w-u\rangle \\ &= \|w-(u-\rho Tu)\|^2. \end{aligned} \tag{7.15}$$

It can be easily shown that the optimal solution of (6.15) is the projection of the point $(u - \rho Tu)$ onto the convex set K, that is,

$$w(u) = P_K[u - \rho Tu], \tag{7.16}$$

which is the fixed-point characterization of the variational inequality (23.3), see Lemma 3.1. Based on the above observations, one can show that the variational inequality (23.3) is equivalent to finding the minimum of the functional $N[u]$ on K in H, where

$$\begin{aligned} N[u] &= -\langle Tu, w(u)-u\rangle - \frac{1}{2}\langle w(u)-u, w(u)-u\rangle \\ &= \frac{1}{2}\{\|\rho Tu\|^2 - \|(w(u)-(u-\rho Tu)\|^2\}, \end{aligned} \tag{7.17}$$

where $w = w(u)$. The function $N[u]$ defined by (7.17) is known as the gap (merit) function associated with the variational inequality (23.3). This equivalence has been used to suggest and analyze a number of methods for solving variational inequalities and nonlinear programming, see, for example, Patriksson [129]. In this direction, we have:

Algorithm 7.5. *For a given $u_0 \in H$, compute the sequence $\{u_n\}$ by the iterative scheme*

$$u_{n+1} = u_n + t_n d_n, \quad n = 0, 1, 2, \ldots,$$

where $d_n = w(u_n) - u_n = P_K[u_n - \rho Tu_n] - u_n$, and $t_n \in [0,1]$ are determined by the Armijo-type rule

$$N[u_n + \beta_l d_n] \leq N[u_n] - \alpha\beta_l\|d_n\|^2.$$

It is worth to note the sequence $\{u_n\}$ generated by

$$\begin{aligned} u_{n+1} &= (1-t_n)u_n + t_n P_K[u_n - \rho Tu_n] \\ &= u_n - t_n R(u_n), \quad n = 0, 1, 2, \ldots, \end{aligned}$$

is very much similar to that generated by the projection-type Algorithm 3.3. Note that for $t_n = 1$, Algorithm 7.5 reduces to Algorithm 23.1. Based on the above observations and discussion, it is clear that the auxiliary principle approach is quite general and flexible. This approach can be used not only to study the existence theory but also to suggest and analyze various iterative methods for solving variational inequalities. Using the technique of Fukushima [34], one can easily study the convergence analysis of Algorithm 7.5.

We have shown that the auxiliary principle technique can be sued to construct gap (merit) functions for variational inequalities (23.3). We use the gap function to consider an optimal control problem governed by the variational inequalities (23.3). The control problem as an optimization problem is also referred as a generalized bilevel programming problem or mathematical programming with equilibrium constraints. It is known that the techniques of the classical optimal control problems cannot be extended for variational inequalities. This has motivated to develop some other techniques including the notion of conical derivatives, the penalty method and formulating the variational inequality as operator equation with a set-valued operator, see [9,10, 23, 65,137,157]. Furthermore, one can construct a so called gap function associated with a variational inequality, so that the variational inequality is equivalent to a scalar equation of the gap function. Under suitable conditions such a gap function is Frechet differentiable and one may use a penalty method to approximate the optimal control problem and calculate a regularized gap function in the sense of Fukushima [34] to the variational inequality (23.3) and determine their Frechet derivative. This approach has been developed in [10,23]. Following this approach one can develop the similar results for the variational inequalities. We only give the basic properties of the optimal control problem and the associated gap functions to give an idea of the approach.

We now consider the following problem of optimal control for the variational inequalities (23.3), that is, to find $u \in K, z \in U$ such that

$$\mathbf{P}. \quad \min I(u,z), \quad \langle T(u,z), v-u\rangle \geq 0, \quad \forall v \in K,$$

where H and U are Hilbert spaces. The sets K and E are closed and convex sets in H and U respectively. Here H is the space of state and $K \subset H$ is the set of state constraints for the problem. U is the space of control and closed convex set $E \subset U$ is the set of control constraints. $T(.,.) : H \times U \longrightarrow H$ is a an operator which is Frechet differentiable. The functional $I(.,.) : H \times U \longrightarrow R \cup \{+\infty\}$ is a proper, convex and lower-semicontinuous function. Also we assume that the problem **P** has at least one optimal solution denoted by $(u^*, z^*) \in H \times U$.

Related to the optimization problem **P**, we consider the regularized gap (merit) function $h_\rho(u,z) : H \times U \longrightarrow R$ as

$$h_\rho(u,z) = \sup_{v \in K}\{\langle -\rho T(u,z), v-u\rangle - \frac{1}{2}\|v-u\|^2\} \quad \forall v \in K. \tag{7.18}$$

We remark that the regularized function (7.18) is a natural generalization of the regularized gap function (7.17) for variational inequalities. It can be shown that the regularized gap function $h_\rho(.,.)$ defined by (7.18) has the following properties. The analysis is in the spirit of [10,23].

Theorem 7.3. *The gap function $h_\rho(.,.)$ defined by (7.18) is well-defined and*

(*i*). *For all* $v \in K, z \in U, \quad h_\rho(u,z) \geq 0.$

(*ii*). $h_\rho(u,z) = \frac{1}{2}\{\|\rho^2\|T(u,z) - d_K^2(u-\rho T(u,z))\},$ *where* d_K *is the distance to* K

(*iii*). $h_\rho(u,z) = -\rho\langle T(u,z), u_K - u\rangle - \frac{1}{2}\|u_K - u\|^2,$

where

$$u = P_K[u - \rho T(u,z)].$$

Proof. It is well-known that

$$d_K^2 = \min_{v \in K}\|v-u\|^2 = \|u - P_K[u_K]\|^2$$

Take $v = u$ in (6.18). Then clearly (i) is satisfied.

Let $(u,z) \in H \times U$. Then

$$\begin{aligned} h_\rho(u,z) &= \rho\langle T(u,z),u\rangle - \frac{1}{2}\|u\|^2 + \sup_{v\in K}\left[\langle -\rho T(u,z),v\rangle - \frac{1}{2}\|v\|^2 + \langle u,v\rangle\right] \\ &= \rho\langle T(u,z),u\rangle - \frac{1}{2}\|u\|^2 + \inf_{v\in K}\left[\frac{1}{2}\|v\|^2 - \langle u-\rho T(u,z),v\rangle\|^2\right] \\ &= \rho\langle T(u,z),u\rangle - \frac{1}{2}\|u\|^2 - \frac{1}{2}\inf_{v\in K}\|v-(u-\rho T(u,z))\|^2 \\ &\quad + \frac{1}{2}\|u-\rho T(u,z)\|^2 \\ &= \frac{\rho^2}{2}\|T(u,z)\|^2 - \frac{1}{2}d_K^2(u-\rho T(u,z)). \end{aligned}$$

Setting $u_K = P_K[u-\rho T(u,z)]$, we have

$$\begin{aligned} h_\rho(u,z) &= \frac{\rho^2}{2}\|T(u,z)\|^2 - \frac{1}{2}\|u-\rho T(u,z)-u_K\|^2 \\ &= -\rho\langle T(u,z),v-u\rangle - \frac{1}{2}\|u_K-u\|^2. \end{aligned}$$

□

Theorem 7.4. *If the set K is convex in H, then the following are equivalent.*

$$\begin{aligned} &(i). && h_\rho(u,z)=0, \quad \forall u\in K, z\in U \\ &(ii). && \langle T(u,z),v-u\rangle \geq 0, \quad \forall u,v\in K, z\in U. \\ &(iii). && u = P_K[u-\rho T(u,z)]. \end{aligned}$$

Proof. We show that$(ii) \Longrightarrow (i)$.
Let $u\in H$ and $z\in U$ be a solution of

$$\langle T(u,z),v-u\rangle \geq 0, \quad \forall v\in K.$$

Then we have

$$h_\rho(u,z) = -\rho\langle T(u,z),v-u\rangle - \frac{1}{2}\|v-u\|^2 \leq 0,$$

which implies that

$$h_\rho(u,z)\leq 0.$$

Also for $v\in K$, we know that

$$h_\rho(u,z)\geq 0.$$

From these above inequalities, we have (i), that is, $h_\rho(u,z)=0$.

Conversely, let (i) hold. Then

$$-\rho\langle T(u,z),v-u\rangle - \frac{1}{2}\|v-u\|^2 \leq 0, \quad \forall v\in K. \tag{7.19}$$

Since K is a convex set, so for all $w,u\in K$, $t\in[0,1]$, $g(v_t)=(1-t)u+w\in K$.

Setting $v=v_t$ in (7.19), we have

$$-\rho\langle T(u,z),w-u\rangle - \frac{t}{2}\|w-u\|^2 \leq 0.$$

Letting $t\longrightarrow 0$, we have

$$\langle T(u,z),w-u)\rangle \geq 0, \quad \forall \quad w\in K.,$$

the required (ii).

Thus we conclude that (i) and (ii) are equivalent. Applying Lemma 2.1, we have $(ii) = (iii)$. □

From Theorem 7.3 and Theorem 7.4, we conclude that the optimization problem **P** is equivalent to

$$\min I(u,z), \quad h_\rho(u,z) = 0, \quad \forall u \in K, z \in U,$$

where $h_\rho(u,z)$ is $\mathbf{C}^1$-differentiable in the sense of Frechet, but is not convex.

If the operator T is Frechet differentiable, then the gap function $h_\rho(u,z)$ defined by (7.18) is also Frechet differentiable, see [10,23].

8. Inertial Proximal Methods

In this section, we use the fixed-point alterative formulation to suggest and analyze some classes of proximal methods for solving the variational inequalities (23.3). It is worth mentioning that the proximal (implicit) methods were first introduced by Martinet [64] as regularization techniques in the context of convex optimization. Such type of the proximal methods have been studied extensively in recent years, see the references. Noor [92] used the auxiliary principle technique to study the convergence criteria of the proximal methods for pseudomonotone operators. Alvarez and Attouch [6] have suggested inertial proximal methods in the context of implicit discretization of second order differential equations in time for maximal monotone operators. It is known that the inertial proximal methods include the classical proximal methods as special cases. Noor at el [115] have considered the inertial proximal methods in the context of mixed quasi variational inequalities by using the auxiliary principle technique and have studied the convergence analysis for the pseudomonotone operators. To convey an ideas of these methods, we include the full details.

The fixed-point formulation (23.3) can be used to suggest and analyze the following implicit method for solving the variational inequalities (23.3) and related problems.

Algorithm 8.1. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$u_{n+1} = P_K[u_n - \rho T u_{n+1}], \quad n = 0, 1, 2, \ldots \tag{8.1}$$

Algorithm 8.1 is known as the proximal point algorithm. Noor [99] has studied the convergence analysis of Algorithm 8.1 for pseudomonotone operator. For the sake of completeness, we include its proof.

Lemma 8.1. *Let $\bar{u} \in K$ be a solution of (2.1). and let (u_{n+1} be the approximate solution obtained from Algorithm 8.1. If the operator T is pseudomonotone, then*

$$\|u_{n+1} - \bar{u}\|^2 \leq \|u_n - \bar{u}\|^2 - \|u_{n+1} - u_n\|^2. \tag{8.2}$$

Proof. Let $\bar{u} \in K$ be a solution of (23.3). Then it follows that

$$\langle Tv, v - \bar{u} \rangle \geq 0, \quad \forall v \in K, \tag{8.3}$$

since T is pseudomonotone.

Taking $v = u_{n+1}$ in (8.3), we have

$$\langle Tu_{n+1}, u_{n+1} - \bar{u} \rangle \geq 0. \tag{8.4}$$

Using Lemma 3.1, equation (8.1) be written in the form

$$\langle \rho T u_{n+1} + u_{n+1} - \bar{u}, v - \bar{u} \rangle \geq 0, \quad \forall v \in K. \tag{8.5}$$

Taking $v = \bar{u}$ in (8.5), we have

$$\langle T u_{n+1} + u_{n+1} - u_n, \bar{u} - u_{n+1} \rangle \geq 0,$$

which implies , by using (8.4), that

$$\langle u_{n+1} - u_n, \bar{u} - u_{n+1} \rangle \geq \langle \rho T u_{n+1}, u_{n+1} - \bar{u} \rangle \geq 0. \tag{8.6}$$

From Lemma 7.1, with

$$v = u_{n+1} - u_n, \quad u = \bar{u} - u_{n+1}$$

and (8.6), we can obtain (8.2), the required result. □

Theorem 8.1. *Let $\bar{u} \in K$ be a solution of (23.3) and let u_{n+1} be the approximate solution obtained from Algorithm 8.1. If H is a finite dimensional space, then*

$$\lim_{n \longrightarrow \infty} u_n = \bar{u}.$$

Proof. Its proofs is very much similar to that of Theorem 7.2. □

We now suggest and analyze an inertial proximal method for solving variational inequalities. For a positive constant α, we can rewrite the equation (23.3) as

$$u = P_K[u - \rho T u + \alpha(u - u)]. \tag{8.7}$$

We use this fixed-point formulation to suggest the following iterative method.

Algorithm 8.2. *For a given u_0, compute the approximate solution u_{n+1} by the iterative scheme*

$$u_{n+1} = P_K[u_n - \rho T u_{n+1} + \alpha_n(u_n - u_{n-1}], \quad n = 1, 2, \ldots$$

which can be written as by using Lemma 3.1,

$$\langle \rho_n T u_{n+1} + u_{n+1} - u_n - \alpha_n(u_n - u_{n-1}, v - u_{n+1} \rangle \geq 0, \quad \forall v \in K. \tag{8.8}$$

Algorithm 8.2 is known as the inertial proximal method for variational inequalities (23.3), which is very much similar to the one suggested by Alvarez [5] for maximal monotone operators. Note that for $\alpha_n = 0$, Algorithm 8.2 reduces to Algorithm 8.1 , which is the classical proximal point algorithm.

For the convergence analysis of Algorithm 8.2, we need the following result, which can be proved using the technique of Lemma 8.1.

Theorem 8.2. *Let $\bar{u} \in K$ be a solution of (23.3) and let u_{n+1} be the approximate solution obtained from Algorithm 8.2. If the operator $T : H \longrightarrow H$ is pseudomonotone, then*

$$\begin{aligned} \|u_{n+1} - \bar{u}\|^2 \leq\ & \|u_n - \bar{u}\|^2 + \alpha_n\{\|u_n - \bar{u}\|^2 \\ & - \|\bar{u} - u_{n-1}\|^2 + 2\|u_n - u_{n-1}\|^2\} \\ & - \|u_{n+1} - u_n - \alpha_n(u_n - u_{n-1})\|^2. \end{aligned} \tag{8.9}$$

Proof. Let $\bar{u} \in K$ be a solution of (23.3). Then (8.4) holds, that is,

$$\langle Tu_{n+1}, u_{n+1} - \bar{u}\rangle \geq 0. \tag{8.10}$$

Setting $v = \bar{u}$ in (8.8), we obtain

$$\langle \rho_n Tu_{n+1} + u_{n+1} - u_n - \alpha_n\{u_n - u_{n-1}\}, \bar{u} - u_{n+1}\rangle \geq 0. \tag{8.11}$$

Adding (8.10) and (8.11), we have

$$\langle u_{n+1} - u_n - \alpha_n\{u_n - u_{n-1}\}, \bar{u} - u_{n+1}\rangle \geq 0,$$

which can be written as

$$\langle u_{n+1} - u_n, \bar{u} - u_{n+1}\rangle \geq \alpha_n\{\langle u_n - u_{n-1}, \bar{u} - u_n + u_n - u_{n+1}\}\rangle. \tag{8.12}$$

Using Lemma 7.1 and rearranging the terms of (8.12), one can easily obtain the required result. □

Theorem 8.3. *Let H be a finite dimensional space. Let $\bar{u} \in K$ be a solution of (23.3) and let u_{n+1} be the approximate solution obtained from algorithm 8.2. If there exist a $\alpha \in [0, 1)$ such that $0 \leq \alpha_n \leq \alpha$, for all $n \in N$ and*

$$\sum_{n=1}^{\infty} \alpha_n \|u_n - u_{n-1}\|^2 \leq \infty,$$

then

$$\lim_{n\to\infty} u_n = \bar{u}.$$

Proof. For the case $\alpha_n = 0$, its proof follows from Theorem 8.1. The proof for the case $\alpha > 0$ is exactly the same as in Alvarez and Attouch [6]. □

We can rewrite the Algorithm 8.2 as two-step method as:

Algorithm 8.3. *For a given u_0, compute the approximate solution u_{n+1} by the iterative schemes*

$$\begin{aligned} y_n &= u_n + \alpha_n(u_n - u_{n+1}) \\ u_{n+1} &= P_K[y_n - \rho T y_n], \quad n = 0, 1, 2, \ldots, \end{aligned}$$

which is known as the inertial proximal method for solving the variational inequalities.
In a similar way, one can suggest the following inertial method.

Algorithm 8.4. *For a given u_0, compute the approximate solution u_{n+1} by the iterative schemes*

$$\begin{aligned} y_n &= u_n + \alpha_n(u_n - u_{n+1}) \\ u_{n+1} &= P_K[y_n - \rho T u_n], \quad n = 0, 1, 2, \ldots, \end{aligned}$$

We now suggest and analyze a unified and general algorithm for solving variational inequalities, from which one can derive several projection, splitting, predictor-corrector and proximal point algorithms. We rewrite the equation (23.3) in the following form

$$u = P_K[u - \rho(Tu + e)], \tag{8.13}$$

where e is a sufficiently small error vector depending upon u.This fixed-point formulation allows us to suggest and analyze the following method.

Algorithm 8.5. *For a given u_0, compute the approximate solution u_{n+1} by the iterative scheme*

$$u_{n+1} = P_K[u_n - \rho(Tu_n + e_n)], \quad n = 0, 1, 2, \ldots \tag{8.14}$$

Here the quantities ρ and e_n may depend upon u and may be viewed as algorithm parameters whereby different choices of ρ and e_n lead to different algorithms.
Note that for $e_n = 0$, Algorithm 8.5 is exactly the projection Algorithm 23.1.
For $e_n = Tu_{n+1} - Tu_n$, Algorithm 8.5 is the proximal point Algorithm 8.1.
By taking

$$e_n = TP_K[u_n - \rho Tu_n] - \rho Tu_n + \frac{1}{\rho}\{u_n - P_K[u_n - \rho Tu_n]\}$$

in (7.14), Algorithm 8.5 reduces to Algorithm 6.1.
In a similar way, one can derive other algorithms as a special cases from Algorithm 8.5. This clearly shows that Algorithm 8.5 is a unified algorithm for solving the variational inequalities (23.3). Using the technique of Luo and Tseng [63], one can obtain the error bounds and study the convergence analysis of algorithm 8.5. The connection of Algorithm 8.5 with other algorithms is significant for it enables certain properties of previous algorithms to be used in the analysis of this algorithm.

9. Feasibility Problems

In recent years much attention has been given to study the split feasibility problems, which arise in diverse fields of pure and applied sciences including image reconstruction, medical sciences (medical image) and signal processing. Many iterative projection-type algorithms have been proposed and analyzed for solving split feasibility problems, see Byrne [**?**, **?**], Yang [**?**, **?**], and the references therein. To implement these algorithms, one has to find the projection on the closed convex sets, which is not possible except in simple cases. We would like to mention that these problems can be studied by the variational inequalities approach. In fact, It has been shown [**?**, **?**] that the split feasibility problems are equivalent to the variational inequalities. This alternative approach is more flexible and allows us to improve the convergence analysis of these iterative-type projection algorithms. This equivalence has been used to suggest and analyze several iterative methods for solving variational inequalities and related optimization problems. It is known that a substantial number of numerical methods can be obtained as special cases from this technique. It has been shown [**?**, **?**] that the variational inequality approach is more flexible and provides a natural and unified framework to suggest and analyze iterative methods for solving split feasibility problems.
Inspired and motivated by the research going in this direction, we propose a descent-projection method for solving the split feasibility problems. We also study the global convergence of the proposed method under some mild conditions. Some preliminary computational results are given to illustrate the efficiency of the proposed method. Results obtained in this paper may be viewed as an improvement and refinement of the previously known results in this field.

Let Ω and C be two nonempty, closed and convex sets in R^n and R^m, respectively, and A an $m \times n$ real matrix. The inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. We consider the problem of finding

$$u \in \Omega \quad \text{such that} \quad Au \in C, \tag{9.1}$$

which is known as the split feasibility problem. For the examples and formulation, see [2-5].
There are many applications, especially in the field of image restoration to deal with this problem [**?**].

It is well known that such type of problems arise in the image reconstruction and have applications in medical image and signal processing. It is known [?, ?] that these problems are equivalent to the following variational inequality

$$u^* \in \Omega, \quad \langle F(u^*), u - u^* \rangle \geq 0, \qquad \forall u \in \Omega \tag{9.2}$$

where

$$F(u) = A^T(I - P_C)Au; \tag{9.3}$$

I and P_C denote the identity operator and the projection of R^m on the closed convex set C, respectively. Problems of the type (9.2) are known as variational inequalities, which were introduced and studied by Stampacchia [?] in 1964. It can be shown that the minimum of the function

$$G(u) = \frac{1}{2}\|P_C Au - Au\|^2$$

on the convex set Ω can be characterized by the variational inequality of the type (9.2) with

$$F(u) = G'(u) = A^T(I - P_C)Au,$$

where $G'(u)$ is the differential of the differentiable convex function $G(u)$ (see Proposition 2.1) at $u \in \Omega$.

Proposition 9.1. *$G(u)$ is differentiable and convex for any matrix A.*

Proof. It is easy to prove that

$$G'(u) = A^T(I - P_C)Au, \qquad \text{for any } u \in R^n.$$

In the following, we will prove that $G(u)$ is convex. For any $u, v \in R^n$, we have

$$\begin{aligned}(u-v)^T(G'(u) - G'(v)) &= (u-v)^T A^T[Au - P_C(Au) - Av + P_C(Av)] \\ &= (Au - Av)^T[Au - P_C(Au) - Av + P_C(Av)] \\ &= \|Au - Av\|^2 - (Au - Av)^T[P_C(Au) - P_C(Av)] \\ &\geq \|Au - Av\|^2 - \|Au - Av\|\|P_C(Au) - P_C(Av)\| \\ &\geq 0.\end{aligned}$$

Since $G'(u)$ is monotone, then $G(u)$ is convex. The proof is completed. □

Lemma 9.1. *$u \in \Omega$ is a solution of the variational inequality (9.2), if and only if, $u \in \Omega$ satisfies*

$$u = P_\Omega[u - \rho A^T(I - P_C)Au].$$

Here $\rho > 0$ is arbitrary but fixed constant.

This alternative equivalent formulation has played an important and crucial part in the development of several projection-type iterative algorithms for solving variational inequalities problems and related optimization problems. Byrne [34] proposed the CQ algorithm, which generates the new iterate as follows

$$u^{k+1} = P_\Omega[u^k - \rho T(u^k)];$$

where $\rho \in (0; 2/L), L$ denotes the largest eigenvalue of the matrix A^TA. However, sometimes the projections onto Ω and C are difficult to calculate. In [168], Yang presented a relaxed CQ algorithm (The name CQ algorithm comes from the sets C and Q that are calculated at every step of the algorithm) for solving (9.1), where at $k-$th iteration, the projections onto Ω and C were replaced with the projections onto some halfspaces Ω_k and C_k, respectively. Note that the step length of the

CQ algorithm and the relaxed version relies on the largest eigenvalue of the matrix A^TA. To avoid the estimation of the largest eigenvalue of the matrix A^TA, Qu and Xiu [?] suggested the idea of choosing the controlling parameter ρ_k dynamically in a suitable way:

$$\tilde{u}^k = P_{\Omega_k}[u^k - \rho_k T(u^k)], \tag{9.4}$$

where

$$T(u^k) = A^T(I - P_{C_k})Au^k, \qquad \rho_k\|T(u^k) - T(\tilde{u}^k)\| \le \mu\|u^k - \tilde{u}^k\|, \qquad 0 < \mu < 1, \tag{9.5}$$

and the new iterate u^{k+1} is updated by

$$u^{k+1} = P_{\Omega_k}[u^k - \rho_k T(\tilde{u}^k)]. \tag{9.6}$$

Inspired and motivated by the above research, we suggest and analyze a descent-projection method for solving the split feasibility problems, which can be regarded as an extension of the method in [148, 149] and this is the main motivation of this paper.

The following well-known results will be often used in this paper. We summarize them in the following lemmas. For the complete proofs, the readers can see the references.

Lemma 9.2. *Let Ω be a closed convex set in R^n, we denote $P_\Omega(.)$ as the projection under the Euclidean norm, that is,*

$$P_\Omega(z) = argmin\{\|z - x\| \mid x \in \Omega\}.$$

Then then the following statements hold:

$$1)\langle v - P_\Omega(v), w - P_\Omega(v)\rangle \le 0, \quad \forall v \in R^n \quad and \quad \forall w \in \Omega; \tag{9.7}$$

$$2)\|P_\Omega(v) - w\|^2 \le \|v - w\|^2 - \|v - P_\Omega(v)\|^2, \quad \forall\, v \in R^n \quad and \quad \forall w \in \Omega; \tag{9.8}$$

$$3)\|(I - P_\Omega)(v) - (I - P_\Omega(\tilde{v}))\|^2 \le \langle (I - P_\Omega)(v) - (I - P_\Omega)(\tilde{v}), v - \tilde{v}\rangle, \qquad \forall v, \tilde{v} \in R^n. \tag{9.9}$$

Let T be a mapping from R^n into R^n. For any $u \in R^n$ and $\rho > 0$, define

$$r(u, \rho) = u - P_\Omega[u - \rho T(u)]. \tag{9.10}$$

Lemma 9.3. *[?] For any $u \in R^n$ and $\tilde{\rho} \ge \rho > 0$, we have*

$$\|r(u, \rho)\| \le \|r(u, \tilde{\rho})\| \tag{9.11}$$

and

$$\frac{\|r(u, \tilde{\rho})\|}{\tilde{\rho}} \le \frac{\|r(u, \rho)\|}{\rho}. \tag{9.12}$$

We also need the following, which is essentially due to Byrne [34].

Lemma 9.4. *[34] The function $T(u)$ defined by (9.5) is Lipschitz continuous with constant L, where L is the largest eigenvalue of A^TA.*

In this paper, we assume that the convex sets Ω and C satisfy the following assumptions:

- The set Ω is given by

  $$\Omega = \{u \in R^n | f(u) \le 0\},$$

  where $f : R^n \longrightarrow R$ is a convex (not necessarily differentiable) function and Ω is nonempty. The set C is given by

  $$C = \{v \in R^m | g(v) \le 0\},$$

  where $g : R^m \longrightarrow R$ is a convex (not necessarily differentiable) function and C is nonempty.

- For any $u \in R^n$, at least one subgradient $\xi \in \partial f(u)$ can be calculated, where $\partial f(u)$ is the subdifferential of $f(u)$ at u and is defined as follows:

$$\partial f(u) = \{\xi \in R^n | f(w) \geq f(u) + \langle \xi, w - u \rangle \quad \forall w \in R^n\}.$$

For any $v \in R^m$, at least one subgradient $\eta \in \partial g(u)$ can be calculated, where $\partial g(u)$ is the subdifferential of $g(u)$ at u and is defined as follows:

$$\partial g(v) = \{\eta \in R^m | g(z) \geq g(v) + \langle \eta, z - v \rangle \quad \forall z \in R^m\}.$$

- The solution set of the split feasibility problem (9.1) denoted by Ω^*, is nonempty.

Denote

$$\Omega_k = \{u \in R^n | f(u^k) + \langle \xi^k, u - u^k \rangle \leq 0\} \tag{9.13}$$

where $\xi^k \in \partial f(u^k)$, and

$$C_k = \{v \in R^m | g(Au^k) + \langle \eta^k, v - Au^k \rangle \leq 0\} \tag{9.14}$$

where $\eta^k \in \partial g(u^k)$.

Proposition 9.2. *For every $k \geq 0$, let $u^k \in R^n$, Ω_k and C_k be defined as in (9.13) and (9.14) respectively. Then for any $u \in R^n$, we have*

$$P_{\Omega_k}(u) = \begin{cases} u - \frac{f(u^k)+\langle \xi^k, u-u^k \rangle}{\|\xi^k\|^2}\xi^k, & \text{if} \quad f(u^k) + \langle \xi^k, u - u^k \rangle > 0; \\ u & \text{otherwise.} \end{cases}$$

and

$$P_{C_k}(Au) = \begin{cases} Au - \frac{g(Au^k)+\langle \eta^k, Au-Au^k \rangle}{\|\eta^k\|^2}\eta^k, & \text{if} \quad g(u^k) + \langle \eta^k, u - u^k \rangle > 0; \\ Au & \text{otherwise.} \end{cases}$$

We need the well known result.

Lemma 9.5. *Suppose $h : R^n \to R$ is a finite convex function, then it is subdifferentiable everywhere and its subdifferentials are uniformly bounded on any bounded subset of R^n.*

We now suggest and analyze the proposed method for solving the split feasibility problem (9.1) and we prove some basic properties, which will be used to establish the sufficient and necessary conditions for the convergence of the proposed method.

Algorithm 9.1. *For a given $u^k \in R^n$, find the approximate solution by the following iterative schemes.*
Step 1.

$$\tilde{u}^k = P_{\Omega_k}[u^k - \rho_k T(u^k)], \tag{9.15}$$

where $\rho_k > 0$ satisfies

$$\rho_k \|T(u^k) - T(\tilde{u}^k)\| \leq \delta \|u^k - \tilde{u}^k\|, \qquad 0 < \delta < 1. \tag{9.16}$$

Step 2. *The new iterate u^{k+1} is defined by*

$$u^{k+1} = P_{\Omega_k}[u^k - \alpha_k d(u^k, \rho_k)]. \tag{9.17}$$

where

$$d(u^k, \rho_k) = u^k - \tilde{u}^k + \rho_k T(\tilde{u}^k), \tag{9.18}$$

$$\varepsilon^k = \rho_k (T(\tilde{u}^k) - T(u^k)), \tag{9.19}$$

$$D(u^k, \rho_k) := u^k - \tilde{u}^k + \varepsilon^k, \tag{9.20}$$

$$\phi(u^k, \rho_k) := \langle u^k - \tilde{u}^k, D(u^k, \rho_k)\rangle \tag{9.21}$$

and

$$\alpha_k := \frac{\phi(u^k, \rho_k)}{\|d(u^k, \rho_k)\|^2}. \tag{9.22}$$

Remark 9.1. *Denote*

$$r_k(u^k, \rho_k) = u^k - P_{\Omega_k}[u^k - \rho_k T(u^k)].$$

Using Cauchy-Schwartz inequality (9.16) implies that

$$|\langle u^k - \tilde{u}^k, \varepsilon^k\rangle| \leq \|u^k - \tilde{u}^k\|\|\varepsilon^k\| \leq \delta\|r_k(u^k, \rho_k)\|^2, \qquad 0 < \delta < 1. \tag{9.23}$$

Remark 9.2. *In general the Step 1 is implementable. Sometimes we can get $\tilde{u}^k$ directly via choosing a suitable small $\rho_k > 0$. For example, if ρ_k satisfies $0 < \rho_k \leq \frac{\delta}{L}$ (L is the largest eigenvalue of the matrix $A^T A$), then from Lemma 2.4, we have*

$$\rho_k\|T(u^k) - T(\tilde{u}^k)\| \leq \rho_k L\|u^k - \tilde{u}^k\| \leq \delta\|u^k - \tilde{u}^k\|,$$

then the condition (3.2) is satisfied. Without loss of generality, we can assume that $\inf_{k=0}^{\infty} \rho_k > 0$.

Lemma 9.6. *For given $u^k \in R^n, u^* \in \Omega^*$ and $\rho_k > 0$, we have*

$$\langle u^k - u^*, d(u^k, \rho_k)\rangle \geq \phi(u^k, \rho_k). \tag{9.24}$$

Proof. Since $u^* \in \Omega^* \subseteq \Omega_k$, it follows from $Au^* \in C \subseteq C_k$ that $T(u^*) = 0$. Using the monotonicity of T, we obtain

$$\langle \rho_k T(\tilde{u}^k), \tilde{u}^k - u^*\rangle \geq 0. \tag{9.25}$$

Substituting $v = u^k - \rho_k T(u^k)$ and $w = u^*$ into (9.7), and using the definition of $r_k(u^k, \rho_k)$, we get

$$\langle r_k(u^k, \rho_k) - \rho_k T(u^k), \tilde{u}^k - u^*\rangle \geq 0. \tag{9.26}$$

Adding (9.25) and (9.26), we have

$$\langle r_k(u^k, \rho_k) - \rho_k[T(u^k) - T(\tilde{u}^k)], \tilde{u}^k - u^*\rangle \geq 0,$$

which can be rewritten as

$$\langle r_k(u^k, \rho_k) - \rho_k[T(u^k) - T(\tilde{u}^k)], u^k - u^* - r_k(u^k, \rho_k)\rangle \geq 0,$$

then

$$\begin{aligned}\langle u^k - u^*, r_k(u^k, \rho_k) + \rho_k T(\tilde{u}^k)\rangle \quad &\geq \quad \langle u^k - u^*, \rho_k T(u^k)\rangle + \|r_k(u^k, \rho_k)\|^2 \\ &\quad -\rho_k\langle r_k(u^k, \rho_k), T(u^k) - T(\tilde{u}^k)\rangle.\end{aligned}$$

Using the monotonicity of T, the first term in the right side of the above inequality is positive, we obtain

$$\langle u^k - u^*, d(u^k, \rho_k)\rangle \geq \|r_k(u^k, \rho_k)\|^2 - \rho_k\langle r_k(u^k, \rho_k), T(u^k) - T(\tilde{u}^k)\rangle,$$

which is the required result. □

Remark 9.3. *From (9.23) and (9.24), we have*

$$\langle u^k - u^*, d(u^k, \rho_k)\rangle \geq \phi(u^k, \rho_k) \geq (1-\delta)\|r_k(u^k, \rho_k)\|^2. \tag{9.27}$$

Then $d(u^k, \rho_k)$ is a descent direction of the distance function at u^k, so along $d(u^k, \rho_k)$, one can find a new iterate which is closer to the solution set. Due to this fact, we construct the new iterate as in Step 2.

For the convergence analysis of the proposed Algorithm 9.1, we need the following results.

Lemma 9.7. *For given $u^k \in R^n$ and $\rho_k > 0$, we have*

$$\phi(u^k, \rho_k) \geq (1-\delta)\|r_k(u^k, \rho_k)\|^2 \tag{9.28}$$

and

$$\alpha_k \geq c, \tag{9.29}$$

where $c > 0$ is a constant.

Proof. It follows from (9.20) and (9.23) that

$$\begin{aligned}\phi(u^k, \rho_k) &:= \langle u^k - \tilde{u}^k, D(u^k, \rho_k)\rangle \\ &= \|u^k - \tilde{u}^k\|^2 + \langle u^k - \tilde{u}^k, \varepsilon^k\rangle \\ &\geq (1-\delta)\|u^k - \tilde{u}^k\|^2.\end{aligned}$$

On the other hand from (9.16), we have

$$\begin{aligned}\phi(u^k, \rho_k) &:= \langle u^k - \tilde{u}^k, D(u^k, \rho_k)\rangle \\ &= \|u^k - \tilde{u}^k\|^2 + \langle u^k - \tilde{u}^k, \varepsilon^k\rangle \\ &\geq \tfrac{1}{2}\|u^k - \tilde{u}^k\|^2 + \langle u^k - \tilde{u}^k, \varepsilon^k\rangle + \tfrac{1}{2}\|\varepsilon^k\|^2 \\ &= \tfrac{1}{2}\|D(u^k, \rho_k)\|^2.\end{aligned}$$

Then, we can find a constant $c > 0$ such that

$$\alpha_k \geq \frac{\|D(u^k, \rho_k)\|^2}{2\|d(u^k, \rho_k\|^2} \geq c,$$

whenever $u^k \neq \tilde{u}^k$. The proof is completed. □

In this section we focus on investigating the convergence of the proposed method. The following theorem concerns the contractive property of the sequence generated by Algorithm 9.1.

Theorem 9.1. *Let u^* be a solution of problem (9.1) and let u^{k+1} be the sequence obtained from algorithm 9.1. Then u^k is bounded and*

$$\|u^{k+1} - u^*\|^2 \leq \|u^k - u^*\|^2 - \gamma c(1-\delta)\|r_k(u^k, \rho_k)\|^2. \tag{9.30}$$

Proof. Let $u^* \in \Omega$ be a solution of problem (9.1) and $\Theta(\alpha_k) := \|u^k - u^*\|^2 - \|u^{k+1}(\alpha_k) - u^*\|^2$. Then, from (9.8) and (9.17), we have

$$\begin{aligned}\Theta(\alpha_k) &\geq \|u^k - u^*\|^2 - \|u^k - u^* - \alpha_k d(u^k, \rho_k)\|^2 \\ &= \alpha_k\langle u^k - u^*, d(u^k, \rho_k)\rangle - \alpha_k^2\|d(u^k, \rho_k)\|^2 \\ &\geq \alpha_k\phi(u^k, \rho_k) - \alpha_k^2\|d(u^k, \rho_k)\|^2.\end{aligned} \tag{9.31}$$

Let

$$\Phi(\alpha_k) = \alpha_k \phi(u^k, \rho_k) - \alpha_k^2 \|d(u^k, \rho_k)\|^2$$

Note that $\Phi(\alpha)$ is a concave quadratic function of α and it reaches its maximum at

$$\alpha_k^* = \frac{\phi(u^k, \rho_k)}{\|d(u^k, \rho_k)\|^2}$$

and

$$\Phi(\alpha_k^*) = \alpha_k^* \phi(u^k, \rho_k).$$

According to (9.31), it follows that

$$\Theta(\alpha_k^*) \geq \alpha_k^* \phi(u^k, \rho_k)$$

For fast convergence, we take the step-size $\alpha_k = \gamma \alpha_k^*$ where $\gamma \in [1, 2)$ is a relaxation factor. From (9.28) and (9.31), we obtain

$$\begin{aligned} \|u^{k+1} - u^*\|^2 &= \|u^k - u^*\|^2 - \Theta(\gamma \alpha_k^*) \\ &\leq \|u^k - u^*\|^2 - \gamma \alpha_k^* \phi(u^k, \rho_k) \\ &\leq \|u^k - u^*\|^2 - \gamma c(1 - \delta) \|r_k(u^k, \rho_k)\|^2, \end{aligned}$$

where the last inequality follows from (9.29). Since $\gamma > 0$ and $\delta \in (0, 1)$ we have

$$\|u^{k+1} - u^*\| \leq \|u^k - u^*\| \leq \ldots \leq \|u^0 - u^*\|.$$

This shows that the sequence u^k is bounded. □

Theorem 9.2. *If* $\inf_{k=0}^{\infty} \rho_k = \beta > 0$, *then the sequence* $\{u^k\}$ *generated by the proposed method converges to a solution point of problem (9.1).*

Proof. It follows from (9.30) that

$$\sum_{k=0}^{\infty} \|r_k(u^k, \rho_k)\|^2 < \infty,$$

which means that

$$\lim_{k \to \infty} \|r_k(u^k, \rho_k)\| = 0, \tag{9.32}$$

Since $\{u^k\}$ is bounded, it has a cluster point $\bar{u}$ and a subsequence $\{u^{k_i}\}$ converges to $\bar{u}$.
Since $\tilde{u}^k \in \Omega_{k_i}$, then by definition of Ω_{k_i}, we have

$$f(u^{k_i}) + \langle \xi^{k_i}, \tilde{u}^{k_i} - u^{k_i} \rangle \leq 0.$$

Passing onto the limit in the above inequality and from (9.32) and Lemma 2.5, we obtain that

$$f(\bar{u}) \leq 0.$$

Then $\bar{u} \in \Omega$.
Next, we show that $A\bar{u} \in C$. It follows from Lemma 2.3 and (9.32) that

$$\begin{aligned} \lim_{k_i \to \infty} \|r_{k_i}(u^{k_i}, 1)\| &\leq \lim_{k_i \to \infty} \frac{\|r_{k_i}(u^{k_i}, \rho_{k_i})\|}{\min\{1, \rho_{k_i}\}} \\ &\leq \lim_{k_i \to \infty} \frac{\|r_{k_i}(u^{k_i}, \rho_{k_i})\|}{\min\{1, \beta\}} \\ &= 0 \quad . \end{aligned} \tag{9.33}$$

Since $u^* \in \Omega_{k_i}$ and from (9.7), we get

$$\langle u^{k_i} - T(u^{k_i}) - P_{\Omega_{k_i}}[u^{k_i} - T(u^{k_i})], u^* - P_{\Omega_{k_i}}[u^{k_i} - T(u^{k_i})]\rangle \leq 0,$$

which can be rewritten as

$$\langle r_{k_i}(\tilde{u}^{k_i}, 1) - T(u^{k_i}), u^{k_i} - u^* - r_{k_i}(\tilde{u}^{k_i}, 1)\rangle \geq 0. \tag{9.34}$$

Since $Au^* \in C_{k_i}$ and from (9.9), we have

$$\begin{aligned}\langle T(u^{k_i}), u^{k_i} - u^*\rangle &= \langle T(u^{k_i}) - T(u^*), u^{k_i} - u^*\rangle \\ &= \langle A^T(I - P_{C_{k_i}})Au^{k_i} - A^T(I - P_{C_{k_i}})Au^*, u^{k_i} - u^*\rangle \\ &= \langle (I - P_{C_{k_i}})Au^{k_i} - (I - P_{C_{k_i}})Au^*, Au^{k_i} - Au^*\rangle \\ &\geq \|(I - P_{C_{k_i}})Au^{k_i} - (I - P_{C_{k_i}})Au^*\|^2 \\ &= \|(I - P_{C_{k_i}})Au^{k_i}\|^2. \end{aligned} \tag{9.35}$$

Combining (9.34) and (9.35), we obtain

$$\begin{aligned}\langle r_{k_i}(\tilde{u}^{k_i}, 1), u^{k_i} - u^*\rangle &\geq \|r_{k_i}(\tilde{u}^{k_i}, 1)\|^2 - \langle T(u^{k_i}), r_{k_i}(\tilde{u}^{k_i}, 1)\rangle + \langle T(u^{k_i}), u^{k_i} - u^*\rangle \\ &\geq \|r_{k_i}(\tilde{u}^{k_i}, 1)\|^2 - \langle T(u^{k_i}), r_{k_i}(\tilde{u}^{k_i}, 1)\rangle + \|(I - P_{C_{k_i}})Au^{k_i}\|^2. \end{aligned} \tag{9.36}$$

Since

$$\|T(u^{k_i})\| = \|T(u^{k_i}) - T(u^*)\| \leq \frac{\delta}{\rho_k}\|u^{k_i} - u^*\| \leq \frac{\delta}{\beta}\|u^{k_i} - u^*\|$$

and $\{u^{k_i}\}$ is bounded, the sequence $\{T(u^{k_i})\}$ is also bounded. Therefore from (9.33) and (9.36)

$$\lim_{k_i\to\infty} \|(I - P_{C_{k_i}})Au^{k_i}\| = 0,$$

then

$$\lim_{k_i\to\infty} P_{C_{k_i}}(Au^{k_i}) - Au^{k_i} = 0. \tag{9.37}$$

Since $P_{C_{k_i}}(Au^{k_i}) \in C_{k_i}$, we have

$$g(Au^{k_i}) + \langle \eta_{k_i}, P_{C_{k_i}}(Au^{k_i}) - Au^{k_i}\rangle \leq 0.$$

Passing onto the limit in the above inequality and from (9.37) and Lemma 2.5, we obtain that

$$g(A\bar{u}) \leq 0,$$

then $A\bar{u} \in C$. Therefore $\bar{u}$ is a solution of the split feasibility problem (9.1).
On the other hand, since $\{u^{k_i}\}$ converges to $\bar{u}$, it follows from (9.30) that for an arbitrary $\epsilon > 0$, there exists a k_0 such that

$$\|u^k - \bar{u}\| \leq \|u^{k_0} - \bar{u}\| \leq \epsilon \qquad \forall k \geq k_0.$$

This implies that the sequence $\{u^k\}$ converges to $\bar{u}$ which is a solution of the split feasibility problem (9.1). □

The detailed algorithm is as follows.

Algorithm 9.2. *Step 0. Let* $\rho_0 = 1, \delta := 0.9 < 1, \gamma \in [1, 2), \epsilon > 0, k = 0$ *and* $u^0 \in R^n$.
Step 1. If $\|\|r(u^k, \rho_k)\|_\infty \leq \epsilon$*, then stop. Otherwise, go to Step 2.*
Step 2.
$\tilde{u}^k = P_{\Omega_k}[u^k - \rho_k T(u^k)], \qquad \varepsilon^k = \rho_k(T(\tilde{u}^k) - T(u^k)),$
$r = \frac{\|\varepsilon^k\|}{\|u^k - \tilde{u}^k\|}.$

While $(r > \delta)$

$\rho_k = 0.8 * \rho_k * min(1, 1/r), \qquad \tilde{u}^k = P_{\Omega_k}[u^k - \rho_k T(u^k)],$

$\varepsilon^k = \rho_k(T(\tilde{u}^k) - T(u^k)), \qquad r = \frac{\|\varepsilon^k\|}{\|u^k - \tilde{u}^k\|}.$

end While

Step 3. Set

$d(u^k, \rho_k) = u^k - \tilde{u}^k + \rho_k T(\tilde{u}^k),$

$D(u^k, \rho_k) := u^k - \tilde{u}^k + \varepsilon^k,$

$\phi(u^k, \rho_k) := \langle u^k - \tilde{u}^k, D(u^k, \rho_k)\rangle, \qquad \alpha_k := \frac{\phi(u^k,\rho_k)}{\|d(u^k,\rho_k)\|^2},$

$u^{k+1} = P_{\Omega_k}[u^k - \gamma\alpha_k d(u^k, \rho_k)].$

Step 4. $\rho_{k+1} = \begin{cases} \frac{\rho_k * 0.9}{r} & \text{if } \; r \le 0.5; \\ \rho_k & \text{otherwise.} \end{cases}$

Step 5. k:=k+1; go to Step 1.

Remark 9.4. *Condition (9.16) ensures that* $\rho_k\|T(u^k) - T(\tilde{u}^k)\|$ *is smaller than* $\|u^k - \tilde{u}^k\|$*, however, too small* $\rho_k\|T(u^k) - T(\tilde{u}^k)\|$ *leads to slow convergence. In this situation, enlarging* ρ_k *for the next iteration is necessary (see Step 4). From numerical point of view, it is necessary to attach a relax factor* γ *to the optimal step length* α_k *to achieve faster convergence.*

We present some numerical results for the proposed method. The two examples used here are taken from the test problems in [148].
In all tests we take $\epsilon = 10^{-10}, \delta = 0.9$ and $\gamma = 1.8$. All iterations start with $\rho_0 = 1$. All codes are written in Matlab and run on a T400-MZ6 note book computer. The iteration numbers and the computational time for algorithm 9.1 and the method in [148] with different starting points are given in tables 5-1-5-3.

Example 9.1 (A convex feasibility problem).
Let

$$\Omega = \{u(u_1, u_2, u_3) \in R^3 : \quad u_2^2 + u_3^2 - 4 \le 0\},$$

$C = \{u(u_1, u_2, u_3) \in R^3 : \quad u_3 - u_1^2 - 1 \le 0\}$. Find some point $u \in \Omega \bigcap C$.

Table 9-1: Numerical results for Example 9.1

	The method in [148]			The proposed method		
Starting point	k	CPU(s)	Approximate solution	k	CPU(s)	Approximate solution
$(1,2,3)^T$	5	0.0125	$(1.0000, 1.1094, 1.6641)^T$	1	0.187	$(2.2818, 0.0772, -0.5251)^T$
$(1,1,1)^T$	0	0.016	$(1,1,1)^T$	0	0.015	$(1,1,1)^T$
$rand(3,1) * 10$	125	0.063	$(0.8670, 0.6374, 1.7518)^T$	1	0.047	$(1.9674, 0.7665, 1.1845)^T$

Example 9.2 (A split feasibility problem). Let $A = \begin{pmatrix} 2 & -1 & 3 \\ 4 & 2 & 5 \\ 2 & 0 & 2 \end{pmatrix}$,

$$\Omega = \{u(u_1, u_2, u_3) \in R^3 : \quad u_1 + u_2^2 + 2u_3 \le 0\},$$

$C = \{u(u_1, u_2, u_3) \in R^3 : \quad u_1^2 + u_2 - u_3 \le 0\}$. Find some point $u \in \Omega$ with $Au \in C$.

Table 9-2: Numerical results for Example 9.2

	The method in [148]			The proposed method		
Initial point	k	CPU(s)	Approximate solution	k	CPU(s)	Approximate solution
$(1,2,3)^T$	62	0.14	$(-0.402, 0.067, 0.197)^T$	15	0.15	$(-0.935, -0.638, -0.246)^T$
$(1,1,1)^T$	79	0.063	$(0.357, 0.034, -0.265)^T$	11	0.062	$(0.240, 0.082, -0.277)^T$
$Rnd(3,1) * 10$	100	0.063	$(0.874, 0.079, -0.687)^T$	39	0.063	$(-0.385, -1.695, -1.245)^T$

Example 9.3 Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}$,

$$\Omega = \{u(u_1, u_2, u_3) \in R^3 : \quad u_1 + u_2^2 + 2u_3 \leq 0\},$$

$C = \{u(u_1, u_2, u_3) \in R^3 : \quad u_1^2 + u_2 - u_3 \leq 0\}$. Find some point $u \in \Omega$ with $Au \in C$.

Table 9-3: Numerical results for Example 9.3

	The method in [148]			The proposed method		
Starting point	k	CPU(s)	Approximate solution	k	CPU(s)	Approximate solution
$(1,2,3)^T$	89	0.15	$(-0.0706, -0.2340, 0.0079)^T$	6	0.15	$(0.516, 0.167, -1.561)^T$
$(1,1,1)^T$	4	0.063	$(0.3988, 0.0763, -0.2023)^T$	3	0.062	$(1.282, -0.101, -1.382)^T$
$rand(3,1)*10$	106	0.063	$(0.0733, 0.1122, -0.0430)^T$	12	0.063	$(0.161, -0.799, -0.765)^T$

Conclusions

In this section, we have presented an iterative projection method for solving the feasibility problems in conjunction with a descent direction, which can be viewed as a refinement and improvement of some existing projection descent methods. Convergence analysis of the proposed method is analyzed under some weak and suitable conditions. The numerical results showed that the new method is attractive for solving some practical problems.

10. Dynamical Systems

In this section, we consider the projected dynamical system associated with the general variational inequalities. Using the fixed-point formulation of the variational inequalities, Dupuis and Nagurney [26] introduced and considered the projected dynamical systems, which the right hand side of the ordinary differential equation is a projected operator associated with variational inequalities. The innovative and novel feature of a projected dynamical system is that its set of stationary points corresponds to the set of solutions of the corresponding variational inequality problem. Hence, equilibrium and nonlinear problems arising in various branches in pure and applied sciences, which can be formulated in the setting of the variational inequalities, can now be studied in the more general setting of dynamical systems. It has been shown [25,26,32,33,69,70,95,96,106,152,162] that the dynamical systems are useful in developing some efficient numerical techniques for solving variational inequalities and related optimization problems. In recent years, much attention has been given to study the globally asymptotic stability of these projected dynamical systems. In Section 3, we have shown that the general variational inequalities are equivalent to the fixed-point and the Wiener-Hopf equations. We use this equivalence to suggest and analyze the projected dynamical system associated with the variational inequalities (23.3).

$$\frac{du}{dt} = \lambda\{P_K[u - \rho Tu] - u\}, \quad u(t_0) = u_0 \in H, \tag{10.1}$$

where λ is a parameter. The system of type (10.1) is called the projected general dynamical system. Here the right hand side is related to the projection operator and is discontinuous on the boundary. It is clear from the definition that the solution to (10.1) always stays in the constraint set. This implies that the qualitative results such as the existence, uniqueness and continuous dependence of the solution on the given data can be studied.

Using the fixed-point formulation (23.3), we can suggest the following projected dynamical system

$$\frac{du}{dt} = \lambda\{P_K[u - \alpha\{\eta R(u) - \rho T(u - \eta R(u))\}] - u\}, \quad u(t_0) = u_0 \in H, \tag{10.2}$$

where α and η are positive constants. For $\alpha = 1$ and $\eta = 1$, we can obtain several new projected-type dynamical systems associated with variational inequalities, which are quite different from the previously known ones considered in [25,26,32,33,69,70,95,96,106,152,162].

From Lemma 4.1, it follows that the variational inequalities are equivalent to the Wiener-Hopf equations (2.17). This equivalence is used to suggest the following dynamical system associated with the variational inequalities (23.3) as:

$$\begin{aligned} \frac{du}{dt} &= \lambda\{P_K[u - \rho Tu] - \rho T P_K[u - \rho Tu] \\ &\quad + \rho Tu - u\}, \quad u(t_0) = u_0 \in H, \end{aligned} \tag{10.3}$$

which is called the Wiener-Hopf dynamical system. Note that the dynamical system (10.3) is different from (10.1) and (10.2). The dynamical system (10.3) was introduced and studied by Noor [106].

The equilibrium points of the dynamical system (10.1) are naturally defined as follows.

Definition 10.1. *An element $u \in K$ is an equilibrium point of the dynamical system (10.1), if $\frac{du}{dt} = 0$, that is,*

$$P_K[u - \rho Tu] - u = 0,$$

Thus it is clear that $u \in K$ is a solution of the variational inequality (23.3), if and only if, $u \in K$ is an equilibrium point. In a similar way, one can define the concept of equilibrium points for other dynamical systems.

Definition 10.2. *The dynamical system (10.1) is said to converge to the solution set S^* of (23.3), if , irrespective of the initial point, the trajectory of the dynamical system satisfies*

$$\lim_{t \to \infty} dist(u(t), S^*) = 0, \tag{10.4}$$

where

$$dist(u, S^*) = inf_{v \in S^*} \|u - v\|.$$

It is easy to see, if the set S^* has a unique point u^*, then (10.3) implies that

$$\lim_{t \to \infty} u(t) = u^*.$$

If the dynamical system is still stable at u^* in the Lyapunov sense, then the dynamical system is globally asymptotically stable at u^*.

Definition 10.3. *The dynamical system is said to be globally exponentially stable with degree η at u^*, if, irrespective of the initial point, the trajectory of the system satisfies*

$$\|u(t) - u^*\| \leq \mu_1 \|u(t_0) - u^*\| exp(-\eta(t - t_0)), \quad \forall \quad t \geq t_0,$$

where μ_1 and η are positive constants independent of the initial point. It is clear that the globally exponentially stability is necessarily globally asymptotically stable and the dynamical system converges arbitrarily fast.

We now study the main properties of the dynamical systems (10.3) and analyze the global stability of the systems. First of all, we discuss the existence and uniqueness of the dynamical system (10.3) and this is the main motivation of our next result.

Theorem 10.1. *Let the operator T be a Lipschitz continuous operator. Then, for each $u_0 \in H$, there exists a unique continuous solution $u(t)$ of dynamical system (10.3) with $u(t_0) = u_0$ over $[t_0, \infty]$.*

Proof. Let

$$G(u) = \lambda\{P_K[u - \rho Tu] - \rho T P_K[u - \rho Tu] + \rho Tu - u\},$$

where $\lambda > 0$ is a constant. $\forall u, v \in R^n$, we have

$$\begin{aligned} \|G(u) - G(v)\| &\leq \lambda\{\|P_K[u - \rho Tu] - P_K[v - \rho Tv] - \rho A P_K[u - \rho Tu] + \rho T P_K[v - \rho Tv]\| \\ &+ \rho\|Tu - Tv\| + \|u - v\|\} \\ &\leq \lambda\{(1 + \rho\beta)\|u - v\| + (1 + \rho\beta)\|(u - v) - \rho(Tu - Tv)\|\} \\ &\leq \lambda(1 + \rho\beta)(2 + \rho\beta)\|u - v\|, \end{aligned}$$

where $\beta > 0$ is a Lipschitz constant of the operator T. This implies that the operator $G(u)$ is a Lipschitz continuous in H. So, for each $u_0 \in H$, there exists a unique and continuous solution $u(t)$ of the dynamical system of (10.3), defined in a interval $t_0 \leq t < T_1$ with the initial condition $u(t_0) = u_0$. Let $[t_0, T_1)$ be its maximal interval of existence; we show that $T_1 = \infty$. Consider

$$\begin{aligned} \|G(u)\| &= \lambda\|P_K[u - \rho Tu] - \rho A P_K[u - \rho Tu]] + \rho Tu - u\| \\ &\leq \lambda\|P_K[u - \rho Tu] - u\| + \lambda\rho\beta\|P_K[u - \rho Tu] - u\| \\ &= \lambda(1 + \rho\beta)\|P_K[u - \rho Tu] - u]\| \\ &\leq \lambda(1 + \rho\beta)\{\|P_K[u - \rho Tu] - P_K[u]\| + \rho\|P_K[u] - P_K[u^*]\| + \|P_K[u^*] - u\|\} \\ &\leq \lambda(1 + \rho\beta)\{\rho\|Tu\| + \|u - u^*\| + \|p_K[u^*] - u\| \\ &\leq \lambda(1 + \rho\beta)\{(2 + \rho\beta)\|u\| + \|P_K[u^*]\| + \|u^*\|\} \\ &= \lambda(1 + \rho\beta)(2 + \rho\beta)\|u\| + \lambda(1 + \rho\beta)\{\|P_K[u^*]\| + \|u^*\|\}, \end{aligned}$$

for any $u \in R^n$, then

$$\begin{aligned} \|u(t)\| &\leq \|u_0\| + \int_{t_0}^{t} \|Tu(s)\| ds \\ &\leq (\|u_0\| + k_1(t - t_0)) + k_2 \int_{t_0}^{t} \|u(s)\| ds, \end{aligned}$$

where $k_1 = \lambda(1 + \rho\beta)\{\|u^*\| + \|P_K[u^*]\|\}$ and $k_2 = \lambda(2 + \rho\beta)(1 + \rho\beta)$. Hence, by invoking Gronwall Lemma [4], we have

$$\|u(t)\| \leq \{\|u_0\| + k_1(t - t_0)\}e^{k_2(t - t_0)}, \quad t \in [t_0, T_1]$$

This shows that the solution $u(t)$ is bounded on $[t_0, T_1)$. So $T_1 = \infty$. □

Theorem 10.2. *Let T be a pseudomonotone Lipschitz continuous operator. Then the Wiener-Hopf dynamical system (10.3) is stable in the sense of Lyapunov and globally converges to the solution subset of (23.3).*

Proof. Since the operator T is a Lipschitz continuous operator, it follows from Theorem 10.1, that the Wiener-Hopf dynamical system (8.3) has a unique continuous solution $u(t)$ over $[t_0, T_1)$ for any fixed $u_0 \in K$. Let $u(t) = u(t, t_0; u_0)$ be the solution of the initial value problem (10.3). For a given $u^* \in K$, consider the following Lyapunov function

$$L(u) = \|u - u^*\|^2, \quad u \in R^n. \tag{10.5}$$

It is clear that $\lim_{n \longrightarrow \infty} L(u_n) = +\infty$ when ever the sequence $\{u_n\} \subset K$ and $\lim_{n \longrightarrow \infty} u_n = +\infty$. Consequently, we conclude that the level sets of L are bounded. Let $u^* \in K$ be a solution of (23.3). Then

$$\langle Tu^*, v - u^* \rangle \geq 0, \forall v \in K,$$

which implies that

$$\langle Tv, v - u^* \rangle \geq 0, \forall v \in K, \tag{10.6}$$

since the operator T is pseudomonotone.

Taking $v = P_K[u - \rho Tu]$ in (10.6), we have

$$\langle TP_K[u - \rho Tu], P_K[P_K[u - \rho Tu] - u^*\rangle \geq 0. \tag{10.7}$$

Setting $v = u^*$, $u = P_K[u - \rho Tu]$, and $z = u - \rho Tu$ in (2.15), we have

$$\langle P_K[u - \rho Tu] - u + \rho Tu, u^* - P_K[u - \rho Tu]\rangle \geq 0. \tag{10.8}$$

Adding (10.7), (10.8) and using (23.4), we obtain

$$\langle -R(u) + \rho Tu - \rho TP_K[u - \rho Tu], u^* - u + R(u)\rangle \geq 0,$$

which implies that

$$\begin{aligned} \langle u - u^*, R(u) - \rho Tu + \rho TP_K[u - \rho Tu]\rangle &\geq \|R(u)\|^2 \\ &- \rho\langle R(u), Tu - TP_K[u - \rho Tu]\rangle \\ &\geq (1 - \delta\rho)\|R(u)\|^2, \end{aligned} \tag{10.9}$$

where we have used the fact that the operator T is Lipschitz continuous with constant $\delta > 0$. Thus, from (10.3) and (10.5), we have

$$\begin{aligned} \frac{d}{dt}L(u) &= \frac{dL}{du}\frac{du}{dt} \\ &= 2\lambda\langle u - u^*, -R(u) + \rho Tu - \rho TP_K[u - \rho Tu]\rangle \\ &\leq -2\lambda(1 - \delta\rho)\|R(u)\|^2 \leq 0. \end{aligned}$$

This implies that $L(u)$ is a global Lyapunov function for the system (10.3) and the Wiener-Hopf dynamical system (10.3) is stable in the sense of Lyapunov. Since $\{u(t) : t \geq t_0\} \subset K_0$, where $K_0 = \{u \in K : L(u) \leq L(u_0)\}$ and the function $L(u)$ is continuously differentiable on the bounded and closed set K, it follows from LaSalle's invariance principle that the trajectory will converge to Ω, the largest invariant subset of the following subset:

$$E = \{u \in K; \frac{dL}{dt} = 0\}.$$

Note that, if $\frac{dL}{dt} = 0$, then

$$\|u - P_K[u - \rho Tu]\|^2 = 0,$$

and hence u is an equilibrium point of the dynamical system (10.3), that is,

$$\frac{du}{dt} = 0.$$

Conversely, if $\frac{du}{dt} = 0$, then it follows that $\frac{dL}{dt} = 0$. Thus , we conclude that

$$E = \{u \in K : \frac{du}{dt} = 0\} = K_0 \cap K^*,$$

which is a nonempty, convex and invariant set contained in the solution set K^*. So

$$\lim_{t \longrightarrow \infty} \text{dis}(u(t), E) = 0.$$

Therefore, the dynamical system (10.3) converges globally to the solution set of (23.3). In particular, if the set $E = \{u^*\}$, then

$$\lim_{t \longrightarrow \infty} u(t) = u^*.$$

Hence the system (10.3) is globally asymptotically stable. □

Theorem 10.3. *Let the operator T be Lipschitz continuous with a constant $\beta > 0$. If $\lambda < 0$, then the Wiener-Hopf dynamical system (10.3) converges globally exponentially to the unique solution of the variational inequality (23.3).*

Proof. From Theorem 10.1, we see that there exists a unique continuously differentiable solution of the dynamical system (10.3) over $[t_0, \infty)$. Then

$$\begin{aligned}\frac{dL}{dt} &= 2\lambda\langle u(t)-u^*, P_K[u(t)-\rho Tu(t)] - \rho TP_K[u(t)-\rho Tu(t)] + \rho Tu(t) - u(t)\rangle \\ &= -2\lambda\|u(t)-u^*\|^2 \\ &+ 2\lambda\langle u(t)-u^*, P_K[u(t)-\rho Tu(t)] - \rho TP_K[u(t)-\rho Tu(t)] + \rho Tu(t) - u^*\rangle, \end{aligned} \quad (10.10)$$

where $u^* \in K$ is the solution of the variational inequality (23.3). Thus

$$u^* = P_K[u^* - \rho Tu^*] - \rho TP_K[u^* - \rho Tu^*] + \rho Tu^*.$$

Now, using the nonexpansivity of P_K and Lipschitz continuity of the operator T, we have

$$\begin{aligned}\|P_K[u-\rho Tu] - \rho TP_K[u-\rho Tu] + \rho Tu &- P_K[u^*-\rho Tu^*] + \rho TP_K[u^*-\rho Tu^*] - \rho Tu^*\| \\ &\le \|P_K[u-\rho Tu] - P_K[u^*-\rho Tu^*]\| \\ &\quad +\rho\|TP_K[u-\rho Tu] - TP_K[u^*-\rho Tu^*]\| \\ &\quad +\rho\|Tu - Tu^*\| \\ &\le (1+\rho\beta)\|P_K[u-\rho Tu] - P_K[u^*-\rho Tu^*]\| \\ &\quad +\rho\beta\|u-u^*\| \\ &\le (1+\rho\beta)\{\|u-u^*\| - \rho(Tu-Tu^*)\|\} \\ &\quad +\rho\beta\|u-u^*\| \\ &\le \{(1+\rho\beta)^2 + \rho\beta\}\|u-u^*\|. \end{aligned} \quad (10.11)$$

From (10.10) and (10.11), we have

$$\frac{d}{dt}\|u(t)-u^*\|^2 \le +2\alpha\lambda\|u(t)-u^*\|^2,$$

where

$$\alpha = (3+\rho\beta)\rho\beta.$$

Thus, for $\lambda = -\lambda_1$, where λ_1 is a positive constant, we have

$$\|u(t)-u^*\| \le \|u(t_0) - u\|e^{-\alpha\lambda_1(t-t_0)},$$

which shows that the trajectory of the solution of the dynamical system (10.3) will globally exponentially converge to the unique solution of the variational inequality (23.3). □

We now introduce the locally projected dynamical system for the variational inequalities (23.3), the origin of which can be traced to Dupuis and Nagurney [26].

Given $u \in K$, and $v \in R^n$, we define the projection of vector v at u (with respect K) by

$$P_K(u,v) := \lim_{\delta\to 0}\{\frac{P_K(u+\delta v) - u}{\delta}\},$$

where P_K is defined as

$$P_K(u) := \text{argmin}_{z\in K}\|u-z\|.$$

It is known that $P_K(u,v)$ defined above is equivalent to the Euclidean projection of v on the tangent cone to K at u, that is,

$$P_K(u,v) = P^{T_K(u)}(v)$$

where $T_K(u)$ denotes the tangent cone to K at u.

Without loss of generality, we denote by P^t the tangent projection of $-Tu$ at $u \in K$. Let $P^t(u)$ be the tangent projection of $-Tu$ at $u \in K$. Then

$$P^t[-Tu] := P^{T_K(u)}[-Tu] = \text{argmin}\{\|v+Tu\| : v \in T_K(u)\}.$$

The equation of the type

$$P^t[-Tu] := P^t(u) = 0 \tag{10.12}$$

is called the tangent projection equation [153]. It has been shown in [153] the variational inequality (23.3) are equivalent to the tangent projection equations (10.12). Using this equivalent formulation, we can suggest the following dynamical system for the variational inequalities (23.3).

$$\frac{du}{dt} = P^t[u] = 0, \qquad u(t_0) \in H.$$

For the classical variational inequalities (23.3), see Patriksson [129] and Pappalardo and Passacantando [128]. Using the concept of the above projection, we can define the locally projected dynamical system for the variational inequalities (23.3) as:

$$\frac{du}{dt} = P_K[u, -Tu], \qquad u(t_0) \in H, \tag{10.13}$$

where K is a closed convex set. The dynamical system (10.13) is exactly the same as introduced and considered by Dupuis and Nagurney [26].

We use the projected dynamical system (10.1) to suggest some iterative for solving variational inequalities (23.3). These methods can be viewed in the sense of Koperlevich [7] and Noor [10] involving the double projection operator.

For simplicity, let $\lambda = 1$. Then the dynamical system(10.1) becomes

$$\frac{du}{dt} + u = P_K[u - \rho Tu], \quad u(t_0) = \alpha. \tag{10.14}$$

We construct the implicit iterative method using the forward difference scheme. Discretizing (10.14), we have

$$\frac{u_{n+1} - u_n}{h} + u_{n+1} = P_K[u_n - \rho T u_{n+1}], \tag{10.15}$$

where h is the step size. Now, we can suggest the following implicit iterative method for solving the variational inequality (23.3).

Algorithm 10.1. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = P_K\left[u_n - \rho T u_{n+1} - \frac{u_{n+1} - u_n}{h}\right], \quad n = 0, 1, 2, \ldots.$$

This is an implicit method and is quite different from the implicit method of [7]. Using Lemma 2.1, Algorithm 10.1 can be rewritten in the equivalent form as:

Algorithm 10.2. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$\langle \rho T u_{n+1} + \{\frac{(1+h)u_{n+1} - (1+h)u_n}{h}\}, v - u_{n+1}\rangle \geq 0, \forall v \in K. \tag{10.16}$$

We now study the convergence analysis of algorithm 10.1 under some mild conditions.

Theorem 10.4. *Let $u \in K$ be a solution of variational inequality (23.3). Let u_{n+1} be the approximate solution obtained from (10.16). If T is monotone, then*

$$\|u - u_{n+1}\|^2 \leq \|u - u_n\|^2 - \|u_n - u_{n+1}\|^2. \tag{10.17}$$

Proof. Let $u \in K$ be a solution of (23.3). Then

$$\langle Tv, v - u\rangle \geq 0, \quad \forall v \in K, \tag{10.18}$$

since T is a monotone operator.
Set $v = u_{n+1}$ in (10.18), to have

$$\langle Tu_{n+1}, u_{n+1} - u\rangle \geq 0. \tag{10.19}$$

Take $v = u$ in equation (10.17), we have

$$\langle \rho Tu_{n+1} + \{\frac{(1+h)u_{n+1} - (1+h)u_n}{h}\}, u - u_{n+1}\rangle \geq 0. \tag{10.20}$$

From (10.19) and (10.20), we have

$$\langle (1+h)u_{n+1} - (1+h)u_n, u - u_{n+1}\rangle \geq 0. \tag{10.21}$$

From (10.21) and using $2\langle a, b\rangle = \|a+b\|^2 - \|a\|^2 - \|b\|^2, \quad \forall a, b \in H$, we obtain

$$\|u_{n+1} - u\|^2 \leq \|u - u_n\|^2 - \|u_{n+1} - u_n\|^2. \tag{10.22}$$

the required result. □

Theorem 10.5. *Let $u \in K$ be the solution of variational inequality (23.3). Let u_{n+1} be the approximate solution obtained from (10.16). If T is a monotone operator, then u_{n+1} converges to $u \in H$ satisfying (23.3).*

Proof. Let T be a monotone operator. Then, from (10.16), it follows the sequence $\{u_i\}_{i=1}^{\infty}$ is a bounded sequence and

$$\sum_{i=1}^{\infty} \|u_n - u_{n+1}\|^2 \leq \|u - u_0\|^2,$$

which implies that

$$\lim_{n\to\infty} \|u_{n+1} - u_n\|^2 = 0. \tag{10.23}$$

Since sequence $\{u_i\}_{i=1}^{\infty}$ is bounded, so there exists a cluster point $\hat{u}$ to which the subsequence $\{u_{ik}\}_{k=1}^{\infty}$ converges. Taking limit in (10.16) and using (10.23), it follows that $\hat{u} \in K$ satisfies

$$\langle T\hat{u}, v - \hat{u}\rangle \geq 0, \quad \forall v \in K,$$

and

$$\|u_{n+1} - u\|^2 \leq \|u - u_n\|^2.$$

Using this inequality, one can show that the cluster point $\hat{u}$ is unique and

$$\lim_{n\to\infty} u_{n+1} = \hat{u}.$$

□

We now suggest an other implicit iterative method for solving (23.3).
Discretizing (10.14), we have

$$\frac{u_{n+1} - u_n}{h} + u_{n+1} = P_K[u_{n+1} - \rho Tu_{n+1}], \tag{10.24}$$

where h is the step size.
This formulation enable us to suggest the following iterative method.

Algorithm 10.3. *For a given $u_0 \in K$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = P_K\left[u_{n+1} - \rho Tu_{n+1} - \frac{u_{n+1} - u_n}{h}\right], \quad n = 0, 1, 2, \ldots.$$

Using lemma 2.1, Algorithm 10.3 can be rewritten in the equivalent form as:

Algorithm 10.4. *For a given $u_0 \in K$, compute u_{n+1} by the iterative scheme*

$$\langle \rho Tu_{n+1} + \{\frac{u_{n+1} - u_n}{h}\}, v - u_{n+1}\rangle \geq 0, \quad \forall v \in K. \tag{10.25}$$

Again using the dynamical systems, we can suggested some iterative methods for solving the variational inequalities and related optimization problems.

Algorithm 10.5. *For a given $u_0 \in K$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = P_K\left[\frac{(h+1)u_n - u_{n+1}}{h} - \rho T u_n\right], \quad n = 0, 1, 2, \ldots,$$

which can be written in the equivalent form as

Algorithm 10.6. *For a given $u_0 \in K$, compute u_{n+1} by the iterative scheme*

$$\langle \rho T u_n + \{\frac{h+1}{h}(u_{n+1} - u_n)\}, v - u_{n+1}\rangle \geq 0, \quad \forall v \in K. \tag{10.26}$$

For appropriate and suitable choice of the discretizing (10.14), one can suggest and analyze a wide class of iterative methods for solving variational inequalities. This is an interesting problem for future research.

We now suggest a second order projected dynamical system associated with the general variational inequalities (2.1) as:

$$\frac{d^2u}{dt^2} + \gamma\frac{du}{dt} = P_K[u - \rho T u] - u, \quad u(t_0) = u_0, \quad u'(t_0) = v_0 \in H. \tag{10.27}$$

One can show that the solution of the second order projected dynamical system (10.27) converges weakly to the solution of the variational inequality (23.3) using the technique of Attouch and Alvarez [7]. It is an open problem to study the asymptotic and stability analysis of the such type of the projected dynamical system in the context of variational inequalities.

11. Fractional Wiener-Hopf dynamical systems

Zeng et al [**?**] have investigated the fractional dynamical systems associated with linear variational inequalities. They have investigated the criteria for the asymtotically stability of the equilibrium points. In this section, we propose and consider a fractional Wiener-Hopf dynamical system associated with the variational inequalities. We show that the fractional Wiener-Hopf dynamical system is exponentially stable and converges to its unique equilibrium point. We would like to point out that our results are more general than the results of Zeng et al [**?**]. These ideas and techniques may inspire the interested readers for further research in this area.

We now define a residue vector $R(u)$ as:

$$R(u) = u - P_K[u - \rho T u]. \tag{11.1}$$

It is known that a solution $u \in K$, if and only if, $u \in K$ is a zero of the equation

$$R(u) = 0.$$

It is well known that the variational inequalities are also equivalent to Wiener Hopf equations. The Wiener Hopf equations technique has been used to develop some efficient and powerful iterative methods for solving variational inequalities and complementarity problems. We now use the Wiener Hopf equations technique to suggest and analyze another dynamical system. For the sake of completeness, we include all the details.

Let $Q_K = I - P_K$, where I is the identity operator and P_K is the projection of H onto the closed and convex set K. For given nonlinear operator $T : H \to H$, consider a problem of finding $z \in H$ such that

$$TP_K[z] + \rho^{-1}Q_K[z] = 0. \tag{11.2}$$

Problem (11.2) is called Wiener Hopf equations associated with the variational inequalities (23.3).

Lemma 11.1. *The problem (23.3) has a solution $u \in K$, if and only if, problem* (11.2) *has a solution $z \in H$, where*

$$u = P_K[z], \tag{11.3}$$

$$z = u - \rho T u, \tag{11.4}$$

where $\rho > 0$ is a constant.

Using Lemma 11.1, the Wiener-Hopf equation (11.2) can be written as

$$u - \rho T u - P_K[u - \rho T u] + \rho T P_K[u - \rho T u] = 0, \tag{11.5}$$

which is equivalent to

$$R(u) - \rho T u + \rho T P_K[u - \rho T u] = 0, \tag{11.6}$$

Thus it is clear from Lemma 11.1 that $u \in K$ is a solution of problem (23.3), if and only if, $u \in K$ satisfies the equation (11.6).

We now suggest a new fractional dynamical system

$$D_t^{\alpha} u(t) = \gamma \{-R(u) - \rho T P_K[u - \rho T u] + \rho T u\}, \quad u(t_0) = u_0 \in K, \tag{11.7}$$

where $0 < \alpha < 1$ and γ is a constant, associated with problem (23.3).

We now discuss some special cases of problem (11.7).

(1) If $\alpha = 1$, then the fractional Wiener-Hopf dynamical system (11.7) reduces to following dynamical system

$$\begin{aligned} \frac{du}{dt} &= \gamma \{P_K[u - \rho T u] - \rho T P_K[u - \rho T u] + \rho T u - u\}, \\ u(t_0) &= u_0 \in K, \end{aligned} \tag{11.8}$$

which was introduced and considered by Noor [**?**].

(2) In the affine case, that is, if $Tu = Au + b$, where $A = a_{ij}$ is a real $n \times n$ matrix and $b = b_j$ is an n-dimensional vector, then the system (11.7) reduces to:

$$\begin{aligned} D_t^{\alpha} u(t) &= \gamma \{P_K[u - \rho A u - \rho b] - \rho A P_K[u - \rho A u - \rho b] + \rho A u - u\}, \\ u(t_0) &= u_0 \in K, \end{aligned} \tag{11.9}$$

which is called linear fractional Wiener-Hopf dynamical system.

We also need the following well-known fundamental results and concepts.

Definition 11.1. *The fractional integral (or, Riemann-Liouville integral) with order $\alpha \in R_+$ of continuous function $u(t)$ is defined as:*

$$I_t^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - \tau)^{\alpha - 1} u(\tau)\, d\tau, \quad t > t_0.$$

Definition 11.2. *The Caputo derivative with order $\alpha \in R_+$ of continuous function $u(t) \in C^n([t_0, +\infty], R)$ is defined as:*

$$D_t^{\alpha} u(t) = I_t^{n-\alpha} u^{(n)}(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - \tau)^{n - \alpha - 1} u^{(n)}(\tau)\, d\tau, \quad t > t_0,$$

where n is positive integer such that $n - 1 < \alpha < n$.

Definition 11.3. *[?] The point u^* is said to be an equilibrium point of the fractional projected dynamical system* (11.1)*, if u^* satisfies the following*

$$P_K\left[u^*(t)-\rho T u^*(t)\right]-u^*(t)=0,\quad i=1,2,\ldots,n.$$

Definition 11.4. *The dynamical system* (11.1) *is said to be α-exponentially stable with degree λ at u^* if, for any two solutions $u(t)$ and $v(t)$ of* (11.1) *with different initial values by u_0 and v_0 satisfies*

$$\|u(t)-v(t)\|\leq\eta\|u_0-v_0\|e^{-\lambda t^\alpha},\quad\forall t\geq t_0,$$

where $\eta>0$ is a constant.

Lemma 11.2. *(Gronwall's Lemma) Let u and v be real valued non-negative continuous functions with domain $\{t:t\geq t_0\}$ and let $\alpha(t)=\alpha_0|t-t_0|$, where α_0 is a monotone increasing function. If, for $t\geq t_0$,*

$$u(t)\leq\alpha(t)+\int_{t_0}^{t}u(s)v(s)\,ds,$$

then

$$u(t)\leq\alpha(t)\cdot\exp\left(\int_{t_0}^{t}v(s)\,ds\right).$$

Lemma 11.3. *Let n is a positive integer such that $n-1<\alpha<n$. If $u(t)\in C^n[a,b]$, then*

$$I_t^\alpha\ D_t^\alpha u(t)=u(t)-\sum_{k=0}^{n-1}\frac{u^{(k)}(a)}{k!}(t-a)^k.$$

In particular, if $0<\alpha\leq1$ and $u(t)\in C^1[a,b]$

$$I_t^\alpha\ D_t^\alpha u(t)=u(t)-u(a).\tag{11.10}$$

Lemma 11.4. *Let $u(t)$ be a continuous function on $[0,+\infty)$ and satisfies*

$$D_t^\alpha u(t)\leq\theta.u(t),\tag{11.11}$$

where $0<\alpha<1$ and θ is a constant. Then

$$u(t)\leq u(0).\exp\left(\frac{\theta.t^\alpha}{\Gamma(\alpha+1)}\right).$$

Proof. For a nonnegative continuous function $h(t)$, relation (11.11) can be written as:

$$D_t^\alpha u(t)+h(t)=\theta.u(t).\tag{11.12}$$

By taking the fractional integral of order α of (11.12), we have

$$I_t^\alpha\ D_t^\alpha u(t)+I^\alpha h(t)=I^\alpha\theta.u(t).\tag{11.13}$$

Using Lemma 11.3, we have

$$I_t^\alpha\ D_t^\alpha u(t)=u(t)-u(0).\tag{11.14}$$

Since $h(t)$ is a nonnegative continuous function, therefore by the definition of fractional integral we have

$$I_t^\alpha h(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}h(\tau)\,d\tau\geq0,\quad t>0,\tag{11.15}$$

and

$$I_t^\alpha\theta.u(t)=\frac{\theta}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}u(\tau)\,d\tau,\quad t>0.\tag{11.16}$$

Combining (11.13) – (11.16), we have

$$u(t) - u(0) + 0 \leq \frac{\theta}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(\tau)\, d\tau, \quad t > 0,$$

which implies

$$\begin{aligned} u(t) &\leq u(0) + \frac{\theta}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(\tau)\, d\tau, \quad t > 0 \\ &\leq u(0) . \exp\left(\frac{\theta}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} d\tau \right) \\ &= u(0) . \exp\left(\frac{\theta . t^{\alpha}}{\Gamma(\alpha+1)} \right), \end{aligned}$$

where we have used Lemma 11.2. This is the desired result. □

Lemma 11.5. *[?] Consider the system*

$$D_t^{\alpha} u(t) = g(t, x), \quad t > t_0, \tag{11.17}$$

with initial condition $u(t_0)$, where $0 < \alpha \leq 1$ and $g : [t_0, \infty) \times \Omega \to H$, $\Omega \subset H$. If $g(t,x)$ satisfies the locally Lipschitz condition with respect to x, then there exists a unique solution of (11.1) *on* $[t_0, \infty) \times \Omega$.

Lemma 11.6. *[?] For the real valued continuous function $\tau(t,x)$, defined in* (11.17)*, we have $\|I_t^{\alpha} g(t,x)\| \leq I_t^{\alpha} \|g(t,x)\|$, where $\alpha \geq 0$ and $\|\cdot\|$ denotes an arbitrary norm.*

We now study the main properties of the dynamical systems (11.7) and analyze the global stability of the systems.

Theorem 11.1. *Let the operator T is Lipschitz continuous with constant $\beta > 0$. If $\gamma > 0$, then there exists a unique solution $u(t) \in H$ of problem (11.7) with $u(t_0) = u_0$, that is defined for all $t \in [t_0, \infty) \geq 0$.*

Proof. Let

$$\mathcal{G}(u(t)) = \gamma \{P_K [u(t) - \rho T u(t)] - \rho T P_K [u(t) - \rho T u(t)] + \rho T u(t) - u(t)\}.$$

To prove that $\mathcal{G}(u(t))$ is Lipschitz continuous for all $u(t), v(t) \in H$, we have to consider

$$\begin{aligned} &\|\mathcal{G}(u(t)) - \mathcal{G}(v(t))\| \\ &= \gamma \|P_K [u(t) - \rho T u(t)] - \rho T P_K [u(t) - \rho T u(t)] + \rho T u(t) - u(t) \\ &\quad - P_K [v(t) - \rho T v(t)] + \rho T P_K [v(t) - \rho T v(t)] - \rho T v(t) + v(t)\| \\ &\leq \gamma \|P_K [u(t) - \rho T u(t)] - P_K [v(t) - \rho T v(t)]\| + \gamma\rho \|T u(t) - T v(t)\| \\ &\quad + \gamma\rho \|T P_K [u(t) - \rho T u(t)] - T P_K [v(t) - \rho T v(t)]\| + \gamma \|u(t) - v(t)\| \\ &\leq \gamma \|u(t) - v(t) - \rho (T u(t) - T v(t))\| + \gamma\rho\beta \|u(t) - v(t)\| \\ &\quad + \gamma\rho\beta \|u(t) - v(t) - \rho (T u(t) - T v(t))\| + \gamma \|u(t) - v(t)\| \\ &\leq 2\gamma \|u(t) - v(t)\| + 3\gamma\rho\beta \|u(t) - v(t)\| + \gamma\rho^2\beta^2 \|u(t) - v(t)\| \\ &= \gamma \left(2 + 3\rho\beta + \rho^2\beta^2\right) \|u(t) - v(t)\|, \end{aligned}$$

where we have used Lipschitz continuity of operator T with constant $\beta > 0$. This implies that operator $G(u)$ is a Lipschitz continuous in H. Thus from Lemma 11.5, it is clear that there exists a unique solution $u(t)$ of problem (11.7). Let $[t_0, \infty)$ be its maximal interval of existence; we show that $T_1 = \infty$. Consider

$$\begin{aligned}
\|D_t^\alpha u(t)\| &= \|G(u(t))\| \\
&= \gamma \|P_K[u(t) - \rho T u(t)] - \rho T P_K[u(t) - \rho T u(t)] + \rho T u(t) - u(t)\| \\
&\leq \gamma \|P_K[u(t) - \rho T u(t)] - u(t)\| + \gamma\rho\beta \|P_K[u(t) - \rho T u(t)] - u(t)\| \\
&= \gamma(1+\rho\beta) \|P_K[u(t) - \rho T u(t)] - P_K[u(t)] \\
&\quad + P_K[u(t)] - P_K[u^*(t)] + P_K[u^*(t)] - u(t)\| \\
&\leq \gamma(1+\rho\beta) \{\|P_K[u(t) - \rho T u(t)] - P_K[u(t)]\| \\
&\quad + \|P_K[u(t)] - P_K[u^*(t)]\| + \|P_K[u^*(t)] - u(t)\|\} \\
&\leq \gamma(1+\rho\beta) \{\|u(t) - \rho T u(t) - u(t)\| + \|u(t) - u^*(t)\| \\
&\quad + \|P_K[u^*(t)]\| + \|u(t)\|\} \\
&\leq \gamma(1+\rho\beta) \{(2+\rho\beta)\|u(t)\| + \|u^*(t)\| + \|P_K[u^*(t)]\|\} \\
&= \gamma(1+\rho\beta)\{\|u^*(t)\| + \|P_K u^*(t)\|\} + \gamma(1+\rho\beta)(2+\rho\beta)\|u(t)\| \\
&= k_1 + k_2\|u(t)\|, \qquad (11.18)
\end{aligned}$$

where we have used the Lipschitz continuity of operator T, with constant $\beta > 0$.

Taking the fractional integral of (11.18), we have

$$\begin{aligned}
I_t^\alpha \|D_t^\alpha u(t)\| &\leq I_t^\alpha \{k_1 + k_2\|u(t)\|\} \\
&= \frac{k_1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} d\tau + \frac{k_2}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} u(\tau)\, d\tau \\
&= \frac{k_1 (t-t_0)^\alpha}{\Gamma(\alpha+1)} + \frac{k_2}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} u(\tau)\, d\tau.
\end{aligned}$$

Using Lemma 11.3 and Lemma 11.6, we have

$$\|u(t) - u(t_0)\| \leq \frac{k_1 (t-t_0)^\alpha}{\Gamma(\alpha+1)} + \frac{k_2}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} u(\tau)\, d\tau.$$

From which by using Lemma 11.2, we obtain

$$\begin{aligned}
\|u(t)\| &\leq \left\{\|u(t_0)\| + \frac{k_1 (t-t_0)^\alpha}{\Gamma(\alpha+1)}\right\} + \frac{k_2}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} u(\tau)\, d\tau \\
&= \left\{\|u(t_0)\| + \frac{k_1 (t-t_0)^\alpha}{\Gamma(\alpha+1)}\right\} \exp\left\{\frac{k_2 (t-t_0)^\alpha}{\Gamma(\alpha+1)}\right\}, \qquad (11.19)
\end{aligned}$$

where

$$\begin{aligned}
k_1 &= \gamma(1+\rho\beta)\{\|u^*(t)\| + \|P_K u^*(t)\|\} \\
k_2 &= \gamma(1+\rho\beta)(2+\rho\beta) > 0.
\end{aligned}$$

Thus from (11.19), it follows that the solution is bounded on $[t_0, \infty)$. □

Theorem 11.2. *Let the operator T be pseudomonotone and Lipschitz continuous. If $\gamma > 0$, then the dynamical system (11.7) is stable in the sense of Lyapunov and globally converges to the solution of variational inequality (23.3).*

Proof. Since the operator is Lipschitz continuous, it follows from Theorem 11.1 that the dynamical system (11.7) has a unique continuous solution $u(t)$ over $[t, T_1)$ for any fixed $u_0 \in K$. Let $u(t, t_0; u_0)$ be he solution of the initial value problem (11.7). For a given $u^* \in K$, consider the following Lyapunov function

$$L(u(t)) = \|u(t) - u^*\|^2, \quad u \in K, \tag{11.20}$$

It is clear that $\lim_{n\to\infty} L(u_n) = \infty$, whenever the sequence $\{u_n\} \subset K$ and $\lim_{n\to\infty} u_n = \infty$. Consequently, we conclude that the level sets of L are bounded. Let $u^* \in K$ be a solution of the variational inequality (23.3). Then

$$\langle Tu^*, v - u^* \rangle \geq 0, \quad \forall v \in K,$$

which implies, using pseudomonotonicity of operator T,

$$\langle Tv, v - u^* \rangle \geq 0, \quad \forall v \in K. \tag{11.21}$$

Setting $v = P_K[u - \rho Tu]$ in (11.21), we have

$$\langle TP_K[u - \rho Tu], P_K[u - \rho Tu] - u^* \rangle \geq 0. \tag{11.22}$$

Now, setting $v = u^*$, $u = P_K[u - \rho Tu]$ and $z = u - \rho Tu$ in (2.17), we have

$$\langle P_K[u - \rho Tu] - u + \rho Tu, u^* - P_K[u - \rho Tu] \rangle \geq 0. \tag{11.23}$$

Adding (11.22)and (11.23), we have

$$\langle P_K[u - \rho Tu] - u + \rho Tu - \rho TP_K[u - \rho Tu], u^* - P_K[u - \rho Tu] \rangle \geq 0, \tag{11.24}$$

using (11.1), (11.24) can be written as

$$\langle -R(u) + \rho Tu - \rho TP_K[u - \rho Tu], u^* - u + R(u) \rangle \geq 0,$$

which implies that

$$\begin{aligned} \langle u - u^*, R(u) - \rho Tu + \rho TP_K[u - \rho Tu] \rangle &\geq \|R(u)\|^2 - \rho \langle R(u), Tu - TP_K[u - \rho Tu] \rangle \\ &\geq \|R(u)\|^2 - \rho \|R(u)\| \|Tu - TP_K[u - \rho Tu]\| \\ &\geq \|R(u)\|^2 - \rho\delta \|R(u)\| \|u - P_K[u - \rho Tu]\| \\ &= (1 - \rho\delta) \|R(u)\|^2, \end{aligned} \tag{11.25}$$

where we have used the fact that the operator T is Lipschitz continuous with constant $\delta > 0$.

Thus from (11.7), (11.20) and (11.25), we have

$$\begin{aligned} D_t^\alpha L(u(t)) &= (D_u^\alpha L(u(t)))(D_t^\alpha u(t)) \\ &= \frac{\Gamma(3)}{\Gamma(3 - \alpha)} \langle u - u^*, D_t^\alpha u \rangle \\ &= \frac{2\gamma}{\Gamma(3 - \alpha)} \langle u - u^*, -R(u) + \rho Tu - \rho TP_K[u - \rho Tu] \rangle \\ &\leq \frac{-2\gamma(1 - \rho\delta)}{\Gamma(3 - \alpha)} \|R(u)\|^2 \leq 0. \end{aligned}$$

This implies that $L(u)$ is a global Lyapunov function for the system (11.7) and the system is stable in the sense of Lyapunov. Since $\{u(t) : t \geq t_0\} \subset K_0$, where $K_0 = \{u \in K : L(u) \leq L(u_0)\}$ and the function is differentiable on the bounded and closed set K, then it follows from LaSalle's invariance principle that the trajectory will converge to Ω, the largest invariant subset of the following subset:

$$E = \{u \in K : D_t^\alpha L(u(t)) = 0\}.$$

Note that, if $D_t^\alpha L(u(t)) = 0$, then

$$\|u - P_K[u - \rho Tu]\| = 0,$$

and hence u is the equilibrium point of the dynamical system (11.1), that is,

$$D_t^\alpha u(t) = 0.$$

Conversely, if $D_t^\alpha u(t) = 0$, then it follows that $D_t^\alpha L(u(t)) = 0$. Thus, we conclude that

$$E = \{u \in K : D_t^\alpha u(t) = 0\} = K_0 \cap K^*,$$

which is nonempty, convex and invariant set containing the solution set K^*. So

$$\lim_{t\to\infty} \operatorname{dist}(u(t), E) = 0.$$

Therefore the dynamical system (11.7) converges globally to the solution set of the variational inequalities (23.3). In particular, if the set $E = \{u^*\}$, then

$$\lim_{t\to\infty} u(t) = u^*.$$

Hence the dynamical system (11.7) is globally asymptotically stable. □

We now discuss the stability and existence of the equilibrium point for the dynamical system (11.7) under some suitable conditions.

Theorem 11.3. *Let the operator T be Lipschitz continuous with constant $\beta > 0$. If $\gamma < 0$, then fractional dynamical system* (11.7) *is α-exponentially stable.*

Proof. Let $u(t)$ and $v(t)$ be any two solutions of dynamical system (11.7) with initial values $u(0) = u_0$ and $v(0) = v_0$ respectively.

Let

$$e(t) = u(t) - v(t),$$

then $e(0) \neq 0$ and taking the fractional derivative of above equation, we have

$$\begin{aligned}
D_t^\alpha e(t) &= D_t^\alpha u(t) - D_t^\alpha v(t) \\
&= \gamma\{P_K[u(t) - \rho Tu(t)] - \rho TP_K[u(t) - \rho Tu(t)] + \rho Tu(t) - u(t)\} \\
&\quad -\gamma\{P_K[v(t) - \rho Tv(t)] - \rho TP_K[v(t) - \rho Tv(t)] + \rho Tv(t) - v(t)\} \\
&= \gamma\{P_K[u(t) - \rho Tu(t)] - P_K[v(t) - \rho Tv(t)]\} + \gamma\rho\{Tu(t) - Tv(t)\} \\
&\quad -\gamma\rho\{TP_K[u(t) - \rho Tu(t)] - TP_K[v(t) - \rho Tv(t)]\} - \gamma e(t), \qquad (11.26)
\end{aligned}$$

where $0 < \alpha < 1$, $t \geq 0$. From Theorem 11.3, $u(t)$ and $v(t)$ are uniquely determined solutions. Therefore $e(t)$ is the uniquely determined solution of error system (11.26) with initial value $e(0) = e_0$.

We claim that if $e(0) > 0$, then $e(t) \geq 0$ for $t \geq 0$, if $e(0) < 0$, then $e(t) \leq 0$ for $t \geq 0$. In fact, if $e(0) > 0$, there exists t_1, such that $e(t) < 0$ for $t \geq t_1$, so there must be $0 < t_0 < t_1$ such that $e(t_0) = 0$. It means that dynamical system (11.7) has two different solutions with initial value t_0 to $e(t_0) \leq 0$ for $t \geq t_0$, which contradicts to Theorem 11.3. In a similar way, we can prove that if $e(0) < 0$, then $e(t_0) \leq 0$ for $t \geq 0$.

So, we can have

$$\begin{aligned}
D_t^\alpha \mathcal{G}(t) &= D_t^\alpha \|e(t)\| \\
&= sgn(e(t))(D_t^\alpha e(t)) \\
&= \gamma sgn(e(t))\{P_K[u(t) - \rho Tu(t)] - P_K[v(t) - \rho Tv(t)]\} \\
&\quad -\gamma\rho sgn(e(t))\{TP_K[u(t) - \rho Tu(t)] - TP_K[v(t) - \rho Tv(t)]\} \\
&\quad +\gamma\rho sgn(e(t))\{Tu(t) - Tv(t)\} - \gamma sgn(e(t))e(t) \\
&\leq \gamma\|P_K[u(t) - \rho Tu(t)] - P_K[v(t) - \rho Tv(t)]\| \\
&\quad +\gamma\rho\|TP_K[u(t) - \rho Tu(t)] - TP_K[v(t) - \rho Tv(t)]\| \\
&\quad +\gamma\rho\|Tu(t) - Tv(t)\| + \gamma\|e(t)\| \\
&\leq \gamma(1 + \rho\beta)\|u(t) - v(t) - \rho(Tu(t) - Tv(t))\| \\
&\quad +\gamma\rho\beta\|u(t) - v(t)\| + \gamma\|e(t)\| \\
&\leq \gamma(1 + \rho\beta)\|u(t) - v(t)\| + \gamma\rho\beta(1 + \rho\beta)\|u(t) - v(t)\| + \gamma(1 + \rho\beta)\|e(t)\| \\
&= \gamma(1 + \rho\beta)(2 + \rho\beta)\|e(t)\|,
\end{aligned}$$

which implies

$$\begin{aligned} D_t^{\alpha}\mathcal{G}(t) &\leq \gamma(1+\rho\beta)(2+\rho\beta)\mathcal{G}(t) \\ &= \theta_1\mathcal{G}(t), \end{aligned}$$

thus by using Lemma 11.4, we have

$$\mathcal{G}(t) \leq \mathcal{G}(0)\exp\left(\frac{\theta_1 t^{\alpha}}{\Gamma(\alpha+1)}\right),$$

where

$$\theta_1 = \gamma(1+\rho\beta)(2+\rho\beta) < 0.$$

Let $\theta_1 = -\theta_2$, where θ_2 is a positive constant. Then

$$\mathcal{G}(t) \leq \mathcal{G}(0)\exp\left(\frac{-\theta_2 t^{\alpha}}{\Gamma(\alpha+1)}\right),$$

which shows that the dynamical system (11.7) is α-exponentially stable. □

12. Merit Functions

In recent years, much attention has been given to reformulate the variational inequality as an optimization problem. A function which can constitute an equivalent optimization problem is called a merit (gap) function. Merit functions turn out to be very useful in designing new globally convergent algorithms and in analyzing the rate of convergence of some iterative methods. Various merit (gap) functions for variational inequalities and complementarity problems have been suggested and proposed by many authors, see [7-14] and the references therein. Error bounds are functions which provide a measure of the distance between a solution set and an arbitrary point. Therefore, error bounds play an important role in the analysis of global or local convergence analysis of algorithms for solving variational inequalities. In this section, we consider normal residue merit functions, regularized merit functions, difference merit functions and dual merit functions for variational inequalities and some other related aspects. We also obtain error bounds for the solution of the variational inequalities under some weaker conditions. Our results can be viewed as refinement of the previously known results for variational inequalities.

Remark 12.1. *We would like to point out that, if the operator T is strongly monotone with a constant $\alpha > 0$, then*

$$\alpha\|u-v\|^2 \leq \langle Tu - Tv, u - v\rangle \leq \|Tu - Tv\|\|u-v\|,$$

implies that

$$\|Tu - Tv\| \geq \alpha\|u-v\|, \quad \forall u, v \in H.$$

In this case, we say that the operator T is strongly nonexpanding with a constant $\alpha > 0$. Note that the strongly monotonicity implies nonexpandicity, but not conversely.

We now studies those conditions under which the variational inequality (23.3) has a unique solution, which is the main motivation our next result.

Theorem 12.1. *Let T be a strongly monotone with constant $\alpha > 0$ and Lipschitz continuous operator with constant $\beta > 0$. If there exists a constant $\rho > 0$ such that*

$$0 < \rho < \frac{2\alpha}{\beta^2}, \tag{12.1}$$

then the variational inequality (23.3) has a unique solution.

Proof. **(a). Uniqueness.**

Let $u_1 \neq u_2 \in H$ be two solutions of (23.3). Then, we have

$$\begin{aligned} \langle Tu_1, v - u_1 \rangle &\geq 0, \quad \forall v \in K, & (12.2) \\ \langle Tu_2, v - u_2 \rangle &\geq 0, \quad \forall v \in K. & (12.3) \end{aligned}$$

Taking $v = u_2$ in (12.2) and $v = u_1$ in (12.3), adding the resultants, we have

$$\langle Tu_1 - Tu_2, u_1 - u_2 \rangle \leq 0.$$

Since T is strongly g-monotone and g is strongly nonexpanding, there exists a constant $\alpha > 0$, such that

$$\alpha \|u_1 - u_2\|^2 \leq \langle Tu_1 - Tu_2, u_1 - u_2 \rangle \leq 0,$$

which implies that $u_1 = u_2$, the uniqueness of the solution of (23.3).

(b). Existence. We now use the auxiliary principle technique to prove the existence of a solution of (23.3). For a given $u \in K$ satisfying (23.3), we consider the problem of finding a unique $w \in K$ such that

$$\langle \rho Tu + w - u, v - w \rangle \geq 0, \quad \forall v \in K, \tag{12.4}$$

where $\rho > 0$ is a constant.

The inequality of type (12.4) is called the auxiliary variational inequality associated with the problem (23.3). It is clear that the relation (12.4) defines a mapping $u \longrightarrow w$. It is enough to show that the mapping $u \longrightarrow w$ defined by the relation (12.4) has a fixed point belonging to H satisfying the variational inequality (23.3).

Let $w_1 \neq w_2$ be two solutions of (12.4) related to $u_1, u_2 \in H$ respectively. It is sufficient to show that for a well chosen $\rho > 0$,

$$\|w_1 - w_2\| \leq \theta \|u_1 - u_2\|,$$

with $0 < \theta < 1$, where θ is independent of u_1 and u_2. Taking $v = w_2$(respectively w_1) in (12.4) related to u_1 (respectively u_2), adding the resultant, we have

$$\langle w_1 - w_2, w_1 - w_2 \rangle \leq \langle u_1 - u_2 - \rho(Tu_1 - Tu_2), w_1 - w_2 \rangle,$$

from which we have

$$\begin{aligned} \|w_1 - w_2\|^2 &\leq \|u_1 - u_2) - \rho(Tu_1 - Tu_2)\|^2 \\ &\leq \|u_1 - u_)\|^2 - 2\rho\alpha\|u_1 - u_2\|^2 + \rho^2\beta^2\|u_1 - u_2\|^2 \\ &\leq (1 - 2\rho\alpha + \rho^2\beta^2)\|u_1 - u_2\|^2, \end{aligned}$$

since T is both strongly monotone and Lipschitz continuous operator with constants $\alpha > 0$ and $\beta > 0$ respectively. Consequently, we have

$$\|w_1 - w_2\| \leq \theta \|u_1 - u_2\|,$$

where

$$\theta = \sqrt{(1 - 2\rho\alpha + \rho^2\beta^2)}.$$

From (12.1), it follows that $\theta < 1$ showing that the mapping defined by (12.4) has a fixed point belonging to H, which is the solution of (23.3), the required result. □

We note that if the operator T is symmetric and positive, then the solution of the auxiliary variational inequality (12.4) is equivalent to finding the minimum of the functional $I[w]$ on the closed convex set K in H, where

$$I[w] = \frac{1}{2}\langle w - u, w - u \rangle + \rho \langle Tu, w - u \rangle, \quad \forall u \in H \tag{12.5}$$

which is a differentiable functional associated with the inequality (12.4). This auxiliary functional can be used to construct a gap (merit) function, whose stationary points solve the variational inequality (23.3). In fact, one can easily show that the variational inequality (23.3) is equivalent to the optimization problem. This approach is used to suggest and analyze some descent iterative methods for solving variational inequalities.

Definition 12.1. *A function $M : H \longrightarrow R\cup\{+\infty\}$ is called a merit (gap) function for the variational inequalities (23.3), iff,*
(i). $M(u) \geq 0, \quad \forall u \in H$.
(ii). $M(\overline{u}) = 0$, *if and only if,* $\overline{u} \in H$ *is the solution of the variational inequality (23.3).*

We now consider some merit functions associated with the variational inequalities (23.3). Using these merit functions, we obtain some error bounds for problem (23.3).
We now consider the residue vector

$$R_\rho(u) \equiv R(u) := u - P_K[u - \rho Tu]. \tag{12.6}$$

It is clear that the problem (23.3) has a solution $u \in H$, if and if, $u \in H$ is a root of the equation

$$R(u) = 0. \tag{12.7}$$

It is known that the normal residue vector $R(u)$ defined by the relation (12.6) is merit function for the variational inequality (23.3). We use this merit function to derive the global error bounds for the solution of (23.3).

Theorem 12.2. *Let $\overline{u} \in H$ be a solution of (23.3). If the operator T is both strongly monotone and Lipschitz continuous with constants $\alpha > 0$ and $\beta > 0$, respectively, then*

$$(1/k_1)\|R(u)\| \leq \|u - \overline{u}\| \leq k_2\|R(u)\|, \quad \forall u \in H, \tag{12.8}$$

where k_1, k_2 are generic constants.

Proof. Let $\overline{u} \in K$ be solution of (23.3). Then

$$\langle T\overline{u}, v - \overline{u}\rangle \quad \geq 0, \quad \forall v \in K. \tag{12.9}$$

Taking $v = P_K[u - \rho Tu]$ in (12.9), we have

$$\langle T\overline{u}, P_K[u - \rho Tu] - \overline{u}\rangle \geq 0. \tag{12.10}$$

Letting $u = P_K[u - \rho Tu], \quad z = u - \rho Tu$ and $v = \overline{u}$ in (2.17), we have

$$\langle \rho Tu + P_K[u - \rho Tu] - u, \overline{u} - P_K[u - \rho Tu]\rangle \geq 0. \tag{12.11}$$

Adding (12.10) and (12.11), we have

$$\langle T\overline{u} - Tu + (1/\rho)(u - P_K[u - \rho Tu]), P_K[u - \rho Tu] - \overline{u}\rangle \geq 0. \tag{12.12}$$

Since T is a strongly monotone, there exists a constant $\alpha > 0$, such that

$$\begin{aligned}
\alpha\|\overline{u} - u\|^2 &\leq T\overline{u} - Tu, \overline{u} - u\rangle \\
&= \langle T\overline{u} - Tu, \overline{u} - P_K[u - \rho Tu]\rangle + \langle T\overline{u} - Tu, P_K[u - \rho Tu] - u\rangle \\
&\leq (1/\rho)\langle u - P_K[u - \rho Tu], P_K[u - \rho Tu] - u \\
&\quad +u - \overline{u}\rangle + \langle T\overline{u} - Tu, P_K[u - \rho Tu] - u\rangle \\
&\leq -(1/\rho)\|R(u)\|^2 + (1/\rho)\|R(u)\|\|u - \overline{u}\| \\
&\quad +\|T\overline{u} - Tu\|\|R(u)\| \\
&\leq (1/\rho)(1 + \beta)\|R(u)\|\|\overline{u} - u\| \\
&\leq (1/\rho)(1 + \beta)\|R(u)\|\|\overline{u} - u\|,
\end{aligned}$$

which implies that

$$\|\overline{u} - u\| \leq k_2 \|R(u)\|, \tag{12.13}$$

the right-hand inequality in (12.8) with $k_2 = (1/\alpha)(1+\beta)$.

Also, we have

$$\begin{aligned}
\|R(u)\| &= \|u - P_K[u - \rho T u]\| \\
&= \|u - \overline{u} + P_K[\overline{u} - \rho T\overline{u}] - P_K[u - \rho T u]\| \\
&\leq \|u - \overline{u}\| + \|u - \overline{u} + \rho(Tu - T\overline{u})\| \\
&\leq \{2 + \rho\beta\}\|u - \overline{u}\| = k_1\|u - \overline{u}\|,
\end{aligned}$$

from which, we have

$$(1/(k_1)\|R(u)\| \leq \|u - \overline{u}\|, \tag{12.14}$$

the left-most inequality in (12.8) with $k_1 = (2 + \rho\beta)$.

Combining (12.14) and (12.14), we obtain the required (12.8). □

Letting $u = 0$ in (12.8), we have

$$(1/k_1)\|R(0)\| \leq \|\overline{u}\| \leq k_2\|R(0)\|. \tag{12.15}$$

Combining (12.8) and (12.15), we obtain a relative error bounds for any point $u \in H$.

Theorem 12.3. *Assume that all the assumptions of Theorem 12.2 hold. If $0 \neq \overline{u} \in H$ is the unique solution of (23.3), then*

$$c_1\|R(u)\|/\|R(0)\| \leq \|u - \overline{u}\|/\|\overline{u}\| \leq c_2\|R(u)\|/\|R(0)\|.$$

Note that the normal residue vector (merit function) $R(u)$ defined by (12.7) is nondifferentiable. To overcome the nondifferentiability, which is a serious drawback of the normal residue merit function, we consider another merit function associated with problem (23.3), which can be viewed as a regularized merit function. From (12.4), we have

$$M_\rho(u) := \max_{v \in K}\{\langle Tu, u - v\rangle - (1/2\rho)\|u - v\|^2\}, \quad u \in K. \tag{12.16}$$

The function $M_\infty(u)$ is commonly called the gap (merit) function associated with the variational inequality (2.1). The function $M_\infty(u)$ has the serious drawback that it is in general nondifferentiable even if T is differentiable and may not be finite-valued. On the other hand, the function $M_\rho(u)$ which is called the regularized merit function, is finite-valued everywhere and is differentiable whenever T is differentiable. which is a mainly due to Fukushima (Ref. 15).

We note that the function $M_\rho(u)$ can be written as

$$\begin{aligned}
M_\rho(u) &= \langle Tu, u - P_K[u - \rho Tu]\rangle \\
&\quad -(1/2\rho)\|u - P_K[u - \rho Tu]\|^2, \quad \forall u \in K.
\end{aligned} \tag{12.17}$$

We now show that the function $M_\rho(u)$ defined by (12.17) is a merit function and this is the main motivation of our next result.

Theorem 12.4. *$\forall u \in H$ and $\rho < 1$, we have*

$$M_\rho(u) \geq (1/2\rho)\|R(u)\|^2. \tag{12.18}$$

Clearly $M_\rho(u) \geq 0, \forall u \in H$. In particular, we have $M_\rho(u) = 0$, if and only if, $u \in H$ is a solution of (23.3).

Proof. Setting $v = u, \quad u = P_K[u - \rho Tu]$ and $z = u - \rho Tu$ in (2.17), we have

$$\langle \rho Tu - (u - P_K[u - \rho Tu]), u - P_K[u - \rho Tu]\rangle \geq 0,$$

which implies that

$$\langle Tu, R(u)\rangle \geq (1/\rho)\|R(u)\|^2. \tag{12.19}$$

Combining (12.14) and (12.19)), we have

$$\begin{aligned} M_\rho(u) &= \langle Tu, R(u)\rangle - (1/2\rho)\|R(u)\|^2 \\ &\geq (1/\rho)\|R(u)\|^2 - (1/2\rho)\|R(u)\|^2 \\ &= (1/2\rho)\|R(u)\|^2, \end{aligned}$$

the required result (12.18). Clearly we have $M_\rho(u) \geq 0, \quad \forall u \in K$.

Now if $M_\rho(u) = 0$, then clearly $R(u) = 0$. Hence by Lemma 3.1, we see that $u \in H$ is a solution of (23.3). Conversely, if $u \in H$ is a solution of (23.3), then $u = P_K[u-\rho Tu]$ by Lemma 3.1. Consequently, we see that $M_\rho(u) = 0$, the required result. □

From Theorem 3.3, we see that the function $M_\rho(u)$ defined by (12.17) is a merit function for the variational inequalities (2.1). It is clear that the regularized merit function is differentiable whenever T is differentiable.
We now derive the error bounds without using the Lipschitz continuity of the T.

Theorem 12.5. *Let $\overline{u} \in H$ be a solution of (23.3). Let T be a strongly monotone with a constant $\alpha > 0$. Then*

$$M_\rho(u) \geq (2\rho)/(2\rho\alpha - 1)\|u - \overline{u}\|^2, \quad \forall u \in H. \tag{12.20}$$

Proof. From (12.17) and the strongly monotonicity of T, we have

$$\begin{aligned} M_\rho(u) &\geq \langle Tu, u - \overline{u}\rangle - (1/2\rho)\|u - \overline{u}\|^2 \\ &\geq \langle T\overline{u}, u - \overline{u}\rangle + \alpha\|u - \overline{u}\|^2 - (1/2\rho)\|u - \overline{u}\|^2. \end{aligned} \tag{12.21}$$

Taking $v = u$ in (3.5), we have

$$\langle T\overline{u}, u - \overline{u}\rangle \geq 0. \tag{12.22}$$

From (12.21),(12.22), we have

$$\begin{aligned} M_\rho(u) &\geq \alpha\|u - \overline{u}\|^2 - (1/2\rho)\|u - \overline{u}\|^2 \\ &= (\alpha - 1/2\rho)\|u - \overline{u}\|^2, \end{aligned}$$

from which the result (12.22) follows. □

We consider another merit function associated with variational inequalities (23.3), which can be viewed as a difference of two regularized merit functions. Such type of the merit functions functions were introduced and studied by many authors for solving variational inequalities and complementarity problems. Here we define the D-merit function by a formal difference of the regularized merit function defined by (3.12). To this end, we consider the following function

$$\begin{aligned} D_{\rho,\mu}(u) &= \max_{v \in H}\{\langle Tu, u - v\rangle + (1/2\mu)\|u - v\|^2 \\ &\quad -(1/2\rho)\|u - v\|^2\}, \quad \forall v \in H \end{aligned} \tag{12.23}$$

which is called the D-merit function associated with the variational inequalities (23.3). The differentiability of $D_{\rho,\mu}(u)$ immediately follows from that of T.

The D-merit function defined by (12.23) can be written as

$$\begin{aligned} D_{\rho,\mu}(u) &= \langle Tu, P_K[u-\mu Tu] - P_K[u-\rho Tu]\rangle \\ &\quad +(1/2\mu)\|u - P_K[u-\mu Tu]\|^2 \\ &\quad -(1/2\rho)\|u - P_K[u-\rho Tu]\|^2 \\ &= \langle Tu, R_\rho(u) - R_\mu(u)\rangle + (1/2\mu)\|R_\mu(u)\|^2 \\ &\quad -(1/2\rho)\|R_\rho(u)\|^2, \quad u \in H. \end{aligned} \tag{12.24}$$

It is clear that the $D_{\rho,\mu}(u)$ is everywhere finite. We now show that the function $D_{\rho,\mu}(u)$ defined by (12.24) is indeed a merit function for the variational inequalities (23.3) and this is the main motivation of our next result.

Theorem 12.6. $\forall u \in H, \rho > \mu > 0$, *we have*

$$(\rho - \mu)\|R_\rho(u)\|^2 \geq 2\rho\mu D_{\rho,\mu}(u) \geq (\rho-\mu)\|R_\mu(u)\|^2. \tag{12.25}$$

In particular, $D_{\rho,\mu}(u) = 0$, if and only if, $u \in H$ solves problem (23.3).

Proof. Taking $v = P_K[u-\mu Tu], u = P_K[u-\rho Tu]$ and $z = u - \rho Tu$ in (2.17), we have

$$\langle P_K[u-\rho Tu] - u + \rho Tu, P_K[u-\mu Tu] - P_K[u-\rho Tu]\rangle \geq 0,$$

which implies that

$$\langle Tu, R_\rho(u) - R_\mu(u)\rangle \geq (1/\rho)\langle R_\rho(u), R_\rho(u) - R_\mu(u)\rangle. \tag{12.26}$$

From (12.24) and (12.26), we have

$$\begin{aligned} D_{\rho,\mu}(u) &\geq (1/\rho)\langle R_\rho(u), R_\rho(u) - R_\mu(u)\rangle + (1/2\mu)\|R_\mu(u)\|^2 \\ &\quad -(1/2\rho)\|R_\rho(u)\|^2 \\ &= (1/2)(1/\mu - 1/\rho)\|R_\mu(u)\|^2 + (1/\rho)\langle R_\rho(u), R_\rho(u) - R_\mu(u)\rangle \\ &\quad +(1/2\rho)\|R_\rho(u) - R_\mu(u)\|^2 - (1/\rho)\langle R_\mu(u), R_\rho(u) - R_\mu(u)\rangle \\ &= (1/2)(1/\mu - 1/\rho)\|R_\mu(u)\|^2 + (1/2\rho)\|R_\rho(u) - R_\mu(u)\|^2 \\ &\geq (1/2)(1/\mu - 1/\rho)\|R_\mu(u)\|^2, \end{aligned} \tag{12.27}$$

which implies the right-most inequality in (12.25).

In a similar way, by taking $u = P_K[u-\mu Tu], z = u - \mu Tu$ and $v = P_K[u-\mu Tu]$ in (2.17), we have

$$\langle P_K[u-\mu Tu] - u + \mu Tu, P_K[u-\rho Tu] - P_K[u-\mu Tu]\rangle \geq 0,$$

which implies that

$$\langle Tu, R_\rho(u) - R_\mu(u)\rangle \geq (1/\mu)\langle R_\mu(u), R_\mu(u) - R_\rho(u)\rangle. \tag{12.28}$$

Consequently, from (12.24) and (12.28), we obtain

$$\begin{aligned} D_{\rho,\mu}(u) &\leq (1/\mu)\langle R_\mu(u), R_\rho(u) - R_\mu(u)\rangle + (1/2\mu)\|R_\mu(u)\|^2 \\ &\quad -(1/2\rho)\|R_\rho(u)\|^2 \\ &= (1/2)(1/\mu - 1/\rho)\|R_\rho(u)\|^2 - (1/2\mu)\|R_\rho(u) - R_\mu(u)\|^2 \\ &\leq (1/2)(1/\mu - 1/\rho)\|R_\rho(u)\|^2, \end{aligned} \tag{12.29}$$

which implies the left-most inequality in (12.25).

Combining (12.27) and (12.29, we obtain the required result. □

Using essentially the technique of Theorem 12.6, we can obtain the following result.

Theorem 12.7. *Let $\overline{u} \in K$ be a solution of (23.3). If the operator T is strongly monotone with constant $\alpha > 0$, then*

$$\|u - \overline{u}\|^2 \leq (2\mu\rho)((2\alpha\mu + 1)\rho - \mu)D_{\rho,\mu}, \quad \forall u \in H. \tag{12.30}$$

Proof. Let $\overline{u} \in H$ be a solution of (23.3). Then,

$$\langle T\overline{u}, v - \overline{u}\rangle \geq 0, \forall v \in K.$$

Taking $v = u$ in the above inequality, we have

$$\langle T\overline{u}, -g(\overline{u})\rangle \geq 0. \tag{12.31}$$

Also from (12.24), (12.31) and strongly monotonicity of T, we have

$$\begin{aligned} D_{\rho,\mu}(u) &\geq \langle Tu, u - \overline{u}\rangle + (1/2\mu)\|u - \overline{u}\|^2 - (1/2\rho)\|u - \overline{u}\|^2 \\ &\geq \langle T\overline{u}, u - \overline{u}\rangle + \alpha\|u - \overline{u}\|^2 \\ &\quad +(1/2\mu)\|u - \overline{u}\|^2 - (1/2\rho)\|u - \overline{u}\|^2 \\ &\geq (\alpha + (1/2\mu) - (1/2\rho))\|u - \overline{u}\|^2, \end{aligned}$$

from which the required result (12.30) follows. □

We now consider the *Dual merit functions* for the variational inequalities (23.3) and obtain some error bounds for the solution of the variational inequalities (23.3). For simplicity and without loss of generality, we define

$$\varphi(u, v) := \langle Tu, u - v\rangle, \quad \forall u, v \in H. \tag{12.32}$$

Using this notation, regularized merit function M_ρ can be written as

$$\begin{aligned} M_\rho(u) &= \max_{v \in H: g(v) \in K}\{\langle Tu, u - v\rangle - (1/2\rho)\|u - v\|^2\} \\ &= \max_{v \in H: v \in K}\{\varphi(u, v) - (1/2\rho)\|u - v\|^2\}. \end{aligned} \tag{12.33}$$

Clearly

$$M(u) = \max_{v \in H: v \in K}\{\langle Tu, u - v\rangle\}, \tag{12.34}$$

which is also a merit function for the variational inequalities (23.3). It is clear that the merit function $M(u)$ is not differentiable. The merit function $M(u)$ defined by (12.34) can be viewed as an extension of the merit function considered by Auslender [25] for variational inequalities (23.3). Essentially using the technique of [24], we consider the dual regularized merit function associated with the variational inequality, which is defined as

$$F_\mu(u) := \max_{v \in K}\{-\varphi(v, u) + (1/2\mu)\|u - v\|^2\}, \quad \forall u \in K, \tag{12.35}$$

where $\mu > 0$ is a constant. Clearly $F_\mu(u) \geq 0, \quad \forall u \in H$.

In particular, If T is a pseudomonotone, then the merit function $F(u)$ defined by (12.35) is nonnegative on K and vanishes at any solution of the variational inequality (23.3). Following the technique of Nguyen and Dupuis [24], one can easily prove the following result.

Theorem 12.8. *Let T be pseudomonotone. Then $u \in K$ is a solution of problem (23.3), if and only if, $F_\mu(u) = 0, \forall u \in K$.*

We now show that the functions $F_\mu(u)$ defined by (12.35) is indeed a merit function for the variational inequality (23.3) and this is the main motivation of our next result.

Theorem 12.9. *Let the operator T be strongly monotone with constant $\alpha > 0$. Then $u \in K$ is a solution of (23.3), if and only if, $F_\mu(u) = 0$.*

Proof. Since T is strongly monotone with constant $\alpha > 0$, we have

$$\begin{aligned}\varphi(u,v) &= \langle Tu, u-v\rangle \\ &\geq \langle Tv, v-u\rangle + \alpha\|v-u\|^2 \\ &= -\varphi(v,u) + \alpha\|v-u\|^2 \\ &\geq -\varphi(v,u) + (1/2\mu)\|v-u\|^2 \\ &\geq -\varphi(v,u), \quad \forall u,v \in K,\end{aligned}$$

which implies that

$$M(u) \geq F_\mu(u) \geq F(u), \quad \forall u \in K. \tag{12.36}$$

Let $\overline{u} \in K$ be a solution of (23.3). Then, from Theorem 12.8 and Theorem 12.7, we have $M(\overline{u}) = 0$ and $F(\overline{u}) = 0$. Thus it follows that $F_\mu(\overline{u}) = 0$.

Conversely, let $F_\mu(u) = 0$. Clearly $F_\mu(u) \geq 0$ on K and it follows from (12.35) that $F(u) = 0$. Thus $u \in K$ is a solution of (23.3). □

We now obtain the upper error bound for the dual merit function $F_\mu(u)$.

Theorem 12.10. *Let T be strongly monotone with a constant $\alpha > 0$. Then*

$$F_\mu(u) \geq \alpha\|u-\overline{u}\|^2, \quad \forall u \in K,$$

where $\overline{u} \in H$ is a solution of (23.3).

Proof. Let $\overline{u} \in H$ be a solution of (23.3). Then by Theorem 12.7, it follows that $F_\mu(\overline{u}) = 0$. Let $u \in H$ be an arbitrary. Then

$$\begin{aligned}F_\mu(u) &= \max_{v\in K}\{\langle Tv, u-v\rangle + (1/2\mu)\|v-u\|^2\} \\ &\geq \langle T\overline{u}, v-\overline{u}\rangle + (1/2\mu)\|v-u\|^2 + \alpha\|u-\overline{u}\|^2 \\ &\geq \alpha\|u-\overline{u}\|^2, \quad \forall u,\overline{u} \in H.\end{aligned}$$

□

Using essentially the previous techniques and ideas, we can construct and analyze the dual difference for the variational inequalities (23.3).

Applications. In this section we show that the obtained results can be extended for a class of quasi variational inequalities. If the convex set K depends upon the solution explicitly or implicitly, then variational inequality problem is known as the quasi variational inequality. For a given operator $T : H \longrightarrow H$, and a point-to-set mapping $K : u \longrightarrow K(u)$, which associates a closed convex-valued set $K(u)$ with any element u of H, we consider the problem of finding $u \in K(u)$ such that

$$\langle Tu, v-u\rangle \quad \geq 0, \quad \forall v \in K(u). \tag{12.37}$$

Inequality of type (12.37) is called the quasi variational inequality. To convey an idea of the applications of the quasi variational inequalities, we consider the second-order implicit obstacle boundary value problem of finding u such that

$$\left.\begin{array}{ll} -u'' \geq f(x) & \text{on } \Omega = [a,b] \\ u \geq M(u) & \text{on } \Omega = [a,b] \\ [-u'' - f(x)][u - M(u)] = 0 & \text{on } \Omega = [a,b] \\ u(a) = 0, \quad u(b) = 0. & \end{array}\right\} \tag{12.38}$$

where $f(x)$ is a continuous function and $M(u)$ is the cost (obstacle) function. The prototype encountered is

$$M(u) = k + \inf_i\{u^i\}. \tag{12.39}$$

In (12.39), k represents the switching cost. It is positive when the unit is turned on and equal to zero when the unit is turned off. Note that the operator M provides the coupling between the unknowns $u = (u^1, u^2, \dots, u^i)$, see [26]. We study the problem (12.38) in the framework of quasi variational inequality approach. To do so, we first define the set $K(u)$ as

$$K(u) = \{v : v \in H_0^1(\Omega) : v \geq M(u), \quad \text{on } \Omega\},$$

which is a closed convex-valued set in $H_0^1(\Omega)$, where $H_0^1(\Omega)$ is a Sobolev (Hilbert) space. One can easily show that the energy functional associated with the problem (4.2) is

$$\begin{aligned} I[v] &= -\int_a^b \left(\frac{d^2v}{dx^2}\right) v dx - 2\int_a^b f(x)\,(v)\,dx, \quad \forall v \in K(u) \\ &= \int_a^b \left(\frac{dv}{dx}\right)^2 dx - 2\int_a^b f(x)\,(v)\,dx \\ &= \langle Tv, v\rangle - 2\langle f, v\rangle \end{aligned} \tag{12.40}$$

where

$$\begin{aligned} \langle Tu, v\rangle &= \int_a^b \left(\frac{d^2u}{dx^2}\right)(v)\,dx = \int_a^b \frac{du}{dx}\frac{dv}{dx}dx \qquad (12.41) \\ \langle f, v\rangle &= \int_a^b f(x)(v)dx. \end{aligned}$$

It is clear that the operator T defined by (12.41) is linear, symmetric and positive. Using the technique of Noor [2], one can show that the minimum of the functional $I[v]$ defined by (12.40) associated with the problem (12.38) on the closed convex-valued set $K(u)$ can be characterized by the inequality of type

$$\langle Tu, v-u\rangle \geq \langle f, v-u\rangle, \quad \forall v \in K(u), \tag{12.42}$$

which is exactly the quasi variational inequality (12.37). See also [3,6-20] for the formulation, applications, numerical methods and sensitivity analysis of the quasi variational inequalities.

13. Penalty Function Method

It is well known that variational inequalities arise as a result of minimization of the convex programming problem. The analogies with convex programming has been used to suggest many numerical methods for solving variational inequalities including the penalty method. Penalty methods reduce a constrained problem to a sequence of problems for which the standard numerical methods can be applied. However, the penalty methods are not efficient due to the inherent instabilities. Here, we use another technique, which is very much similar to the penalty methods, but it does not have an inherent instability. This technique is called the penalty function method, which is due to Lewy and Stampacchia [55]. This technique has been used in [1-5, 118,123,124, 127] for solving variational inequalities associated with obstacle boundary value problems. In this method, the solution is required to satisfy some extra continuity conditions on the subintervals in addition to the usual boundary conditions. Due to the extra continuity conditions at the subintervals, all the numerical methods including collocation, finite difference and spline methods which approximate the solution at the knots give first order accurate approximations to the solution regardless the order of the methods being used. The present work aims to compare the methods that have been used for solving such a system of differential equations. Roughly speaking these methods fall into two categories: (1) methods based on consistency relations connecting the values of the solution at the knots and the corresponding values of the second derivative (CRK methods), and (2) methods with consistency relations connecting the values of the solution at the midknots and the corresponding values of the second derivative (CRMK methods). All the results in this section are due to Noor and Al-Said [124].

We now convey an idea of the penalty function method. Now using the penalty function technique of Lewy and Stampacchia [55], the variational inequality (2.1) can be characterized by a system of variational equations such as

$$\langle Tu, v\rangle + \langle\{\nu(u-\psi)\}(u-\psi), v\rangle = 0, \quad \forall v \in H, \tag{13.1}$$

where $\nu(t)$ is the discontinuous function defined by

$$\nu(t) = \begin{cases} 1, & \text{for } t \geq 0, \\ 0, & \text{for } t < 0, \end{cases} \tag{13.2}$$

is known as the penalty function and $\psi \leq 0$ on the boundary is the obstacle function.

We consider a general system of second order boundary value problem of the type

$$u'' = \begin{cases} f(x), & a \leq x \leq c, \\ g(x)u(x) + f(x) + r, & c \leq x \leq d, \\ f(x), & d \leq x \leq b, \end{cases} \tag{13.3}$$

with the boundary conditions

$$u(a) = \alpha_1 \quad \text{and} \quad u(b) = \alpha_2, \tag{13.4}$$

and the continuity conditions of u and u' at c and d. Here, f and g are continuous functions on $[a, b]$ and $[c, d]$, respectively. The parameters r, α_1 and α_2 are real finite constants. Such type of systems arises in the study of obstacle, unilateral, moving and free boundary value problems and has important applications in other branches of pure and applied sciences, see[1-5, 118, 123, 127]. We will discuss and compare some collocation, finite difference and spline methods for solving the system of differential equations (9.3) over the whole interval $[a, b]$. Without loss of generality, we may take $c = \frac{1}{4}(3a + b)$ and $d = \frac{1}{4}(a + 3b)$. For this purpose we discretize the interval $[a, b]$ using the equally spaced grid points

$$\begin{array}{l} x_i = a + ih, \quad i = 0, 1, 2, \ldots, 4n \\ x_0 = a, \quad x_{4n} = b \quad \text{and} \quad h = \frac{b-a}{4n}, \end{array}$$

where n is a positive integer. Also, let $u(x)$ be the exact solution of (9.3) and w_i be an approximation to $u_i = u(x_i)$, for $i = 1, 2, \ldots, 4n - 1$ obtained by the numerical method being used.

CRK Methods: In these methods, we compute the numerical solution using grid values of the second derivative at the knots.

(i) *Collocation method.* The second order cubic spline collocation method developed by Noor and Khalifa [127] is based on the B-splines

$$\check{B}_i(x) = \begin{cases} (x - x_{i-2})^3, & \text{for } x \in [x_{i-2}, x_{i-1}] \\ h^3 + 3h^2(x - x_{i-1}) + 3h(x - x_{i-1})^2 - 3(x - x_{i-1})^3, & \text{for } x \in [x_{i-1}, x_i] \\ h^3 + 3h^2(x_{i+1} - x) + 3h(x_{i+1} - x)^2 - 3(x_{i+1} - x)^3, & \text{for } x \in [x_i, x_{i+1}] \\ 0, & \text{otherwise,} \end{cases}$$

along with the following modifications

$$\begin{aligned} B_0(x) &= \check{B}_0(x) + 3\check{B}_{-1}(x), \quad B_1(x) = \check{B}_1(x) + \check{B}_{-1}(x) \\ B_n(x) &= \check{B}_n(x) + \check{B}_{n+2}(x), \quad B_{n+1}(x) = \check{B}_{n+1}(x) + 3\check{B}_{n+2}(x), \quad B_i(x) = \check{B}_i(x) \quad i = 2, 3, \ldots, n-1. \end{aligned}$$

They compute the approximate solution

$$\check{u}(x) = \sum_{i=0}^{n+1} C_i B_i(x) \tag{13.5}$$

for problem (13.3), where the coefficients C_i are determined by solving a linear system of equations. Their numerical results indicate that (13.5) is a first order approximation to the solution of (13.3).

(ii) *Finite difference methods.* Noor and Tirmizi [123] have developed finite difference schemes based on the second order difference formula

$$h^2 u_i'' = u_{i-1} - 2u_i + u_{i+1} - \frac{h^4}{12} u_i^{(4)} \tag{13.6}$$

and the well known fourth order Numerov formula for solving problem (13.3). Their numerical experiments show that both methods produced approximate solutions with first order accuracy.

(iii) *Spline methods.* The use of second order cubic spline method

$$u_{i-1} - 2u_i + u_{i+1} = \frac{h^2}{6}[u_{i-1}'' + 4u_i'' + u_{i+1}''] - \frac{h^4}{12} u_i^{(4)} \tag{13.7}$$

and the fourth order quintic spline method

$$\left.\begin{array}{l} 4u_0 - 7u_1 + 2u_2 + u_3 = \frac{h^2}{12}[4u_0'' + 41u_1'' + 14u_2'' + u_3''] - \frac{h^6}{48}u_0^{(6)}, \qquad \text{for } i = 1, \\ u_{i-2} + 2u_{i-1} - 6u_i + 2u_{i+1} + u_{i+2} = \frac{h^2}{20}[u_{i-2}'' + 26u_{i-1}'' + 66u_i'' + 26u_{i+1}'' + u_{i+2}''] + \frac{h^6}{120}u_i^{(6)}, \\ \text{for } 2 \le i \le n-2, \\ u_{n-3} + 2u_{n-2} - 7u_{n-1} + 4u_n = \frac{h^2}{12}[u_{n-3}'' + 14u_{n-2}'' + 14u_{n-1}'' + u_n''] - \frac{h^6}{48}u_n^{(6)}, \qquad \text{for } i = n-1, \end{array}\right\} \tag{13.8}$$

has been used by Al-Said, Noor and Al-Shejari [5] to obtain smooth approximations to the solution of (13.3) and its first derivative. They have shown that the computed solutions for problem(13.3) are first order accurate approximations. The first order accuracy also hold for the approximations of the derivatives.

Also, Al-Said [3] has used the quadratic spline function

$$Q_i(x) = a_i(x - x_i)^2 + b_i(x - x_i) + c_i, \tag{13.9}$$

along with the hypotheses

$$Q_i(x_i) = w_i, \quad Q_i(x_{i+1}) = w_{i+1}, \quad Q_i''(x_i) = \frac{1}{2}[F_{i+1} + F_i],$$

to formulate the coefficients

$$a_i = \frac{1}{4}[F_{i+1} + F_i], \quad b_i = \frac{1}{h}[w_{i+1} - w_i] - \frac{h}{4}[F_{i+1} + F_i], \quad c_i = w_i,$$

for $i = 0, 1, 2, \dots, n-1$. These equations are then used to develop the consistency relation

$$w_{i-1} - 2w_i + w_{i+1} = \frac{1}{4}h^2[F_{i-1} + 4F_i + F_{i+1}], \tag{13.10}$$

for $i = 1, 2, \dots, n-1$, where

$$F_i = \begin{cases} f_i, & \text{for } 0 \le i \le \frac{n}{4} \text{ and } \frac{3n}{4} < i \le n, \\ g_i w_i + f_i + r, & \text{for } \frac{n}{4} < i \le \frac{3n}{4}, \end{cases}$$

The local truncation error for associated with (13.10) is $-\frac{1}{6}h^4 u_i^{(4)} + O(h^6)$. However, the numerical results indicate that the smooth approximations produced by this method for both the solution and its first derivative are first order approximations.

CRMK Methods : In these methods we use polynomial functions to develop consistency relations that connecting the values of the solution at the midknots and the corresponding values of the second derivatives. This phenomenon of the consistency relations helps overcome the difficulty of the continuity conditions in our problem. **(i)** The first method we discuss here is the quadratic spline method developed by Al-Said [2], where he used the quadratic spline polynomial (13.9) along with hypotheses

$$Q_i(x_{i+\frac{1}{2}}) = w_{i+\frac{1}{2}}, \quad Q_i'(x_i) = D_i, \quad Q_i''(x_{i+\frac{1}{2}}) = F_{i+\frac{1}{2}},$$

to formulate the coefficients

$$a_i = \frac{1}{2}F_{i+\frac{1}{2}}, \quad b_i = D_i, \quad c_i = w_{i+\frac{1}{2}} - \frac{1}{2}hD_i - \frac{1}{8}h^2F_{i+\frac{1}{2}},$$

for $i = 0, 1, 2, \ldots, n-1$, which are then used to develop the consistency relations

$$\left.\begin{aligned} &2w_0 - 3w_{\frac{1}{2}} + w_{\frac{3}{2}} = \tfrac{1}{8}h^2[\tfrac{8}{3}F_0 + F_{\frac{1}{2}} + \tfrac{7}{3}F_{\frac{3}{2}}], && \text{for } i = 1,\\ &w_{i+\frac{1}{2}} - 2w_{i-\frac{1}{2}} + w_{i-\frac{3}{2}} = \tfrac{1}{8}h^2[F_{i+\frac{1}{2}} + 6F_{i-\frac{1}{2}} + F_{i-\frac{3}{2}}], && \text{for } 2 \le i \le n-1,\\ &w_{n-\frac{3}{2}} - 3w_{n-\frac{1}{2}} + 2w_n = \tfrac{1}{8}h^2[\tfrac{7}{3}F_{n-\frac{3}{2}} + F_{n-\frac{1}{2}} + \tfrac{8}{3}F_n], && \text{for } i = n, \end{aligned}\right\} \tag{13.11}$$

where

$$9.12F_{i+\frac{1}{2}} = \begin{cases} f_{i+\frac{1}{2}}, & \text{for } 0 \le i < \dfrac{n}{4} \text{ and } \dfrac{3n}{4} \le i < n,\\ g_{i+\frac{1}{2}}w_{i+\frac{1}{2}} + f_{i+\frac{1}{2}} + r, & \text{for } \dfrac{n}{4} \le i < \dfrac{3n}{4}, \end{cases} \tag{13.12}$$

The local truncation errors associated with (9.10) are given by

$$t_i = \begin{cases} \frac{9}{64}h^4u_0^{(4)} + O(h^5), & i = 1,\\ \frac{1}{24}h^4u_i^{(4)} + O(h^5), & 2 \le i \le n-1,\\ \frac{9}{64}h^4u_n^{(4)} + O(h^5), & i = n. \end{cases}$$

(ii) In Al-Said [3], the cubic spline function

$$9.13P_i(x) = a_i(x-x_i)^3 + b_i(x-x_i)^2 + c_i(x-x_i) + d_i \tag{13.13}$$

with the hypotheses

$$P_i(x_{i+\frac{1}{2}}) = w_{i+\frac{1}{2}}, \quad P_i'(x_i) = D_i, \quad P_i''(x_{i+\frac{1}{2}}) = F_{i+\frac{1}{2}}, \quad P_i'''(x_i) = T_i,$$

was used to write

$$a_i = \frac{1}{6}T_i, \quad b_i = \frac{1}{2}F_{i+\frac{1}{2}} - \frac{1}{4}hT_i, \quad c_i = D_i, \quad d_i = w_{i+\frac{1}{2}} - \frac{1}{2}hD_i - \frac{1}{8}h^2F_{i+\frac{1}{2}} + \frac{1}{24}h^3T_i,$$

for $i = 0, 1, 2, \ldots, n-1$. These equations are then used to develop the consistency relations

$$\left.\begin{aligned} &2w_0 - 3w_{\frac{1}{2}} + w_{\frac{3}{2}} = \tfrac{1}{24}h^2[15F_{\frac{1}{2}} + 3F_{\frac{3}{2}}], && \text{for } i = 1,\\ 9.14 \quad &w_{i+\frac{1}{2}} - 2w_{i-\frac{1}{2}} + w_{i-\frac{3}{2}} = \tfrac{1}{24}h^2[F_{i+\frac{1}{2}} + 22F_{i-\frac{1}{2}} + F_{i-\frac{3}{2}}], && \text{for } 2 \le i \le n-1,\\ &w_{n-\frac{3}{2}} - 3w_{n-\frac{1}{2}} + 2w_n = \tfrac{1}{24}h^2[3F_{n-\frac{3}{2}} + 15F_{n-\frac{1}{2}}], && \text{for } i = n, \end{aligned}\right\} \tag{13.14}$$

where $F_{i+\frac{1}{2}}$, $i = 0, 1, 2, \ldots, n-1$ are as defined in (13.10).

The local truncation errors associated with (**??**) are given by

$$t_i = \begin{cases} \frac{1}{64}h^4u_0^{(4)} + O(h^5), & i = 1,\\ \frac{1}{24}h^4u_i^{(4)} + O(h^5), & 2 \le i \le n-1,\\ \frac{1}{64}h^4u_n^{(4)} + O(h^5), & i = n. \end{cases}$$

(iii) Al-Said and Noor [4] have used Taylor polynomial to develop the Modified Numerov scheme

9.15
$$\left.\begin{aligned} &2w_0 - 3w_{\frac{1}{2}} + w_{\frac{3}{2}} = \tfrac{1}{240}h^2[-2F_0 + 150F_{\frac{1}{2}} + 35F_{\frac{3}{2}} - 3F_{\frac{5}{2}}], && i = 1,\\ &w_{i+\frac{1}{2}} - 2w_{i-\frac{1}{2}} + w_{i-\frac{3}{2}} = \tfrac{1}{12}h^2[F_{i+\frac{1}{2}} + 10F_{i-\frac{1}{2}} + F_{i-\frac{3}{2}}], && 2 \le i \le n-1,\\ &w_{n-\frac{3}{2}} - 3w_{n-\frac{1}{2}} + 2w_n = \tfrac{1}{240}h^2[-3F_{n-\frac{5}{2}} + 35F_{n-\frac{3}{2}} + 150F_{n-\frac{1}{2}} - 2F_n], && i = n, \end{aligned}\right\} \quad (13.15)$$

where $F_0 = f(x_0)$, $F_n = f(x_n)$ and $F_{i+\frac{1}{2}}$, $i = 0, 1, 2, \ldots, n-1$ are as defined in (9.10).

The local truncation errors are given by

$$t_i = \begin{cases} \frac{2}{539}h^6 u_0^{(6)} + O(h^7), & i = 1,\\ -\frac{1}{240}h^6 u_i^{(6)} + O(h^7), & 2 \le i \le n-1,\\ \frac{2}{539}h^6 u_n^{(6)} + O(h^7), & i = n. \end{cases}$$

However, the numerical experiments indicate that this scheme gives second order approximations for the solution of (13.3) at midknots. When the values of $w_{i-\frac{1}{2}}$, $i = 1, 2, \ldots, n$ are computed using scheme (13.16), we can compute approximations to the solution $u(x)$ at x_i, $i = 1, 2, \ldots, n-1$ using the second order interpolation

$$w_i = \frac{1}{2}[w_{i+\frac{1}{2}} + w_{i-\frac{1}{2}}], \quad (13.16)$$

for $i = 1, 2, \ldots, n-1$, which are second order accurate approximations for the solution of (13.3) at the knots x_i, $i = 1, 2, \ldots, n-1$. Also, the numerical experiments show that this two stage method gives better results than the others. See section 4 for more details.

We now discuss the convergence criteria of the cubic spline methods.

We first let $\mathbf{U} = (u_{i+\frac{1}{2}})$, $\mathbf{S} = (w_{i+\frac{1}{2}})$, $\mathbf{C} = (\bar{c}_i)$, $\mathbf{T} = (t_i)$ and $\mathbf{E} = (e_{i+\frac{1}{2}})$ be n-dimensional column vectors. Here $e_{i+\frac{1}{2}} = u_{i+\frac{1}{2}} - w_{i+\frac{1}{2}}$ is the discretization error. Thus, we can write our the cubic spline method as follow

$$\mathbf{AU} = \mathbf{C} + \mathbf{T}, \quad (13.17)$$
$$\mathbf{AS} = \mathbf{C}, \quad (13.18)$$
$$\mathbf{AE} = \mathbf{T}, \quad (13.19)$$

where

$$\mathbf{A} = \mathbf{A}_0 + \frac{1}{24}h^2\mathbf{BG}, \quad (13.20)$$

$\mathbf{G} = \mathrm{diag}(g_{i-\frac{1}{2}})$, $i = 1, 2, \ldots, n$, with $g_{i-\frac{1}{2}} \neq 0$ for $\dfrac{n}{4} < i \le \dfrac{3n}{4}$ and $\mathbf{A}_0 = (a_{ij})$ is the tridiagonal matrix defined by

$$a_{ij} = \begin{cases} 3, & i = j = 1, n,\\ 2, & i = j = 2, 3, \ldots, n-1,\\ -1, & |i-j| = 1,\\ 0, & \text{otherwise}. \end{cases} \quad (13.21)$$

The tridiagonal matrix $\mathbf{B}$ is given by

$$\mathbf{B} = \begin{bmatrix} 3 & 15 & 0 & \dots & \dots & 0 \\ 1 & 22 & 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & & 0 & 1 & 22 & 1 \\ 0 & & & 0 & 15 & 3 \end{bmatrix}. \tag{13.22}$$

For the vector $\mathbf{C}$, we have

$$\bar{c}_i = \begin{cases} 2\alpha_1 - \frac{1}{24}h^2\left[\bar{F}_1\right], & i = 1, \\ -\frac{1}{24}h^2\left[\bar{F}_i\right], & 2 \le i < \frac{n}{4} \text{ and } \frac{3n}{4} + 1 < i \le n-1, \\ -\frac{1}{24}h^2\left[\bar{F}_i + r\right], & i = \frac{n}{4} \text{ and } \frac{3n}{4} + 1, \\ -\frac{1}{24}h^2\left[\bar{F}_i + 23r\right], & i = \frac{n}{4} + 1 \text{ and } \frac{3n}{4}, \\ -\frac{1}{24}h^2\left[\bar{F}_i + 24r\right], & \frac{n}{4} + 2 \le i \le \frac{3n}{4} - 1, \\ 2\alpha_2 - \frac{1}{24}h^2\left[\bar{F}_n\right], & i = n, \end{cases} \tag{13.23}$$

where $$\bar{F}_i = \begin{cases} 3f_{\frac{1}{2}} + 15f_{\frac{3}{2}}, & i = 1, \\ f_{i+\frac{1}{2}} + 22f_{i-\frac{1}{2}} + f_{i-\frac{3}{2}}, & 2 \le i \le n-1, \\ 15f_{n-\frac{3}{2}} + 3f_{n-\frac{1}{2}}, & i = n. \end{cases}$$

Note that the first equation of the linear system (13.18) is

$$3u_{\frac{1}{2}} - u_{\frac{3}{2}} = 2\alpha_1 - \frac{1}{24}h^2\left[3u''_{\frac{1}{2}} + 15u''_{\frac{3}{2}}\right] + t_1,$$

the ith equation is

$$-u_{i+\frac{1}{2}} + 2u_{i-\frac{1}{2}} - u_{i-\frac{3}{2}} = -\frac{1}{24}h^2\left[u''_{i+\frac{1}{2}} + 24u''_{i-\frac{1}{2}} + u''_{i-\frac{3}{2}}\right] + t_i, \quad \text{for } i = 2, 3, \dots, n-1,$$

and the nth equation is

$$-u_{n-\frac{3}{2}} + 3u_{n-\frac{1}{2}} = 2\alpha_2 - \frac{1}{24}h^2\left[15u''_{n-\frac{3}{2}} + 3u''_{n-\frac{1}{2}}\right] + t_n,$$

where $t_i, i = 1, 2, \dots, n$ are the local truncation errors given by

$$t_i = \begin{cases} \frac{1}{64}h^4u_0^{(4)} + O(h^5), & i = 1, \\ \frac{1}{24}h^4u_i^{(4)} + O(h^5), & 2 \le i \le n-1, \\ \frac{1}{64}h^4u_n^{(4)} + O(h^5), & i = n. \end{cases} \tag{13.24}$$

Thus,

$$\|\mathbf{T}\| = \frac{1}{24}h^4M_4, \quad M_4 = \max_x |u(x)^{(4)}|, \tag{13.25}$$

where $\|\cdot\|$ represent the ∞-norm in matrix vector.

Our main purpose now is to derive a bound on $\|\mathbf{E}\|$. From equation (9.19) we have

$$\mathbf{E} = \mathbf{A}^{-1}\mathbf{T} = (\mathbf{A}_0 + \frac{1}{24}h^2\mathbf{BG})^{-1}\mathbf{T} = (\mathbf{I} + \frac{1}{24}h^2\mathbf{A}_0^{-1}\mathbf{BG})^{-1}\mathbf{A}_0^{-1}\mathbf{T}. \tag{13.26}$$

Now using the technique of [3] and the fact that $\|\mathbf{B}\| = 24$ and $\|\mathbf{G}\| \leq |g(x)|$, we get

$$\|\mathbf{E}\| \leq \frac{\lambda M_4 h^2}{24[1 - \lambda|g(x)|]} \cong O(h^2), \tag{13.27}$$

where $\lambda = \frac{1}{8}[(b-a)^2 + h^2]$. The relation (9.26) shows that the cubic spline method is a second order convergent method.

To illustrate the application of the numerical method developed in the previous sections, we consider the obstacle boundary value problem of finding u such that

$$\left.\begin{array}{ll} u''(x) \geq f(x) & \text{on } \Omega = [0, \pi] \\ u(x) \geq \psi(x) & \text{on } \Omega = [0, \pi] \\ [u''(x) - f(x)]\,[u(x) - \psi(x)] = 0 & \text{on } \Omega = [0, \pi] \\ u(0) = u(\pi) = 0, & \end{array}\right\} \tag{13.28}$$

where $f(x)$ is a given force acting on the string and $\psi(x)$ is the elastic obstacle. It can be shown that, see [50], the problem (13.28) is equivalent to the variational inequality problem

$$a(u, v - u) \geq \langle f, v - u \rangle \quad \text{for all } v \in K.y \tag{13.29}$$

This equivalence has been used to study the existence of a unique solution of (9.27) see, for example, [8,50,73].

Following the idea and technique of Lewy and Stampacchia [55], the variational inequality (9.28) can be written as

$$\left.\begin{array}{ll} u'' - \nu\{(u - \psi)\}(u - \psi) = 0, & 0 < x < \pi, \\ u(0) = u(\pi) = 0, & \end{array}\right\} \tag{13.30}$$

where $\nu(t)$ is the penalty function defined in (9.2), and ψ is the given obstacle function defined by

$$\psi(x) = \begin{cases} -1, & \text{for } 0 \leq x \leq \frac{\pi}{4}, \\ 1, & \text{for } \frac{\pi}{4} \leq x \leq \frac{3\pi}{4}, \\ -1, & \text{for } \frac{3\pi}{4} \leq x \leq \pi. \end{cases} \tag{13.31}$$

From equations (13.30), (13.2) and (13.31), we obtain the following system of differential equations

$$u'' = \begin{cases} f, & \text{for } 0 \leq x \leq \frac{\pi}{4} \text{ and } \frac{3\pi}{4} \leq x \leq \pi, \\ u + f - 1, & \text{for } \frac{\pi}{4} \leq x \leq \frac{3\pi}{4}, \end{cases} \tag{13.32}$$

with the boundary conditions

$$u(0) = u(\pi) = 0, \tag{13.33}$$

and the condition of continuity of u and u' at $x = \dfrac{\pi}{4}$ and $\dfrac{3\pi}{4}$.

Example 13.1. *We consider the system of differential equation (9.31) when $f = 0$.*

$$u'' = \begin{cases} 0, & \textit{for } 0 \leq x \leq \frac{\pi}{4} \textit{ and } \frac{3\pi}{4} \leq x \leq \pi, \\ u - 1, & \textit{for } \frac{\pi}{4} \leq x \leq \frac{3\pi}{4}, \end{cases} \tag{13.34}$$

with the boundary conditions (13.33). The analytical solution for this problem (13.34) is

$$u(x) = \begin{cases} \frac{4}{\gamma_1} x, & 0 \leq x \leq \frac{\pi}{4}, \\ 1 - \frac{4}{\gamma_2} \cosh(\frac{\pi}{2} - x), & \frac{\pi}{4} \leq x \leq \frac{3\pi}{4}, \\ \frac{4}{\gamma_1}(\pi - x), & \frac{3\pi}{4} \leq x \leq \pi, \end{cases} \tag{13.35}$$

where $\gamma_1 = \pi + 4\coth\frac{\pi}{4}$ and $\gamma_2 = \pi\sinh\frac{\pi}{4} + 4\cosh\frac{\pi}{4}$. The problem (13.34) was solved using the methods described in this section with a variety of h values. The observed maximum errors (in absolute value) associated with u_i, are given in Table 9.1 and Table 9.2. When using the spline methods we also compute the maximum errors associated with u_i', and these errors are listed in Table 9.3 and Table 9.4.

Table 9.1: Observed maximum errors in absolute values associated with u_i.

h	Quadratic spline [1]	Cubic spline [2]	Modified Numerov [4]
$\pi/20$	2.20×10^{-3}	1.94×10^{-3}	1.65×10^{-3}
$\pi/40$	5.87×10^{-4}	4.99×10^{-4}	4.33×10^{-4}
$\pi/80$	1.51×10^{-4}	1.27×10^{-4}	1.11×10^{-4}

Table 9.2: Observed maximum errors in absolute values associated with u_i.

h	scheme (2.4) [123]	Numerov [123]	Quintic spline [127]	Colloc-cubic [127]
$\pi/20$	2.50×10^{-2}	2.32×10^{-2}	1.82×10^{-2}	1.40×10^{-2}
$\pi/40$	1.29×10^{-2}	1.21×10^{-2}	9.17×10^{-3}	7.71×10^{-3}
$\pi/80$	6.58×10^{-3}	6.17×10^{-3}	4.61×10^{-3}	4.04×10^{-3}

Table 9.3: Observed maximum errors in absolute values associated with u_i'.

h	Quadratic spline (2.8) [3]	Cubic spline (2.4) [5]	Quintic spline [5]
$\pi/20$	2.78×10^{-2}	2.75×10^{-2}	9.05×10^{-2}
$\pi/40$	1.40×10^{-2}	1.39×10^{-2}	4.69×10^{-3}
$\pi/80$	7.04×10^{-3}	7.02×10^{-3}	2.44×10^{-2}

Table 9.4: Observed maximum errors in absolute values associated with u_i'.

h	Quadratic spline (2.9) [1]	Cubic spline (2.12) [2]
$\pi/32$	1.18×10^{-3}	3.26×10^{-4}
$\pi/64$	3.03×10^{-4}	8.15×10^{-5}
$\pi/128$	6.89×10^{-5}	2.04×10^{-5}

14. Sensitivity Analysis

In recent years variational inequalities are being used as mathematical programming models to study a large number of equilibrium problems arising in finance, economics, transportation, operations research and engineering sciences. The behaviour of such equilibrium problems as a result of changes in the problem data is always of concern. In this section, we study the sensitivity analysis of the variational inequalities (23.3), that is, examining how solutions of such problems change when the data of the problems are changed. We like to mention that sensitivity analysis is important for several reasons. First, estimating problem data often introduces measurement errors, sensitivity analysis helps in identifying sensitive parameters that should be obtained with relatively high accuracy. Second, sensitivity analysis may help to predict the future changes of the equilibrium as a result of changes in the governing system. Third, sensitivity analysis provides useful information for designing or planning various equilibrium systems. Furthermore, from mathematical and engineering point of view, sensitivity analysis can provide new insight regarding problems being studied can stimulate new ideas and techniques for problem solving. due to these and other reasons, there has been increasing interest in studying the sensitivity analysis of variational inequalities and related optimization problems. Sensitivity analysis for variational inequalities has been studied by many authors including Dafermos [20], Kyparisis [52,53], Qiu and Magnanti [133], Yen [159], Noor and Noor [114], Moudafi and Noor [68] and Liu [58] using quite different techniques. The techniques suggested so far vary with the problem being studied. Dafermos [20] used the equivalence between the variational inequalities and the fixed-point problem to study the sensitivity analysis of the classical variational inequalities. This technique has been modified and extended by many authors for studying the sensitivity analysis of various other classes of variational inequalities. This approach has strong geometrical flavor. It is well known that the variational inequalities are equivalent to the Wiener-Hopf equations, see Noor [79,80]. This fixed-point equivalence is obtained by a suitable and appropriate rearrangement of the Wiener-Hopf equations. The Wiener-Hopf equation approach is quite general, flexible unified and provides us with a new technique to study the sensitivity analysis of variational inequalities without assuming the differentiability of the given data. Our analysis is in the spirit of Noor [83] and Noor and Noor [114].

We now consider the parametric versions of the problems (23.3) and (2.17). To be more precise, let M be an open subset of H in which the parameter λ takes values. Let $T(u,\lambda)$ be a given operator defined on $H \times M$ and takes values in H. Assume that $\{K_\lambda : \lambda \in M\}$ is a family of closed convex subsets of H. From now onward, we denote $T_\lambda(.) := T(.,\lambda)$ unless otherwise specified. The parametric general variational inequality problem is to find $(u,\lambda) \in H \times M$ such that

$$\langle T_\lambda(u), v - u \geq 0, \quad \forall v \in K_\lambda. \tag{14.1}$$

We also assume that the parametric general variational inequality (14.1) has a unique solution $\bar{u}$ for some $\bar{\lambda} \in M$.

Related to the parametric general variational inequality (14.1), we consider the parametric Wiener-Hopf equations. We consider the problem of finding $(z,\lambda) \in H \times M$ such that

$$T_\lambda P_{K_\lambda} z + \rho^{-1} Q_{K_\lambda} z = 0, \tag{14.2}$$

where $\rho > 0$ ia constant and $Q_{K_\lambda} \equiv I - P_{K_\lambda}$, is defined on the set of (z,λ) with $\lambda \in M$ and takes values in H. The equations of the type (10.2) are called the parametric Wiener-Hopf equations.

Using Lemma 4.1, one can easily establish the equivalence between problems (14.1) and (14.2).

Lemma 14.1. *The parametric variational inequality (14.1) has solution* $(u,\lambda \in H \times M$ *if and only if the parametric Wiener-Hopf equation (14.2) has a solution* (z,λ), *if*

$$u = P_{K_\lambda} z, \tag{14.3}$$

$$z = u - \rho T_\lambda(u). \tag{14.4}$$

From Lemma 014.1, we see that the problems (14.1) and (14.2) are equivalent. We use this equivalence to study the sensitivity analysis of the variational inequalities (14.1). We assume that for some $\bar{\lambda} \in M$, problem (14.2) has a unique solution $\bar{z}$ and X is a closure of a ball in H centered at $\bar{z}$. We want to investigate those conditions under which for each λ in a neighborhood of $\bar{z}$, problem (14.2) has a unique solution $z(\lambda)$ near $\bar{z}$ and the function $z(\lambda)$ is Lipschitz continuous and differentiable.

Definition 14.1. *let* $T_\lambda(.)$ *be an operator on* $X \times M$. *Then,* $\forall \lambda \in M, \quad u, v \in X$, *the operator* T_λ *is said to be:*

(a) locally strongly monotone, if there exists a constant $\alpha > 0$ *such that*

$$\langle T_\lambda(u) - T_\lambda(v), u - v\rangle \geq \alpha \|u - v\|^2.$$

(b) locally Lipschitz continuous, if there exists a constant $\beta > 0$ *such that*

$$\|T_\lambda(u) - T_\lambda(v)\| \leq \beta \|u - v\|.$$

From (a) and (b), it follows that $\alpha \leq \beta$.

We now consider the case, when the solutions of the parametric Wiener-Hopf equations (14.2) lie in the interior of X. Following the ideas and technique of Noor [83] and Noor and Noor [114], we consider the map

$$\begin{aligned} F_\lambda(z) &= P_{K_\lambda} z - \rho T_\lambda(u), \quad \forall \quad (z,\lambda) \in X \times M, \\ &= u - \rho T_\lambda(u), \end{aligned} \tag{14.5}$$

where

$$u = P_{K_\lambda} z. \tag{14.6}$$

We have to show that the map $F_\lambda(z)$ defined by (14.5) has a fixed point, which is solution of the Wiener-Hopf equation (14.2). We have to show that the map $F_\lambda(z)$ defined by (14.5) is a contraction map with respect to z uniformly in $\lambda \in M$.

Lemma 14.2. *Let $T_\lambda(.)$ be a locally strongly monotone with constant $\alpha > 0$ and locally Lipschitz continuous with constant $\beta > 0$. Then*

$$\|F_\lambda(z_1) - F_\lambda(z_2)\| \leq \theta\|z_1 - z_2\|,$$

where

$$\theta = \sqrt{1 - 2\alpha\rho + \beta^2\rho^2} < 1 \quad \textit{for} \quad 0 < \rho < \frac{2\alpha}{\beta^2}. \tag{14.7}$$

Proof. $\forall \quad z_1, z_2 \in H$, and $\lambda \in M$, we have

$$\begin{aligned} \|F_\lambda(z_1) - F_\lambda(z_2)\| &\leq \|u_1 - u_2 - (g(u_1) - g(u_2))\| \\ &\quad + \|u_1 - u_2 - \rho(T_\lambda(u_1) - T_\lambda(u_2))\|. \end{aligned} \tag{14.8}$$

Using the locally strongly monotonicity and locally Lipschitz continuity of the operator T, we have

$$\|u_1 - u_2 - \rho(T_\lambda(u_1) - T_\lambda(u_2))\|^2 \leq (1 - 2\rho\alpha + \beta^2\rho^2)\|u_1 - u_2\|^2. \tag{14.9}$$

Also, from (14.6), we have

$$\|u_1 - u_2\| = \|P_{K_\lambda} z_1 - P_{K_\lambda} z_2| \leq \|z_1 - z_2\|, \tag{14.10}$$

since P_{K_λ} is a nonexpansive operator.

From (14.6) and (14.12), we obtain

$$\|F_\lambda(z_1) - F_\lambda(z_2)\| \leq \sqrt{1 - 2\rho\alpha + \beta^2\rho^2}\|z_1 - z_2\| \leq \theta\|z_1 - z_2\|, \quad \text{using (14.7)}.$$

Thus $\theta = \sqrt{1 - 2\rho\alpha + \beta^2\rho^2} < 1$ for $0 < \rho < \frac{2\alpha}{\beta^2}$ and consequently, the map $F_\lambda(z)$ defined by (14.5) is a contraction map and has a fixed point $z(\lambda)$, which is a solution of the Wiener-Hopf equation (14.2). □

Remark 14.1. *From Lemma 14.2, we see that the map $F_\lambda(z)$ defined by (14.5) has a unique fixed point $z(\lambda)$, that is, $z(\lambda) = F_\lambda(z)$. Also, by assumption,, the function $\bar{z}$, for $\lambda = \bar{\lambda}$ is a solution of the parametric Wiener-Hopf equation (14.2). Again using Lemma 14.1, we see that $\bar{z}$, for $\lambda = \bar{\lambda}$, is a fixed point of $F_\lambda(z)$ and it is also a fixed point of $F_{\bar{\lambda}}(z)$. Consequently, we conclude that*

$$z(\bar{\lambda}) = \bar{z} = F_{\bar{\lambda}}(z(\bar{\lambda})).$$

Using Lemma 14.2 and technique of Noor and Noor [114], we can prove the continuity of the solution $z(\lambda)$ of the parametric Wiener-Hopf equation (14.2). We include its proof to convey an idea of the technique.

Lemma 14.3. *Let that the operator $T_\lambda(.)$ be a locally Lipschitz continuous with respect to the parameter λ. If the operators $T_\lambda(.)$ is locally Lipschitz continuous and the map $\lambda \longrightarrow P_{K_\lambda}$ is continuous (or Lipschitz continuous), then the function $z(\lambda)$ satisfying (14.2) is (Lipschitz) continuous at $\lambda = \bar{\lambda}$.*

Proof. For all $\lambda \in M$, invoking Lemma 14.2 and the triangle inequality, we have

$$\begin{aligned} \|z(\lambda) - z(\bar{\lambda})\| &\leq \|F_\lambda(z(\lambda)) - F_{\bar{\lambda}}(z(\bar{\lambda}))\| + \|F_\lambda(z(\bar{\lambda}) - F_{\bar{\lambda}}(z(\bar{\lambda}))\| \\ &\leq \theta\|z(\lambda) - z(\bar{z})\| + \|F_\lambda(z(\bar{\lambda})) - F_{\bar{\lambda}}(z(\bar{\lambda}))\|. \end{aligned} \tag{14.11}$$

From (14.5) and the fact that the operator $T_\lambda(.)$ is a locally Lipschitz continuous with respect to the parameter λ, we have

$$\begin{aligned} \|F_\lambda(z(\bar{\lambda})) - F_{\bar{\lambda}}(z(\bar{\lambda}))\| &= \|u(\bar{\lambda}) - u(\bar{\lambda}) + \rho(T_\lambda(u(\bar{\lambda})) - T_{\bar{\lambda}}(u(\bar{\lambda})))\| \\ &\leq \rho\mu\|\lambda - \bar{\lambda}\|. \end{aligned} \tag{14.12}$$

Combining (14.11) and (14.12), we obtain

$$\|z(\lambda) - z(\bar{\lambda})\| \leq \frac{\rho\mu}{1-\theta}\|\lambda - \bar{\lambda}\|, \quad \forall \lambda, \bar{\lambda} \in M,$$

from which the required result follows. □

We now state and prove the main result of this section and is the motivation of our next result.

Theorem 14.1. Theorem 10.1. *Let $\bar{u}$ be the solution of the parametric variational inequality (14.1) and $\bar{z}$ be the solution of the parametric Wiener-Hopf equation (14.2) for $\lambda = \bar{\lambda}$. Let $T_\lambda(.)$ be the locally strongly monotone Lipschitz continuous operator. If the map $\lambda \longrightarrow P_{K_\lambda}$ is (Lipschitz) continuous at $\lambda = \bar{\lambda}$, then there exists a neighborhood $N \subset K$ of $\bar{\lambda}$ such that for $\lambda \in N$, the parametric Wiener-Hopf equation (14.2) has a unique solution $z(\lambda)$ in the interior of X, $z(\bar{\lambda}) = \bar{z}$ and $z(\lambda)$ is (Lipschitz) continuous at $\lambda = \bar{\lambda}$.*

Proof. Its proof follows from Lemmas 10.2, 10.3 and Remark 14.1. □

15. Finite Convergence Analysis

In this section, we analyze local convergence behaviour of the projection-type methods for variational inequalities (23.3). This study is extremely useful in the design of algorithms for the solution of variational inequalities. We show that the sequences generated by the iterative methods terminates at a solution of the concerned problem or enter and remain in relative interior of the optimal face and hence the subproblem reduces to a simpler form under conditions. Our analysis is in the spirit of Burke and More [35]. Our results can be viewed as a nice and novel applications of the techniques developed for variational inequalities (23.3).

First of all, we recall some well-known concepts.

Given $u \in K$, we say that a direction v is feasible at a point $u \in K$, if $u + \mu v$ belongs to K for all sufficiently small $\mu > 0$. The tangent cone $T_K(u)$ is the closure of the cone of all feasible directions. Since $T_K(u)$ is a nonempty closed convex set, $-T(u)$ has a unique projection on $T_K(u)$ with the following form :

$$\begin{aligned} P^t[-Tu] &:= P^t(u) \\ &:= P^{T_K(u)}[-T(u)] = \text{argmin}\{\|v + T(u)\| : \quad v \in T_K(u)\}. \end{aligned}$$

This projection has the following properties.

Lemma 15.1. *Let $P^t(u)$ be the tangent projection of $-T(u)$ at $u \in K$. Then*

$$(a) \qquad \langle T(u), P^t(u)\rangle = -\|P^t(u)\|^2$$
$$(b) \qquad min\{\langle T(u), v\rangle : \quad v \in T_K(u), \quad \|v\|v \leq 1\} = -\|P^t u\|.$$

Using Lemma 15.1, we establish the equivalence between the variational inequality (23.3) and the tangent projection equations (2.3).

Lemma 15.2. *The variational inequality (23.3) has solution $u \in H$, if and only if, u solves the tangent projection equation (2.3), that is,*

$$P^{tg}[u] = 0.$$

Proof. Let $u \in K$ be a solution of (23.3). Then

$$\langle Tu, y - u\rangle \geq 0. \quad \forall y \in K. \tag{15.1}$$

Let v be any feasible direction at a point $u \in K$. Then $u+\mu v$ must be in K for some small $\mu > 0$. Thus there is a point $y \in H$ such that $y = u + \mu v$. Substituting y into (11.1), we obtain $\langle T(u), v\rangle \geq 0$. Since $T_K(u)$ is the closure of the set of all feasible directions at $u \in K$, we have

$$\langle T(u), w\rangle \geq 0, \quad \forall w \in T_K(u). \tag{15.2}$$

Thus, by Lemma 11.1(b), we obtain

$$P^t(u) := P^t[-Tu] = 0,$$

the required tangent projection equation (2.3).

Conversely, let $u \in K$ be a solution of the tangent projection equation (2.3). Invoking Lemma 15.1(b), we obtain the inequality (15.2). Since $y - u \in T_K(u)$ for any $y \in K$, from (15.2), we obtain (15.1), and this shows that u is a solution of the variational inequality (23.3). □

Lemma 15.2 implies that the problems (23.3) and (2.3) are equivalent. We use this equivalent formulation to derive a characterization of a wide class of iterative methods which identify the optimal face in a finite number of iterations. We show that some explicit and proximal projection-type methods have such a characterization.

For a set $S \subseteq H$, the affine hull *affine*(S) is the smallest affine set which contains S, and the relative interior $ri(S)$ is the interior of S relative to *aff*(S). For a cone $K \subseteq H$, the linearity *lin*$\{K\}$ of the cone is the largest subspace contained in K. For a nonempty closed set $K \subseteq H$, a convex set $K_{face} \subseteq K$ is said to be a face K if the endpoints of any closed line segment in K whose relative interior intersects K_{face} are contained in K_{face}. Thus if x and y are in K and $\theta x + (1 - \theta)y$ lies in K_{face} for some $0 < \theta < 1$, then x and y must also belong to K_{face}.

Lemma 15.3. *If K_{face} of the convex set K, then the normal cone $N(u)$ is independent of u for any $u \in ri(K_{face})$. A face K_{face} of a convex set K is said to be quasi-polyhedral, if aff$(K_{face}) = u + lin\{T(u)\}$ for any $u \in ri(K_{face})$.*

We denote the tangent cone $T_c(u)$ for $u \in ri(K_{face})$ by $T_c(K_{face})$ and $N(u)$ by $N(K_{face})$.

Definition 15.1. *A solution $u \in K$ of problem (23.3) is said to nondegenerate if*

$$-T(u) \in ri(N(u)).$$

Definition 15.1 is due to Dunn, see, for example, [35]. To see this, we consider the variational inequality (23.3) with linear constraints, which is to find $u \in R^n$ such that $u \in X$ and

$$\langle Tu, y - u\rangle \geq 0, \quad \text{for all } y \in X, \tag{15.3}$$

where

$$X = \{u \in R^n : A^T u \geq b\}, \quad A = (a_1, a_2, \ldots, a_n) \in R^{n\times m}$$

and $b = (b_1, b_2, \ldots, b_m)^T \in R^m$. Its KKT system is

$$\begin{aligned} T(u) &= A\lambda \\ \langle \lambda, A^T u - b\rangle &= 0, \quad \lambda \geq 0, \quad A^T u - b \geq 0. \end{aligned} \tag{15.4}$$

If u is a solution of (15.3), then there is a vector $\lambda^* \in R^m$, the Lagrange multiplier, such that (u, λ^*) is a solution of (11.4). According to definition 15.1, u is nondegenerate if λ^* and $A^T u - b$ are strictly complementary.

Now we consider algorithm 8.4 which generate the sequence $\{u_n\} \subseteq K$. By using only the knowledge of face geometry (which is independent of any algorithm), Burke and More [35] proved the following result.

Lemma 15.4. *Assume that K_{face} is a quasi-polyhedral face of the convex set K, with $u \in ri(K_{face})$ and $d^* \in ri(N(K_{face}))$. If $u_n \in K$ and $d_n \in N(u_n)$ for all n, and the sequence $\{u_n\}$ and $\{d_n\}$ converge to u and d^* respectively, then $u_n \in ri(K_{face})$ for all n sufficiently large.*

In view of Lemma 15.3 and 15.4, we state and prove the following result.

Theorem 15.1. *Assume that* $\{u_n\} \in K$ *is a sequence which converges to a nondegenerate solution* u *of the variational inequality (23.3). If* K_{face} *is a quasi-polyhedral face of* K *with* $u \in ri(K_{face})$*, then* $u \in ri(K_{face})$ *for all sufficiently large* n *if and only if* $\{T^t(u_n)\}$ *converges to zero.*

Proof. See [14, Theorem 3.4.]. □

When K is a polyhedral set X, problem (23.3) reduces to (15.3) and Theorem 15.1 can be simplified as follows.

Theorem 15.2. *Assume that* $\{u_n\} \in X$ *is a sequence which converges to a nondegenerate solution* u *of problem (18.3). Then* $\mathbf{A}(u_n) = \mathbf{A}(u)$ *for all sufficiently large* n *if and only if* $\{P^t(u_n)\}$ *converges to zero, where* $\mathbf{A}(u) := \{j : a_j^T u = b_j, \quad j = 1, 2, \ldots, m\}$ *is the active set at* u*.*

We now consider some applications of Theorem 15.1 and Theorem 15.2 to projection-type Algorithm 8.4 suggested and analyzed in section 7.

Theorem 15.3. *If Algorithm 8.4 produces a sequence* $\{u_n\}$ *which converges to a solution* u *of problem (23.3), then* $\{P^t(u_n)\}$ *converges to zero. In addition, if* u *is nondegenerate, then*
(a). $u_n \in ri(K_{face}(u))$*, for all* n *sufficiently large.*
(b). For $K = X$, $\mathbf{A}(u_n) = \mathbf{A}(u)$*, for all* n *sufficiently large.*

Proof. Let $\epsilon > 0$ be given and let v_n be a feasible direction at $u_n \in K$ with $\|v_n\| \leq 1$ such that

$$\|P^t(u_n)\| \leq \langle -Tu_n, v_n\rangle + \epsilon. \tag{15.5}$$

Since

$$u_{n+1} = P_K[u_n - \rho Tu_n + \rho_n e_n]$$

can be written in the form of the variational inequality as

$$-\rho_n\langle u_{n+1} - u_n + \rho_n\langle(Tu_n + e_n), w - u_{n+1}\rangle \geq 0, \quad \forall w \in K,$$

from which it follows that

$$\rho_n\langle Tu_n, w - u_{n+1}\rangle - \rho_n\langle e_n, w - u_{n+1}\rangle \leq \|u_{n+1} - u_n\| \times \|w - u_{n+1}\|.$$

Take some $\mu > 0$ such that $w := u_{n+1} + \mu v_{n+1} \in K.$, Then from the above inequality, $\|v_{n+1}\| \leq 1$, $\lim_{n\to\infty} e_n = 0$, we have

$$\begin{aligned}\limsup_{n\to\infty} -\langle Tu_{n+1}, v_{n+1}\rangle \quad &\leq \quad \limsup_{n\to\infty} \frac{\|u_n - u_{n+1}\|\|v_{n+1}\|}{\rho_n} \\ &\quad + \limsup_{n\to\infty} |\langle e_n, v_{n+1}\rangle| \leq 0.\end{aligned}$$

This together with the continuity of T, and $\|u_n - u_{n+1}\| \to 0$, implies that

$$\begin{aligned}\limsup_{n\to\infty} -\langle Tu_{n+1}, v_{n+1}\rangle \quad &\leq \quad \limsup_{n\to\infty} -\langle Tu_n, v_{n+1}\rangle \\ &\quad + \limsup_{n\to\infty} \|T(u_n) - T(u_{n+1})\| \leq 0.\end{aligned} \tag{15.6}$$

Combining (15.5) with (15.6), we have

$$\limsup_{n\to\infty} \|P^t(u_{n+1}\| \leq \epsilon.$$

Since $\epsilon > 0$ is arbitrary, the sequence $\{P^t(u_{n+1})\}$ converges to zero. and hence u_n is a solution of problem (23.3). If u is degenerate, then by invoking Theorem 15.1 and Theorem 15.2, we obtain the required results (a) and (b). □

By using part (a) or (b) of Theorem 15.3, the sequence $\{u_n\}$ eventually enters and remains in relative interior of the optimal face or computation of the orthogonal projection $P_X[u_n - \rho_n(Tu_n + e_n)]$ eventually reduces to solving an equality constraint subproblem

$$\min \|u - (u_n - \rho_n(T(u_n) + e_n))\|^2$$
$$\text{such that } a_j^T w = b_j, \quad j \in \mathbf{A}(u).$$

the solution of which can be easily computed. In a similar way, one can study the finite convergence criteria of iterative projection-type methods including the inertial proximal point methods

16. Variational-like Inequalities

In recent years the concept of convexity has been generalized in many directions, which has potential and important applications in various field including nonlinear optimization, network equilibrium, economics, finance and operations research. A significant generalization of convex functions is the introduction of invex and preinvex functions. Various kind of necessary and sufficient conditions in which invex and preinvex were closely related have been studied. It has been shown that the minimum of preinvex functions on the invex sets can be characterized by a class of variational inequalities, known as variational-like inequalities involving the function $\eta(.,.)$. Due to the presence of the function $\eta(.,.)$ in the variational-like inequalities, it is not possible to prove that the variational-like inequalities are equivalent to the fixed-point problems, which have been obtained by using the projection and resolvent operator techniques. To overcome this drawback, one usually uses the auxiliary principle technique, which has been discussed in the previous sections. We, here, again use the auxiliary principle technique to suggest and analyze a proximal point method for solving variational-like inequalities. We show that the convergence of the proximal method requires only pseudomonotonicity, which is a weaker condition than monotonicity. As special cases, we obtain various known results discussed in the previous sections.

We now recall the following known concepts.

Definition 16.1. *A set $K_\eta \subseteq H$ is said to be invex, if there exists a bifunction $\eta(.,.) : K_\eta \times K_\eta \longrightarrow H$ such that*

$$u + t\eta(v,u) \in K_\eta, \quad \forall u, v \in K_\eta, t \in [0,1].$$

Note that every convex set is invex but the converse is not true. This implies that invex sets include convex sets as special cases.

Definition 16.2. *Let $K_\eta \subseteq H$ be an invex set with respect to the bifunction $\eta(.,.) : K_\eta \times K_\eta \longrightarrow H$. The function $F : K_\eta \longrightarrow R$ is said to be preinvex, if*

$$F(u + t\eta(v,u)) \leq (1-t)F(u) + tF(v), \quad \forall u, v \in K_\eta, t \in [0,1].$$

Every convex function is a preinvex function, but the converse is not true.

From now onward, we assume that the set K_η is an invex set with respect to the bifunction $\eta(.,.) : K_\eta \times K_\eta \longrightarrow H$, unless otherwise specified.

For $t = 1$, we have

$$F(u + \eta(v,u)) \leq F(v), \quad \forall u, v \in K_\eta,$$

which is called the condition A.

Definition 16.3. *The function $F : K \longrightarrow H$ is called quasi preinvex, if*

$$F(u + t\eta(v,u)) \leq \max\{F(u), F(v)\} \quad \forall u, v \in K_\eta, \quad t \in [0,1].$$

The function F is called the strictly quasi semi preinvex, if strict inequality holds $\forall \quad u, v \in K_\eta, u \neq v$. The function F is said to be quasi semi preincave if $-F$ is quasi semi preinvex. A function which is both quasi semi preinvex and quasi semi preincave is called the quasimonotone

Definition 16.4. *The function $F : H \longrightarrow H$ is said to be logarithmic semi preinvex on the invex set K_η with respect to the bifunction $\eta(,.)$, if*

$$F(u + t\eta(v,u)) \leq (F(u))^{1-t}(F(v))^t, \quad \forall u, v \in K_\eta \quad t \in [0,1],$$

where $F(.) > 0$.

For a function $F : K_\eta \longrightarrow H$, it follows that
logarithmic semi preinvexity $\Longrightarrow$ semi preinvexity $\Longrightarrow$ quasi semi preinvexity.

For appropriate and suitable choice of the operators. bifunction $\eta(,.,)$ and spaces, one can obtain several classes of generalized convexity.

Noor [] proved that the minimum of the preinvex function on the invex set K_η in H can be characterized by a class of variational inequalities, which is called variational-like inequality.

Theorem 16.1. *Let $F : K_\eta \longrightarrow H$ be a differentiable and preinvex function with respect to η. Then $u \in K_\eta$ is a minimum of F on K, if and only if, $u \in K_\eta$ satisfies*

$$\langle F'(u), \eta(v,u)\rangle \geq 0, \quad \forall v \in K_\eta, \tag{16.1}$$

where $F'(u)$ is the Frechet derivative of F at $u \in K_\eta$.

Proof. Let $u \in K$ be a minimum of the differentiable preinvex function F. Then

$$F(u) \leq F(v), \quad \forall u, v \in K_\eta. \tag{16.2}$$

Since the set K_η is an invex set, so $\forall u, v \in K_\eta$ and $t \in [0,1]$ such that $v_t = u + \eta(v,u) \in K_\eta$.
Setting $v = v_t$ in (16.2), we have

$$F(u) \leq F(v_t) = F(u + t\eta(v,u)).$$

Dividing the above inequality by t and taking limit as $t \longrightarrow 0$, we have

$$\langle F'(u), \eta(v,u)\rangle \geq 0, \quad \forall u, v \in K_\eta.$$

the required (16.1).
Conversely, let $u \in K_\eta$ satisfy the inequality (16.1). Then using the fact that function F is an preinvex function, we have

$$F(u + t\eta(v,u)) - F(u) \leq t\{F(v) - F(u)\}, \quad \forall u, v \in K_\eta.$$

Dividing the above inequality by t and letting $t \longrightarrow 0$, we have

$$F(v) - F(u) \geq \langle F'(u), \eta(v,u)\rangle \geq 0, \quad \text{using (16.1)}.$$

which implies that

$$F(u) \leq F(v),$$

which shows that $u \in K_\eta$ is a minimum of the preinvex function on the invex set K_η in H. □

Inequalities of the type (16.1) are called the variational-like inequalities. From Theorem 16.1, it follows that the variational-like inequalities (16.1) arise naturally in connection with the minimum of preinvex function over invex sets. In many applications, problems like (16.1) which occur do not arise as a result of minimization. We consider a problem of finding a solution of a more general variational-like inequality of which (16.1) is a special case.

Given (nonlinear) operator $T : H \longrightarrow H$ and bifunction$\eta(,.,) : K_\eta \times K_\eta \longrightarrow H$, where K_η is a nonempty invex set in H, we consider the problem of finding $u \in K_\eta$ such that

$$\langle Tu, \eta(v,u)\rangle \geq 0, \quad \forall u, v \in K_\eta. \tag{16.3}$$

which is known as the variational-like inequality.

If $\eta(v,u) = v - u$, and then the invex set K_η becomes a convex set K. . In this case, problem (16.3) is equivalent to finding $u \in K$ such that

$$\langle Tu, v - u\rangle \geq 0, \quad \forall u, v \in K$$

which is exactly the variational inequality (23.3.

For suitable and appropriate choice of the operators T, the bifunction $\eta(,.,)$, and the invex set, one may derive a wide class of known and new classes variational inequalities as special cases of problem (16.3). It is clear that variational-like inequalities are more general and unifying ones.

We would like to emphasize that the projection (resolvent) method and its variant forms cannot be extended for variational-like inequalities , since it is not possible to find the projection of H onto the invex set K_η, which is not a convex set. Consequently, we cannot establish the equivalence between the variational-like inequalities and the projection (resolvent) fixed-point problems. Some authors have used the resolvent operator technique to establish the equivalence between the mixed variational-like inequalities and the related fixed-point problems. This is not correct and is misleading. It is well-known that if the proper and lower-semicontinuous involving the mixed variational-like inequalities is an indicator function of the set K_η, then the mixed variational-like inequality is just the variational-like inequality and the resolvent operator associated with the subdifferential of the function is equivalent to the projection of H onto the convex set K. In this case, one cannot use the projection Lemma to show that the variational-like inequality is equivalent to the fixed-point problem due to the presence of $\eta(v,u)$ in place of $v - u$. This clearly shows that the resolvent technique cannot be used for mixed variational-like inequalities. Almost all the results obtained for variational-like inequalities and their generalizations in the setting of convexity are superficial and wrong. In passing, we would like to point out that variational-like inequalities are only well-defined in the setting of invexity as variational inequalities are well-defined in the setting of convexity. To overcome these difficulties, one usually uses the auxiliary principle technique to suggest and analyze some iterative methods for variational-like inequalities. We now use this technique to consider a proximal method for variational-like inequalities (16.3).

For a given $u \in K_\eta$ satisfying (16.3). consider the problem of finding a solution $w \in K_\eta$ satisfying the auxiliary variational-like inequality

$$\langle \rho Tw + E'(w) - E'(u), \eta(v,w)\rangle \geq 0, \quad \forall v \in K_\eta \tag{16.4}$$

where $\rho > 0$ is a constant and E' is the differential of a strongly preinvex function E. Obviously, if $w = u$, then w is a solution of (16.3). This fact enables us to suggest and analyze the following iterative method for solving (16.3).

Algorithm 16.1. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho Tu_{n+1} + E'(u_{n+1}) - E'(u_n), \eta(v, u_{n+1})\rangle \geq 0, \quad \forall u, v \in K_\eta. \tag{16.5}$$

which is known as the proximal point algorithm for solving variational-like inequalities (16.3).

For $\eta(v,u) = v - u$, the invex set K_η becomes the convex set K and consequently, Algorithm 16.1 reduces to:

Algorithm 16.2. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho Tu_{n+1} + E'(u_{n+1}) - E'(u_n), v - u_{n+1}\rangle \geq 0, \quad \forall \quad v \in K,$$

which is known as the proximal point algorithm for solving variational inequalities (23.3).

Remark 16.1. *The function $B(w,u) = E(w) - E(u) - \langle E'(u), \eta(w,u)\rangle$ associated with the preinvex functions $E(u)$ is called the generalized Bregman distance function. We note that, if $\eta(v,u) = v - u$, then $B(w,u) = E(w) - E(u) - \langle E'(u), v - u\rangle$ is the well known Bregman distance function.*

For the applications of Bregman distance function in solving variational inequalities and related optimization problems, see [29, 164].

We now study the convergence analysis of Algorithm 16.1. For this purpose, we recall the following concepts.

Definition 16.5. $\forall \quad u, v, z \in H$, *an operator $T : H \longrightarrow H$ is said to be:*
(i). *η-pseudomonotone, if*

$$\langle Tu, \eta(v,u)\rangle \geq 0 \quad \Longrightarrow \quad -\langle Tv, \eta(v,u)\rangle \geq 0.$$

(ii). *η-Lipschitz continuous, if there exists a constant $\beta > 0$ such that*

$$\langle Tu - Tv, \eta(u,v)\rangle \leq \beta\|u - v\|^2.$$

(iii). *η-cocoercive, if there exists a constant $\mu > 0$ such that*

$$\langle Tu - Tv, \eta(v,u)\rangle \geq \mu\|Tu - Tv\|^2.$$

(iv). *η-partially relaxed strongly monotone, if there exists a constant $\alpha > 0$ such that*

$$\langle Tu - Tv, \eta(z,v)\rangle \geq \mu\|z - u\|^2.$$

(v). *η-hemicontinuous, if, for $t \in [0,1]$ the mapping $\langle T(u + t(v-u)), \eta(v,u)\rangle$ is continuous.*

For $\eta(v,u) = v - u$, Definition 16.5 reduces to the standard definition of monotonicity, pseudomonotonicity, cocoercivity, hemicontinuity, and partially relaxed strongly monotonicity of the operator T. We note that for $z = u$, partially strongly monotonicity reduces to monotonicity. Using Lemma 2.2, one can easily show that η-cocoercivity implies η-partially relaxed strongly monotonicity, but the converse is not true.

Definition 16.6. *A function F is said to be strongly preinvex function on K_η with respect to the bifunction $\eta(.,.)$ with modulus $\mu > 0$, if,*

$$F(u + t\eta(v,u)) \leq (1-t)F(u) + tF(v) - t(1-t)\mu\|v - u\|^2, \forall u, v \in K_\eta, t \in [0,1].$$

We note that the differentiable strongly preinvex function F is a strongly invex function, that is,

$$F(v) - F(u) \geq \langle F'(u), \eta(v,u)\rangle + \mu\|v - u\|^2, \quad \forall u, v \in K_\eta.$$

but the converse is not true.

Assumption 16.1. $\forall u, v, z \in H$, *the operator $\eta : H \times H \longrightarrow H$ satisfies the condition*

$$\eta(u,v) = \eta(u,z) + \eta(z,v).$$

In particular, from Assumption 16.1 , we obtain

$$\eta(u,v) + \eta(v,u) = 0.$$

This implies that the bifunction $\eta(,.,)$ is skew symmetric and

$$\eta(u,v) = 0 \quad \Leftrightarrow \quad u = v, \forall u, v \in H.$$

Assumption 16.1 has played an important role in studies if variational-like inequalities and related optimization problems.

Theorem 16.2. *Let T be a η-pseudomonotone operator. Let E be a strongly differentiable preinvex function with modulus β and Assumption 16.1 hold. Then the approximate solution u_{n+1} obtained from Algorithm 16.1 converges to a solution of (16.3).*

Proof. Since the function E is strongly preinvex , so the solution u_{n+1} is unique. Let $u \in K_\eta$ be a solution of the variational-like inequality (16.3). Then

$$\langle Tu, \eta(v,u)\rangle \geq 0, \quad \text{for all} \quad v \in K_\eta,$$

which implies that

$$-\langle Tv, \eta(v,u)\rangle \geq 0, \quad \forall v \in K_\eta, \tag{16.6}$$

since T is η-pseudomonotone .

Taking $v = u_{n+1}$ in (16.6), we have

$$-\langle Tu_{n+1}, \eta(u_{n+1}, u)\rangle \geq 0. \tag{16.7}$$

We consider the Bregman function

$$\begin{aligned} B(u,w) &= E(u) - E(w) - \langle E'(u), \eta(v,u)\rangle \\ &\geq \frac{\beta}{2}\|u-w\|^2, \quad \text{using strongly preinvexity.} \end{aligned} \tag{16.8}$$

Now

$$\begin{aligned} B(u,u_n) - B(u,u_{n+1}) &= E(u_{n+1}) - E(u_n) + \langle E'(u_{n+1}), \eta(u,u_{n+1})\rangle \\ &\quad -\langle E'(u_n), \eta(u,u_n)\rangle \end{aligned} \tag{16.9}$$

Using Assumption 16.1, we have

$$\eta(u,u_n) = \eta(u,u_{n+1}) + \eta(u_{n+1},u_n). \tag{16.10}$$

Combining (16.8), (16.9) and (16.10) , we have

$$\begin{aligned} B(u,u_n) - B(u,u_{n+1}) &= E(u_{n+1}) - E(u_n) - \langle E'(u_n), \eta(u_{n+1},u_n) \\ &\quad +\langle E'(u_{n+1}) - E'(u_n), \eta(u,u_{n+1})\rangle \\ &\geq \beta\|u_{n+1}-u_n\|^2 + \langle E'(u_{n+1}) - E'(u_n), \eta(u,u_{n+1})\rangle \\ &\geq \beta\|u_{n+1}-u_n\|^2 + \langle \rho Tu_{n+1}, \eta(u_{n+1},u)\rangle \quad \text{using(12.5) with } v=u \\ &\geq \beta\|u_{n+1}-u_n\|^2, \quad \text{using (16.7).} \end{aligned}$$

If $u_{n+1} = u_n$, then clearly u_n is a solution of the variational-like inequality (16.3). Otherwise, it follows that $B(u,u_n) - B(u,u_{n+1})$ is nonnegative , and we must have

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0.$$

Now using the technique of Zhu and Marcotte [164], one can easily show that the entire sequence $\{u_n\}$ converges to the cluster point $\bar{u}$ satisfying the variational-like inequality (16.3). □

To implement the proximal method, one has to calculate the solution implicitly, which is in itself a difficult problem. We again use the auxiliary principle technique to suggest another iterative method, the convergence of which requires only the η-partially relaxed strongly monotonicity.

For a given $u \in K_\eta$ satisfying (16.3), find a solution $w \in K_\eta$ which satisfies the auxiliary variational-like inequality

$$\langle \rho Tu + E'(w) - E'(u), \eta(v,w)\rangle \geq 0, \quad \forall \quad v \in K_\eta, \tag{16.11}$$

where $E(u)$ is a strongly differentiable preinvex function. It is clear that, if $w = u$, then w is a solution of the variational-like inequality (16.3). This fact allows us to suggest and analyze the following iterative method for solving (16.3).

Algorithm 16.3. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho Tu_n + E'(u_{n+1}) - E'(u_n), \eta(v,u_{n+1})\rangle \geq 0, \quad \textit{for all} \quad v \in K, \tag{16.12}$$

where K is an invex set in H.

Note that, for $\eta(v,u) = v - u$, and $K_\eta = K$, a convex set, Algorithm **??** reduces to:

Algorithm 16.4. *For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho Tu_n + E'(u_{n+1}) - E'(u_n), v - u_{n+1}\rangle \geq 0, \quad \textit{for all} \quad v \in K.$$

for solving the variational inequalities (23.3), which appears to be a new one. In a similar way, one can obtain a number of new and known iterative methods for solving various classes of variational inequalities and complementarity problems.

We now study the convergence analysis of Algorithm 16.3. The analysis is in the spirit of Theorem 16.2. We only give the main points.

Theorem 16.3. *Let T be a η-partially relaxed strongly monotone with a constant $\alpha > 0$. Let E be a strongly differentiable preinvex function with modulus β and Assumption 16.1 hold. If $0 < \rho < \frac{\beta}{\alpha,}$ then the approximate solution u_{n+1} obtained from Algorithm 16.3 converges to a solution of (16.3).*

Proof. Since the function E is strongly preinvex , so the solution u_{n+1} of (16.11) is unique. Let $u \in K_\eta$ be a solution of the variational-like inequality (16.3). Then

$$\langle Tu, \eta(v,u)\rangle \geq 0, \quad \text{for all} \quad v \in K.$$

Taking $v = u_{n+1}$ in the above inequality, we have

$$\langle \rho Tu, \eta(u_{n+1}, u)\rangle \geq 0. \tag{16.13}$$

Combining (16.8), (16.9) and (16.13) , we have

$$\begin{aligned} B(u,u_n) - B(u,u_{n+1}) &= E(u_{n+1}) - E(u_n) - \langle E'(u_n), \eta(u_{n+1}, u_n) \\ &\quad + \langle E'(u_{n+1}) - E'(u_n), \eta(u, u_{n+1})\rangle \\ &\geq \beta\|u_{n+1} - u_n\|^2 + \langle E'(u_{n+1}) - E'(u_n), \eta(u, u_{n+1})\rangle \\ &\geq \beta\|u_{n+1} - u_n\|^2 + \langle \rho Tu_n, \eta(u_{n+1}, u)\rangle, \quad \text{using}(12.12). \\ &\geq \beta\|u_{n+1} - u_n\|^2 + \langle \rho Tu_n - \rho Tu, \eta(u_{n+1}, u)\rangle, \quad (12.13). \\ &\geq (\beta - \rho\alpha)\|u_{n+1} - u_n\|^2. \end{aligned}$$

If $u_{n+1} = u_n$, then clearly u_n is a solution of the variational-like inequality (16.3) .Otherwise, the assumption $0 < \rho < \frac{\alpha}{\beta}$, implies that the sequence $B(u,u_n) - B(u,u_{n+1})$ is nonnegative , and we must have

$$\lim_{n \longrightarrow \infty} \|u_{n+1} - u_n\| = 0.$$

Now by using the technique of Zhu and Marcotte [164], it can be shown that the entire sequence $\{u_n\}$ converges to the cluster point $\bar{u}$ satisfying the variational-like inequality (16.3). □

We now show that the solution of the auxiliary variational-like inequality (16.11) is the minimum of the functional $I[w]$ on the invex set K_η, where

$$I[w] = E(w) - E(u) - \langle E'(u) - \rho Tu, \eta(w,u)\rangle = B(w,u) - \rho\langle Tu, \eta(w,u)\rangle, \tag{16.14}$$

is known as the auxiliary energy functional associated with the auxiliary variational-like inequality (16.11), where $B(w,u)$ is a Bregman function. We now prove that the minimum of the functional $I[w]$, defined by (16.14), can be characterized by the variational-like inequality (16.11). The analysis is in the spirit of Noor [105].

Theorem 16.4. *Let E be a differentiable preinvex function. If Assumption 16.1 holds and the bifunction$\eta(.,.)$ is prelinear in the first argument, then the minimum of $I[w]$, defined by (16.14), can be characterized by the auxiliary variational-like inequality (16.11).*

Proof. Let $w \in K$ be the minimum of $I[w]$ on K_η. Then

$$I[w] \leq I[v], \quad \forall \quad v \in K_\eta.$$

Since K_η is an invex set, so for all $w, u \in K_\eta, t \in [0,1], v_t = w + t\eta(v,w) \in K_\eta$.

Replacing v by v_t in the above inequality, we have

$$I[w] \leq I[w + t\eta(v,w)]. \tag{16.15}$$

Since $\eta(.,.)$ is prelinear in the first argument, from (16.14) and (16.15), we have

$$\begin{aligned} E(w) - E(u) - \langle E'(u) - \rho Tu, \eta(w,u)\rangle &\leq E(v_t) - E(u) - \langle E'(u) - \rho Tu, \eta(w_t,u)\rangle \\ &\leq E(v_t) - (1-t)\langle E'(u) - \rho Tu, \eta(w,u)\rangle \\ &\quad -t\langle E'(u) - \rho Tu, \eta(v,u)\rangle \end{aligned}$$

which implies that

$$E(w + t\eta(v,w)) - E(w) \geq t\langle E'(u) - \rho Tu, \eta(v,w)\rangle - t\langle E'(u) - \rho Tu, \eta(w,u)\rangle. \tag{16.16}$$

Now using Assumption 16.1, we have

$$\langle E'(u), \eta(v,u)\rangle = \langle E'(u), \eta(v,w)\rangle + \langle E'(u), \eta(w,u)\rangle \tag{16.17}$$

$$\langle Tu, \eta(v,u)\rangle = \langle Tu, \eta(v,w)\rangle + \langle Tu, \eta(w,u)\rangle. \tag{16.18}$$

From (16.15), (16.16), (16.17) and (16.18), we obtain

$$E(w + t\eta(v,w)) - E(w) \geq t\langle E'(u) - \rho Tu, \eta(v,w)\rangle$$

Dividing both sides by t and letting $t \to 0$, we have

$$\langle E'(w), \eta(v,w)\rangle \geq \langle E'(u) - \rho Tu, \eta(v,w)\rangle$$

the required inequality (16.11).

Conversely, let $u \in K$ be a solution of (16.11). Then

$$\begin{aligned} I[w] - I[v] &= E(w) - E(v) - \langle E'(u) - \rho Tu, \eta(w,u)\rangle \\ &\quad +\langle E'(u) - \rho Tu, \eta(v,u)\rangle \\ &\leq -\langle E'(w), \eta(v,w)\rangle + \langle E'(u), \eta(v,u) - \eta(w,u)\rangle \\ &\quad -\rho\langle Tu, \eta(v,u) - \eta(w,u)\rangle \\ &\leq \langle E'(u), \eta(v,w)\rangle - \langle E'(w) - \rho Tu, \eta(v,w)\rangle \\ &\quad +\langle E'(w) + \rho Tu, \eta(v,w)\rangle - \langle E'(u), \eta(v,w)\rangle \\ &\leq 0. \end{aligned}$$

Thus it follows that $I[w] \leq I[v]$, showing that $v \in K$ is the minimum of the functional $I[w]$ on K, the required result. □

In recent years, there is a substantial interest for solving the variational inequalities and complementarity problems via the merit (gap) functions. The main idea is to find the differentiable equivalent optimization problem for variational inequalities and then using this equivalent formulation, one usually suggests the iterative methods for solving the variational inequalities. On the other hand, there is no such merit(gap) function for the variational-like inequalities due to partly the presence of the bifunction η and partly, the variational-like inequalities are closely related to the preinvex functions, which are not convex functions. For recent results, see [105,157].

We now introduce and study a class of merit(gap) functions for variational-like inequalities using the technique of auxiliary principle.

Definition 16.7. *A function $\Psi : H \longrightarrow R \cup \{+\infty\}$ is called a merit(gap) function for the variational-like inequality (16.3), if*
(i). $\Psi(u) \geq 0, \quad \forall u \in K_\eta$.
(ii). $\Psi(\overline{u}) = 0$ *if and only if $\overline{u}$ solves (12.3).*

Using this definition of the merit function, we reformulate the variational-like inequality as an equivalent optimization problem.

$$\text{Minimize} \quad \Psi(u) \quad \text{subject to } u \in K_\eta$$

This approach is due to Fukushima [54], Zhu and Marcotte [177] and Patriksson[129] for variational inequalities and complementarity problems. We consider the function $L : H \times H \longrightarrow R$, with

$$\begin{aligned} L(u,w) &= E(u) - E(w) + \langle \rho Tu - E'(u), \eta(u,w)\rangle \\ &= B(u,w) + \rho\langle Tu, \eta(u,w)\rangle, \quad \forall u,w \in H, \end{aligned} \tag{16.19}$$

where $E : H \longrightarrow R \cup \{+\infty\}$ is a differentiable preinvex function. We now define the function Ψ as:

$$\Psi(u) = \max_{w \in K_\eta} \{L(u,w)\} \tag{16.20}$$

and the optimization problem

$$\inf_{u \in K_\eta} \{\Psi(u)\}.$$

For sake of simplicity, we rewrite (16.20) in the following form

$$\Psi(u) = \inf_{u \in K_\eta} \{\max_{w \in K_\eta} \{\langle \rho Tu, \eta(u,w)\rangle + B(u,w)\}\}, \tag{16.21}$$

where $B(w,u)$ is the Bregman function. Note that $B(u,u) = 0$ and $B(w,u) \geq 0, \quad \forall u \in K_\eta$, since E is differentiable preinvex function.

We now show that the function $\Psi(u)$ defined by (16.21) is a merit function for the variational-like inequalities (16.3) and this is the motivation of our next result.

Theorem 16.5. *If Assumption 16.1 holds and the function $\eta(.,.)$ is prelinear in the first argument, then the function Ψ defined by (16.21) is a merit function for the variational-like inequality (16.3).*

Proof. Since $B(u,u) =, 0$, for each $u \in K_\eta$, it is clear that $\Psi(u) \geq 0$, for all $u \in K_\eta$. Now assume that $u \in K_\eta$ is a solution of the variational-like inequality (16.3). Then

$$\langle Tu, \eta(w,u)\rangle \geq 0, \quad \forall w \in K_\eta.$$

As $B(u,w) \geq 0, \quad \forall w \in K_\eta$,

$$\langle \rho Tu, \eta(w,u)\rangle - B(u,w) \geq 0, \quad \forall \quad w \in K_\eta.$$

This implies , by using the assumption $\eta(w,u) = -\eta(u,w)$, that $\Psi(u) \leq 0$. Thus we conclude that

$$\Psi(u) = 0.$$

Conversely, let $\Psi(u) = 0, \forall u \in K_\eta$. Then by Theorem 16.3, we have

$$\langle E'(u), \eta(w,u)\rangle \geq \langle E'(u) - \rho Tu, \eta(w,u)\rangle,$$

that is

$$\langle Tu, \eta(w,u)\rangle \geq 0,$$

which implies that $u \in K_\eta$ satisfies the variational-like inequality (16.3), the required result. □

It is also possible to establish lower bound for the merit function and this is the motivation of our next result.

Theorem 16.6. *Let T be a η-strongly monotone with constant $\alpha > 0$ and E' be η-Lipschitz continuous with constant $\beta > 0$. Let u be a unique solution of (16.3). If $\beta < \alpha$, then*

$$(\alpha - \beta)\|v - u\|^2 \leq \Psi(v), \quad \forall v \in K_\eta. \tag{16.22}$$

Proof. Let $u \in K_\eta$ be a solution of (16.3). Then

$$\langle Tu, \eta(v, u)\rangle \geq 0, \forall \quad v \in K_\eta,$$

which implies that

$$\langle Tv, \eta(v, u)\rangle \geq \langle Tv - Tu, \eta(v, u)\rangle \geq \alpha\|v - u\|^2, \tag{16.23}$$

since T is η-strongly monotone with constant $\alpha > 0$.

Now using (16.23), invexity of E and η-Lipschitz continuity of T, we have

$$\begin{aligned}
\Psi(v) &= E(v) - E(u) - \langle E'(v) - Tv, \eta(v, u)\rangle \\
&\geq E(v) - E(u) - \langle E'(v), \eta(v, u)\rangle + \alpha\|v - u\|^2 \\
&\geq \langle E'(u) - E'(v), \eta(v, u)\rangle + \alpha\|v - u\|^2 \\
&\geq (\alpha - \beta)\|v - u\|^2,
\end{aligned}$$

from which the required result (16.22) follows. □

Remark 16.2. *Following the techniques and ideas of Patriksson [143] and Zhu and Marcotte [177], one can easily discuss continuity properties of the merit function including the characterization of the solution of the variational-like inequalities (16.3) as stationary points of the problem (16.21). In a similar way, using this merit function Ψ, we can develop the descent framework for solving the variational-like inequalities. The development and implementations of such algorithms is the subject of future research efforts. Some open problems are suggested in [126] for variational-like inequalities and preinvex functions.*

We now consider the optimal control problem for variational-like inequalities (16.3), that is, find $u \in K_\eta, z \in E$ such that

$$\mathbf{P}_1 \quad \min I(u, z), \quad \langle T(u, z), \eta(v, u)\rangle \geq 0, \quad \forall \quad v \in K_\eta,$$

where the sets K_η and E are invex sets in the Hilbert spaces H and U respectively. Here $K_\eta \subset H$ is the set of state of constraints for the problem and $E \subset U$ is the set of control constraints. The functional $I(.,.) : K_\eta \times E \longrightarrow R \cup \{+\infty\}$ is proper, preinvex and lower-semicontinuous.

Related to the optimization control problem, one can consider the gap (merit) function $h_\rho(.,.)$: $H \times U \longrightarrow R$ as

$$h_\rho(u, z) = \sup_{v \in K_\eta} \{\langle -\rho T(u, z), \eta(v, u)\rangle - B(v, u)\}, \quad \forall \quad v \in H, z \in E \subset U. \tag{16.24}$$

The gap (merit) function $h_\rho(.,.)$ defined by (16.24) is a natural generalization of the gap function (7.18). Using the technique of Section 6, one can easily show that the gap function $H_\rho(.,.)$ defined by (16.24) is well-defined. In fact, we have

Theorem 16.7. *If the set K_η is an invex set in H, then the following are equivalent.*

$$\begin{aligned}
&(i). \qquad h_\rho(u, z) = 0, \quad \forall \quad u \in K_\eta, z \in E \\
&(ii). \qquad \langle T(u, z), \eta(v, u)\rangle \geq 0, \quad \forall \quad u, v \in K_\eta, z \in E.
\end{aligned}$$

Proof. Its proof is very much similar to that of Theorem 7.4. □

Theorem 16.8. *Let the operator T be η-pseudomonotone and η-hemicontinuous. If $\eta(u+t\eta(v,u),u) = t\eta(v,u)$, then the variational-like inequality problem (16.3) is equivalent to finding $u \in K_\eta$ such that*

$$\langle Tv, \eta(v,u)\rangle \geq 0, \quad \forall v \in K_\eta. \tag{16.25}$$

Proof. Let $u \in K_\eta$ be a solution of (16.3). Then

$$\langle Tu, \eta(v,u)\rangle \geq 0, \quad \forall v \in K_\eta,$$

which implies that

$$\langle Tv, \eta(v,u)\rangle \geq 0, \quad \forall v \in K_\eta,$$

since T is η-pseudomonotone.

Conversely, let $u \in K$ satisfy (16.25). Since the set K_η is an invex set, so $\forall u, v \in K_\eta, t \in [0,1], \quad v_t = u + t\eta(v,u) \in K_\eta$. Replacing v by v_t in (16.25), we have

$$\begin{aligned} 0 &\leq \langle Tv_t, \eta(v_t,u)\rangle = \langle Tv_t, \eta(u+t\eta(v,u),u)\rangle \\ &= t\langle Tv_t, \eta(v,u)\rangle, \quad \text{by assumption16.1.} \end{aligned}$$

Dividing the above inequality by t and letting $t \longrightarrow 0$, we have

$$\langle Tu, \eta(v,u)\rangle \geq 0, \quad \forall v \in K_\eta,$$

which is the required (16.3). □

If $\eta(v,u) = v-u$, then Theorem 16.8 is known as Minty Lemma. Thus Theorem 16.8 can be viewed as a generalization and extension of a result of Minty. In passing, we remark that the variational-like inequality (16.25) is also known as the dual variational-like inequality, whereas, the variational-like inequality (16.3) is called the primal variational-like inequality. Using (16.25), one can define the gap function as

$$G(u) = \sup\{\langle Tv, \eta(v,u)\rangle : v \in K_\eta\}, \quad u \in H,$$

which is known as the dual gap function. It is clear that $G(u) \geq 0 \quad \forall u \in K_\eta$; and from Theorem 16.7, it follows that $u \in S^*$, if and only if, $G(u) = 0$, where S^* is the solution set of (16.3). Consequently, any solution of the problem (16.3) is a global minimizer for the preinvex optimization problem

$$\min\{G(u) : u \in K_\eta\}$$

with zero optimal value. Note that the set K_η is an invex set, so many properties of the convex sets may not hold for the variational-like inequalities. This implies that the results obtained for the variational inequalities may not hold for the variational-like inequalities. It is an open problem to study and investigate th properties of the dual gap function $G(u)$ associated with the variational-like inequalities (16.25). One should investigate such type of problems in the setting of invexity.

17. Regularized Variational inequalities

In the previous sections, we have studied the variational inequalities over the convex subsets K in the Hilbert space H using the projection method and its variant forms including Wiener-Hopf equations. It is clear that all the techniques are strongly based on the properties of the projection operator over convex sets and these properties may not hold in general, when the set K is noncongeal. In Section 12, we have studied the variational-like inequalities on the invex sets, which are not convex sets. We have seen that the projection method and its variant forms cannot be extended for variational-like inequalities. In this section, we will consider another kind of noncongeal sets, which are called uniformly prox-regular sets. This class of noncongeal is sufficiently large to include the class of convex sets, p-convex sets and many other noncongeal sets, see [38, 145]. We make use of recent techniques and ideas from nonsmooth analysis to reformulate the variational inequalities (23.3) into an equivalent variational problem. For this purpose, we recall some well known concepts from nonsmooth analysis, see [38, 145].

Definition 17.1. *The proximal normal cone of K at u is given by*

$$N^P(K;u) := \{\xi \in H : u \in P_K[u+\alpha\xi]\},$$

where $\alpha > 0$ is a constant and

$$P_K[u] = \{u^* \in S : d_K(u) = \|u - u^*\|\}.$$

Here $d_K(.)$ is the usual distance function to the subset K, that is

$$d_K(u) = \inf_{v\in K} \|v-u\|.$$

The proximal normal cone $N^P(K;u)$ has the following characterization.

Lemma 17.1. *Let K be a closed subset in H. Then $\zeta \in N^P(K;u)$, if and only if, there exists a constant $\alpha > 0$ such that*

$$\langle \zeta, v-u\rangle \leq \alpha\|v-u\|^2, \quad \forall v \in K.$$

Definition 17.2. *The Clarke normal cone, denoted by $N^C(K;u)$, is defined as*

$$N^C(K;u) = \overline{co}[N^P(K;u)],$$

where $\overline{co}$ means the closure of the convex hull.
Clearly $N^P(K;u) \subset N^C(K;u)$, but the converse is not true. Note that $N^P(K;u)$ is always closed and convex, whereas $N^P(K;u)$ is convex, but may not be closed, see [38, 145].

Poliquin et al [145] and Clarke et al [38] have introduced and studied a new class of noncongeal sets, which are also called uniformly prox-regular sets. This class of uniformly prox-regular sets has played an important part in many noncongeal applications such as optimization, dynamic systems and differential inclusions. In particular, we have

Definition 17.3. *For a given $r \in (0,\infty]$, a subset K_r is said to be uniformly r-prox-regular, if and only if, every nonzero proximal normal to K_r can be realized by an r-ball, that is, $\forall u \in K_r$ and $0 \neq \xi \in N^P(K_r;u)$, one has*

$$\langle(\xi)/\|\xi\|, v-u\rangle \leq (1/2r)\|v-u\|^2, \quad \textit{for allv} \in K_r.$$

It is clear that the class of uniformly prox-regular sets K_r is sufficiently large to include the class of convex sets, p-convex sets, $C^{1,1}$submanifolds (possibly with boundary) of H, the images under a $C^{1,1}$ diffeomorphism of convex sets and many other noncongeal sets, see [15,131]. It is clear that if $r = \infty$, then uniform r-prox-regularity of K_r is equivalent to the convexity of K. This fact plays an important part in this paper.

It is known that, if K_r is a uniformly r-prox-regular set, then the proximal normal cone $N^P(K;u)$ is closed as a set-valued mapping. Thus, we have $N^C(K_r;u) = N^P(K_r;u)$.

For sake of simplicity, we denote $N(K_r;u) = N^C(K_r;u) = N^P(K;u)$, unless otherwise specified.

It is well-known that problem (23.3) is equivalent to finding $u \in K_r$ such that

$$T(u) \cap \{-N(K_r;u)\} \neq 0, \tag{17.1}$$

where $N(K_r;u)$ denotes the normal cone of K_r at u in the sense of convex analysis. Problem (17.1) is called the variational problem associated with the variational inequality (23.3).

If K_r is a noncongeal set, then problem (23.3) and problem (17.1) are called noncongeal variational inequality and noncongeal variational problem respectively. We now show that problems (23.3) and

(17.1) are equivalent, if the set K_r is r-prox-regular. In fact, we have the following result, which is proved using Lemma 17.1. We include its proof for sake of completeness and to convey an idea.

Lemma 17.2. *If K_r is r-prox-regular set, then the noncongeal variational problem (17.1) is equivalent to finding $u \in K_r$ such that*

$$\langle Tu, v-u\rangle + (1/2r)\|Tu\|\|v-u\|^2 \geq 0, \quad \forall \quad v \in K_r. \tag{17.2}$$

Proof. Let $u \in K_r$ be a solution of (17.2). Then

$$\langle Tu, v-u\rangle + (1/2r)\|Tu\|\|v-u\|^2 \geq 0, \quad \forall \quad v \in K_r.$$

If $Tu = 0$, then clearly zero vector always belongs to any normal cone. If $Tu \neq 0$, then $\forall u \in K_r$, we have

$$\langle(-Tu)/\|Tu\|, v-u\rangle \leq (1/2r)\|v-u\|^2.$$

Thus by invoking Lemma 17.1, we have

$$\{(-Tu)/\|Tu\|\} \in N(K_r;u) \quad \Longrightarrow \quad -Tu \in N(K_r;u),$$

which shows that $u \in$ is a solution of problem (17.1).

Conversely, if $u \in K_r$ is a solution of problem (17.1), then from Definition 17.3, it follows that $u \in K_r$ is a solution of noncongeal variational inequality (17.2). □

Remark 17.1. *Inequality of type (3.2) is called the regularized (noncongeal) variational inequality. Note that if $r = \infty$, then the regularized variational inequality (17.2) reduces to variational inequality (23.3) and noncongeal variational problem (17.1) becomes the convex variational problem. Consequently, we conclude that problems (23.3) and (17.1) are also equivalent for the convex set K.*

It is well known that the evaluation of the projection of H onto the noncongeal sets K_r is not possible except in very simple cases. To overcome these drawbacks, we again consider the auxiliary principle technique to suggest and analyze some iterative methods for the regularized variational inequalities (17.2).

For a given $u \in K_r$, a r-prox-regular set in H, consider the problem of finding a unique solution $w \in K_r$ such that

$$\langle \rho Tw + w - u, v - w\rangle \geq (-1/2r)\|Tu\|\|v-w\|^2 \quad \forall \quad v \in K_, \tag{17.3}$$

where $\rho > 0$ is a constant. Inequality of type (17.3) is called the auxiliary regularized variational inequality. Note that if $w = u$, then w is a solution of (17.2). This simple observation enables us to suggest the following iterative method for solving the regularized variational inequalities (17.2).

Algorithm 17.1. *For a given $u_0 \in K_r$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho Tu_{n+1} + u_{n+1} - u_n, v - u_{n+1}\rangle \geq (-1/2r)\|Tu_n\|\|u_{n+1} - u_n\|^2, \quad \forall v \in K_r. \tag{17.4}$$

Algorithm 17.1 is called the proximal point algorithm for solving regularized variational inequalities (17.2). In particular, if $r = \infty$, then the r-prox-regular set K_r becomes the standard convex set K, and consequently Algorithm 17.1 reduces to:

Algorithm 17.2. *For a given $u_0 \in K$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho Tu_{n+1} + u_{n+1} - u_n, v - u_{n+1}\rangle \geq 0, \quad \forall v \in K,$$

which is known as the proximal point algorithm for solving variational inequalities (23.3) and has been studied in the previous sections.

We now consider the convergence criteria of Algorithm 17.1. The analysis is in the spirit of Theorem 7.1.

Theorem 17.1. *Let the operator $T : K_r \longrightarrow H$ be pseudomonotone. If u_{n+1} is the approximate solution obtained from Algorithm 17.1 and $u \in K_r$ is a solution of (17.2), then*

$$\{1-(1/r)\|Tu\|\}\|u-u_{n+1}\|^2 \leq \|u-u_n\|^2 - \{1-(1/r)\|Tu_n\|\}\|u_n-u_{n+1}\|^2. \quad (17.5)$$

Proof. Let $u \in K_r$ be a solution of (17.2). Then

$$\langle Tu, v-u\rangle + (1/2r)\|Tu\|\|v-u\|^2 \geq 0, \quad \forall v \in K_r,$$

which implies that

$$\langle Tv, v-u\rangle + (1/2r)\|Tu\|\|v-u\|^2 \geq 0, \quad \forall v \in K_r, \quad (17.6)$$

since T is pseudomonotone.

Taking $v = u_{n+1}$ in (17.6), we have

$$\langle Tu_{n+1}, u_{n+1}-u\rangle + (1/2r)\|Tu\|\|u_{n+1}-u\|^2 \geq 0. \quad (17.7)$$

Setting $v = u$ in (17.4) and using the above inequality, we have

$$\begin{aligned}\langle u_{n+1}-u_n, u-u_{n+1}\rangle &\geq \rho\langle Tu_{n+1}, u_{n+1}-u\rangle - (1/2r)\|Tu_n\|\|u_n-u_{n+1}\|^2\\ &\geq -(1/2r)\|Tu\|\|u-u_{n+1}\|^2\\ &\quad -(1/2r)\|Tu_n\|\|u_n-u_{n+1}\|^2. \end{aligned} \quad (17.8)$$

Using $2ab = \|a+b\|^2 - \|u\|^2 - \|v\|^2, \quad a, b \in H$. we obtain

$$2\langle u_{n+1}-u_n, u-u_{n+1}\rangle = \|u-u_n\|^2 - \|u_n-u_{n+1}\|^2 - \|u-u_{n+1}\|^2. \quad (17.9)$$

From (17.8) and (17.9), we obtain (17.5), the required result. □

Theorem 17.2. *Let H be a finite dimension subspace and let u_{n+1} be the approximate solution obtained from Algorithm 17.1. If $\|Tu\| \leq r$ and $\|Tu_n\| \leq r$, and $u \in K$ is a solution of (17.2), then $\lim_{n\longrightarrow\infty} u_n = u$.*

Proof. Let $u \in K$ be a solution of (17.2). Since $\|Tu\| \leq r$, and $\|Tu_n\| \leq r$, it follows from (17.5) that the sequence $\{u_n\}$ is bounded and

$$\sum_{n=0}^{\infty}\{1-(1/r)\|Tu_n\|\|u_n-u_{n+1}\|^2 \leq \|u_0-u\|^2,$$

which implies that

$$\lim_{n\longrightarrow\infty}\|u_n-u_{n+1}\| = 0. \quad (17.10)$$

Let $\hat{u}$ be a cluster point of the sequence $\{u_n\}$ and let the subsequence $\{u_j\}$ of the sequence $\{u_n\}$ converge to $\hat{u} \in K$. replacing u_n by u_{n_j} in (17.4) and taking the limit $n_j \longrightarrow \infty$ and using (17.10), we have

$$\langle T\hat{u}, v-\hat{u}\rangle + (1/2r)\|T\hat{u}\|\|v-\hat{u}\|^2 \geq 0, \quad \forall v \in K_r,$$

which implies that $\hat{u}$ solves the regularized variational inequality (17.2) and

$$\|u_n-u_{n+1}\|^2 \leq \|\hat{u}-u_n\|^2.$$

Thus it follows from the above inequality that the sequence $\{u_n\}$ has exactly one cluster point $\hat{u}$ and $\lim_{n\longrightarrow\infty} u_n = \hat{u}$. □

We note that for $r = \infty$, the r-prox-regular set K_r becomes a convex set and regularized variational inequality (17.2) collapses to variational inequality (23.3). Thus our results include the previous known results as special cases.

It is well-known that to implement the proximal point methods, one has to calculate the approximate solution implicitly, which is in itself a difficult problem. To overcome this drawback, we suggest another iterative method, the convergence of which requires only the partially relaxed strongly monotonicity, which is a weaker condition that cocoercivity.

For a given $u \in K_r$ satisfying (17.2), consider the problem of finding $w \in K_r$ such that

$$\langle \rho Tu + w - u, v - w\rangle + (1/2r)\|Tu\|\|v - w\|^2 \geq 0, \quad \forall v \in K, \tag{17.11}$$

which is also called the auxiliary variational inequality.
Note that problems (17.3) and (17.11) are quite different. If $w = u$, then clearly w is a solution of the regularized variational inequality (17.2). This fact enables us to suggest and analyze the following iterative method.

Algorithm 17.3. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho Tu_n + u_{n+1} - u_n, v - u_{n+1}\rangle \geq (-1/2r)\|Tu_n\|\|v - u_{n+1}\|^2, \quad \forall v \in K_r.$$

Note that for $r = \infty$, the r-prox-regular set K_r becomes a convex set and Algorithm 17.3 reduces to:

Algorithm 17.4. *For a given $u_0 \in K_r$, calculate the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho Tu_n + u_{n+1} - u_n, v - u_{n+1}\rangle \geq 0, \quad \forall v \in K,$$

or equivalently

$$u_{n+1} = P_K[u_n - \rho Tu_n], \quad n = 0, 1, 2, \ldots,$$

which is known as the projection iterative method for solving variational inequalities (23.3) and have been studied extensively.

Using essentially the technique of Theorem 17.2, one can study the convergence analysis of Algorithm 17.3.

18. Equilibrium Problems

Equilibrium problems theory provides us with a unified, natural, innovative and general framework to study a wide class of problems arising in finance, economics, network analysis, transportation, elasticity and optimization. This theory has witnessed an explosive growth in theoretical advances and applications across all disciplines of pure and applied sciences. As a result of this interaction, we have a variety of techniques to study the existence results for equilibrium problems, see [29,118,124,125,135]. Equilibrium problems include variational inequalities as special cases. In previous sections,, several numerical techniques including projection, resolvent and auxiliary principle have been developed and analyzed for solving variational inequalities. It is well-known and projection and resolvent type methods can not be extended for mixed quasi variational inequalities. To overcome this drawback, one usually uses the auxiliary principle technique. We again use the auxiliary principle technique to suggest and analyze some iterative methods for equilibrium problems. We have studied the convergence criteria of these methods under some mild conditions. As a consequence of this approach, we construct the gap (merit) function for equilibrium problems, which can be used to develop descent-type methods for solving equilibrium problems.

Let K be a nonempty closed convex set in H. Let $T : H \longrightarrow H$ be a nonlinear operator. For a given nonlinear bifunction $F(.,.) : H \times H \longrightarrow H$, consider the problem of finding $u \in K$ such that

$$F(u, v) \geq 0, \quad \forall v \in K, \tag{18.1}$$

which is called the **equilibrium problem,** considered and investigated by Blub and Fettle [29] and Noor and Fettle [125] in 1994. For applications and numerical results, see [29, 118, 124, 125, 135].

If $F(u,v) = \langle Tu, \eta(v,u)\rangle$, where $\eta(.,.) : H \times H \longrightarrow H$ is a single-valued mapping, then problem (18.1) is equivalent to finding $u \in K_\eta$ such that

$$\langle Tu, \eta(v,u)\rangle \geq 0, \quad \forall v \in K_\eta, \tag{18.2}$$

which is the variational-like inequality (16.3). Here the set K_η is invex set, which may not be a convex set. It is well-known that variational-like inequality problems are closely related to the preinvex functions, which are not necessarily convex functions, see [93,105,157,158].

If $F(u,v) \equiv \langle Tu, g(v) - g(u)\rangle$, where $g, T : H \longrightarrow H$ are nonlinear single-valued operators, then problem (18.1) is equivalent to finding $u \in H : g(u) \in K$ such that

$$\langle Tu, g(v) - g(u)\rangle \geq 0, \quad \forall v \in K, \tag{18.3}$$

is called the **general variational inequality.** General variational inequalities (18.3) were introduced by Noor [89] in 1988. It has been shown that a wide class of nonsymmetric, odd-order free, moving, equilibrium and optimization problems can be studied by the general variational inequalities, see [89, 92, 94, 100, 102, 108, 117, 120, 122, 134–136] and the references therein.

If $F(u,v) = \langle Tu, v - u\rangle$, where $T : H \longrightarrow H$ is a nonlinear operator, then problem (18.1) is equivalent to finding $u \in K$ such that

$$\langle Tu, v - u\rangle \geq 0, \quad \text{for all} \quad v \in K, \tag{18.4}$$

which is the variational inequality (23.3). It is well-known that a wide class of obstacle, unilateral, contact, free, moving and equilibrium problems arising in mathematical, engineering, economics and finance can be studied in the unified and general framework of the variational inequalities. For the physical and mathematical formulation of problems (18.1)-(18.4), see [89, 92, 94, 100, 102, 108, 117, 120, 122, 134–136] and the references therein.

Definition 18.1. *The bifunction $F(.,.) : K \times K \longrightarrow H$ is said to be:*
(i). *pseudomonotone, if*

$$F(u,v) \geq 0 \quad \Longrightarrow -F(v,u) \geq 0, \quad \forall u, v \in K.$$

(ii). *partially relaxed strongly monotone, if there exists a constant $\alpha > 0$ such that*

$$F(u,v) + F(v,z) \leq \alpha \|z - u\|^2, \quad \forall u, v, z \in K.$$

(iii). *hemicontinuous, if, for all $u, v \in K$, the mapping $t \in [0,1]$ implies that $F(u + t(v - u), v)$ is continuous.*

Note that for $z = u$, partially relaxed strongly monotonicity reduce to

$$F(u,v) + F(v,u) \leq 0, \quad \forall u, v \in K,$$

which is known as the monotonicity of $F(.,.)$. It is known that monotonicity implies pseudomonotonicity, but the converse is not true.

We suggest and analyze some proximal methods for equilibrium problems (18.1) using the auxiliary principle technique.

For a given $u \in K$ satisfying (18.1, consider the auxiliary problem of finding a unique $w \in K$ such that

$$\rho F(w, v) + \langle w - u + \gamma(u - u), v - w\rangle \geq 0, \quad \forall v \in K, \tag{18.5}$$

where $\rho > 0$ and $\gamma > 0$ are constants.

We note that if $w = u$, then clearly w is solution of the equilibrium problem (18.1). This observation enables us to suggest and analyze the following iterative method for solving (18.1).

Algorithm 18.1. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho F(u_{n+1}, v) + \langle u_{n+1} - u_n + \gamma_n(u_n - u_{n-1}), v - u_{n+1}\rangle \geq 0, \quad \forall v \in K,$$

which is known as the inertial proximal method for solving equilibrium problem (18.1). Such type inertial proximal methods have been considered by Alvarez and Attouch [8] and Noor [96] for solving variational inequalities and equilibrium problems.

For $\gamma_n = 0$, Algorithm 18.1 collapses to:

Algorithm 18.2. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho F(u_{n+1}, v) + \langle u_{n+1} - u_n, v - u_{n+1}\rangle \geq 0, \quad \forall v \in K, \tag{18.6}$$

which is called the proximal method for solving problem (18.1). This shows that the inertial proximal methods include the classical proximal methods as a special case.

If $F(u, v) = \langle Tu, v - u\rangle$, where $T : K \longrightarrow H$ is a nonlinear continuous operator, then Algorithm 18.1 reduces to:

Algorithm 18.3. *For a given $u_0 \in K$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho T u_{n+1} + u_{n+1} - u_n + \gamma_n(u_n - u_{n-1}), v - u_{n+1}\rangle \geq 0, \quad \forall v \in K,$$

which can be written as

$$u_{n+1} = P_K[u_n - \rho T u_{n+1} + \gamma_n(u_n - u_{n-1})], \quad n = 0, 1, 2, \ldots,$$

where P_K is the projection of H onto the convex set K. Algorithm 18.3 is known as the inertial proximal point algorithm for solving variational inequalities and has been studied by Noor [109,117]. In a similar way, one can obtain several iterative methods for variational-like inequalities (18.2), general variational inequalities (18.3) and their special cases.

We now study the convergence analysis of Algorithm 18.2. The analysis is in the spirit of section 7. The convergence analysis of Algorithm 18.1 and Algorithm 18.3 can be studied in a similar way.

Theorem 18.1. *labelthm14.1 Let $\bar{u} \in K$ be a solution of (18.1) and u_{n+1} be the approximate solution obtained from Algorithm 18.2, then*

$$\|u_{n+1} - \bar{u}\|^2 \quad \leq \quad \|u_n - \bar{u}\|^2 - \|u_{n+1} - u_n\|^2. \tag{18.7}$$

Proof. Let $\bar{u} \in K$ be a solution of (18.1). Then

$$F(\bar{u}, v) \geq 0, \quad \text{for all} \quad v \in K,$$

which implies that

$$-F(v, \bar{u}) \geq 0, \quad \text{for all} \quad v \in K, \tag{18.8}$$

since $F(., .)$ is pseudomonotone.

Taking $v = u_{n+1}$ in (18.8), we have

$$-F(u_{n+1}, \bar{u}) \geq 0. \tag{18.9}$$

Now taking $v = \bar{u}$ in (18.6), we obtain

$$\rho F(u_{n+1}, \bar{u}) + \langle u_{n+1} - u_n, \bar{u} - u_{n+1} \rangle \geq 0. \tag{18.10}$$

From (18.9) and (18.10), we have

$$\langle u_{n+1} - u_n, \bar{u} - u_{n+1} \rangle \geq -\rho F(u_{n+1}, \bar{u}) \geq 0. \tag{18.11}$$

Setting $u = \bar{u} - u_{n+1}$ and $v = u_{n+1} - u_n$ in (7.5), we obtain

$$2\langle u_{n+1} - u_n, \bar{u} - u_{n+1} \rangle = \|\bar{u} - u_n\|^2 - \|\bar{u} - u_{n+1}\|^2 - \|u_n - u_{n+1}\|^2. \tag{18.12}$$

Combining (18.11) and (18.12), we have

$$\|u_{n+1} - \bar{u}\|^2 \leq \|u_n - \bar{u}\|^2 - \|u_{n+1} - u_n\|^2,$$

the required result. □

Theorem 18.2. *Let H be a finite dimensional space. If u_{n+1} is the approximate solution obtained from Algorithm 14.2 and $\bar{u} \in K$ is a solution of (14.1), then $\lim_{n\longrightarrow\infty} u_n = \bar{u}$.*

Proof. Let $\bar{u} \in K$ be a solution of (18.1). From (18.7), it follows that the sequence $\{\|\bar{u} - u_n\|\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. Also from (18.7), we have

$$\sum_{n=0}^{\infty} \|u_{n+1} - u_n\|^2 \leq \|u_0 - \bar{u}\|^2,$$

which implies that

$$\lim_{n\longrightarrow\infty} \|u_{n+1} - u_n\| = 0. \tag{18.13}$$

Let $\hat{u}$ be a cluster point of $\{u_n\}$ and the subsequence $\{u_{n_j}\}$ of the sequence $\{u_n\}$ converge to $\hat{u} \in H$. Replacing u_n by u_{n_j} in (18.6) and taking the limit $n_j \longrightarrow \infty$ and using (18.13), we have

$$F(\hat{u}, v) \geq 0, \quad \text{for all} \quad v \in K,$$

which implies that $\hat{u}$ solves the equilibrium problem (1) and

$$\|u_{n+1} - u_n\|^2 \leq \|u_n - \bar{u}\|^2.$$

Thus it follows from the above inequality that the sequence $\{u_n\}$ has exactly one cluster point $\hat{u}$ and

$$\lim_{n\longrightarrow\infty} u_n = \hat{u},$$

the required result. □

It is known that in order to implement the inertial proximal and proximal algorithms, one has to find the approximate solution implicitly, which is itself a difficult problem. To overcome this drawback, we suggest another iterative method for solving equilibrium problem (18.1).

For a given $u \in K$ 18.1), consider the auxiliary problem of finding a unique $w \in K$ such that

$$\rho F(u, v) + \langle w - u, v - w \rangle \geq 0, \quad \forall v \in K, \tag{18.14}$$

where $\rho > 0$ is a constant.

We note that if $w = u$, then clearly w is solution of the equilibrium problem (18.1). Note that problems (18.5) and (18.14) are quite different. In fact, problem (18.14) is equivalent to an optimization problem. This observation enables us to suggest and analyze the following iterative method for solving equilibrium problem (18.1).

Algorithm 18.4. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho F(u_n, v) + \langle u_{n+1} - u_n, v - u_{n+1}\rangle \geq 0, \quad \forall v \in K. \tag{18.15}$$

If $F(u,v) \equiv \langle Tu, v-u\rangle$, then Algorithm 18.4 is equivalent to the following iterative method for solving variational inequalities (23.3).

Algorithm 18.5. *For a given $u_0 \in K$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho Tu_n + u_{n+1} - u_n, v - u_{n+1}\rangle \geq 0, \quad \forall v \in K,$$

or equivalently

$$u_{n+1} = P_K[u_n - \rho Tu_n], \quad n = 0, 1, 2, \ldots$$

where P_K is the projection operator. Algorithm 18.5 has been studied and considered in previous sections.. For suitable and appropriate choice of the function $F(.,.)$ and the space H, one can obtain several iterative schemes for solving problems (18.1)-(4.4) and related optimization problems.

We now study the convergence analysis of Algorithm 18.4.

Theorem 18.3. *Let $\bar{u} \in K$ be a solution of (18.1) and u_{n+1} be the approximate solution obtained from Algorithm refalg14.4. If $F(.,.) : H \times H \longrightarrow H$ is partially strongly monotone with constant $\alpha > 0$, then*

$$\|u_{n+1} - \bar{u}\|^2 \leq \|u_n - \bar{u}\|^2 - (1 - 2\alpha\rho)\|u_{n+1} - u_n\|^2. \tag{18.16}$$

Proof. Let $\bar{u} \in K$ be a solution of (18.1). Then

$$F(\bar{u}, v) \geq 0, \quad \text{for all} \quad v \in K. \tag{18.17}$$

Taking $v = u_{n+1}$ in (18.17), we have

$$F(\bar{u}, u_{n+1}) \geq 0. \tag{18.18}$$

Now taking $v = \bar{u}$ in (18.15), we obtain

$$\rho F(u_n, \bar{u}) + \langle u_{n+1} - u_n, \bar{u} - u_{n+1}\rangle \geq 0. \tag{18.19}$$

From 18.19) and (18.18), we have

$$\begin{aligned} \langle u_{n+1} - u_n, \bar{u} - u_{n+1}\rangle &\geq -\rho\{F(u_n, \bar{u}) + F(\bar{u}, u_{n+1})\} \\ &\geq -\alpha\rho\|u_n - u_{n+1}\|^2, \end{aligned} \tag{18.20}$$

since $F(.,.)$ is partially relaxed strongly monotone with a constant $\alpha > 0$.

Combining (18.12) and (18.20), we have

$$\|u_{n+1} - \bar{u}\|^2 \leq \|u_n - \bar{u}\|^2 - (1 - 2\rho\alpha)\|u_{n+1} - u_n\|^2,$$

the required result. □

Theorem 18.4. *Let H be a finite dimensional space and let $0 < \rho < \frac{1}{2\alpha}$. If u_{n+1} is the approximate solution obtained from Algorithm 18.4 and $\bar{u} \in H$ is a solution of (18.1), then $\lim_{n\longrightarrow\infty} u_n = \bar{u}$.*

Proof. Its proof is similar to Theorem 14.2. □

It is obvious that the auxiliary equilibrium problem (18.14) is equivalent to finding the minimum of the functional $I[w]$ over the convex set K, where

$$I[w] = \frac{1}{2}\langle w - u, w - u\rangle - \rho F(u, w), \quad \forall w \in H, \tag{18.21}$$

which is known as the auxiliary energy (virtual work, potential) function associated with the problem (18.14). Using this functional $I[w]$, one can reformulate the equilibrium problem (18.1) as an equivalent optimization problem:

$$\Psi_\alpha(u) = \max_{w\in K}\{-\rho F(u,w) - (\alpha/2)\|u-w\|^2\}, \tag{18.22}$$

where $\alpha > 0$ is a constant. Function of the type $\Psi(u)$ defined by (18.22) is called the regular gap function for the equilibrium problem. Note that for $\alpha = 0$, and $F(u,v) \equiv \langle Tu, v-u\rangle$, we obtain the original gap function for the variational inequality (23.3), which is due to Fukushima [54]. From the above discussion and observation, it is clear that can obtain the gap (merit) function for the equilibrium problems (18.1) by using the auxiliary principle technique. In passing, we remark this is observation is due to Noor [117, 118], where it has been shown that the auxiliary principle technique can be used to construct gap functions for several variational inequalities. This equivalent optimization formulation of the equilibrium problems can be used to develop some descent-type algorithms for solving equilibrium problems under suitable conditions on the bifunction $F(.,.)$, by using the technique of Fukushima [54].

19. Well-Posed Equilibrium Problems

In recent years, much attention has been given to introduce the concept of well-posedness for variational of variational inequalities, see [59, 77, 78, 117] and the references therein. In this Section, we introduce the similar concepts of well-posedness for equilibrium problems of type (18.1). The results obtained can be considered as a natural generalization of previous results of Luccheti and Patrone [77,78], Goeleven and Mantague [59] and Noor [117]. For this purpose, we define the following:

For a given $\epsilon > 0$, we consider the sets

$$A(\epsilon) = \{u \in K : F(u,v) \geq -\epsilon\|v-u\|, \quad \text{for all} \quad v \in K\}$$

and

$$B(\epsilon) = \{u \in K : F(v,u) \leq \epsilon\|v-u\|, \quad \text{for all} \quad v \in K.\}$$

For a nonempty set $X \subset H$, we define the diameter of X, denoted by $D(X)$, as

$$D(X) = \sup\{\|v-u\|; \quad \text{for all} \quad u,v \in X.\}$$

Definition 15.1. We say that the equilibrium problem (14.1) is *well-posed,* iff

$$A(\epsilon) \neq \phi \quad \text{and} \quad D(A(\epsilon)) \longrightarrow 0, \text{as} \quad \epsilon \longrightarrow 0.$$

For $F(u,v) = \langle Tu, v-u\rangle$, our definition of well-posedness is exactly the same as one introduced by Luccheti and Patrone [56,57] for variational inequalities and extended by Noor [117] and Goeleven and Mantague [59] for variational-like inequalities and hemivariational inequalities respectively.

Theorem 19.1. *Let the bifunction $F(.,.)$ be pseudomonotone, hemicontinuous and convex in the second argument. Then*

$$A(\epsilon) = B(\epsilon).$$

Proof. Let $u \in K$ be such that

$$F(u,v) \geq -\epsilon\|v-u\|, \quad \text{for all} \quad v \in K,$$

which implies that

$$F(v,u) \leq \epsilon\|v-u\|, \quad \text{for all} \quad v \in K, \tag{19.1}$$

since $F(.,.)$ is pseudomonotone.

Thus

$$A(\epsilon) \subset B(\epsilon). \tag{19.2}$$

Conversely, let $u \in K$ such that (15.1) hold. Since K is a convex set, for all $u, v \in K, t \in [0,1]$, $v_t = u + t(v-u) \equiv (1-t)u + tv \in K$.

Taking $v = v_t$ in (15.1), we have

$$F(v_t, u) \leq t\epsilon\|v-u\|. \tag{19.3}$$

Also

$$\begin{aligned} 0 &= F(v_t, v_t) \\ &\leq tF(v_t, v) + (1-t)F(v_t, u) \\ &\leq tF(v_t, v) + (1-t)t\epsilon\|v-u\|, \end{aligned}$$

where we have used (15.3).

Dividing the above inequality by t and letting $t \longrightarrow 0$, we have

$$F(u, v) \geq -\epsilon\|v-u\|, \quad \text{for all} \quad v \in K,$$

which implies that

$$B(\epsilon) \subset A(\epsilon). \tag{19.4}$$

Thus from (15.2) and (15.4), we have

$$A(\epsilon) = B(\epsilon),$$

the required result. □

Theorem 19.2. *The set $B(\epsilon)$ is closed.*

Proof. Let $\{u_n : n \in N\} \subset B(\epsilon)$ be such that $u_n \longrightarrow u$ in K as $n \longrightarrow \infty$. This implies that $u_n \in K$ and

$$F(v, u_n) \leq \epsilon\|v-u_n\|, \quad \text{for all} \quad v \in K.$$

Taking the limit in the above inequality as $n \longrightarrow \infty$, we have

$$F(v, u) \leq \epsilon\|v-u\|, \quad \text{for all} \quad v \in K,$$

which implies that $u \in K$, since K is a closed and convex set. Consequently, it follows that the set $B(\epsilon)$ is closed. □

Using essentially the technique of Goeleven and Mantague [59], we can prove the following results. To convey an idea and for the sake of completeness, we include their proofs.

Theorem 19.3. *Let $F(.,.)$ be pseudomonotone and hemicontinuous. If the equilibrium problem (14.1) is well-posed, then equilibrium problem (14.1) is unique.*

Proof. Let us define the sequence $\{u_k : k \in N\}$ by $u_k \in A(1/k)$. Let $\epsilon > 0$ be sufficiently small and let $m, n \in N$ such that $n \geq m \geq \frac{1}{\epsilon}$. Then

$$A(\frac{1}{n}) \subset A(\frac{1}{m}) \subset A(\epsilon).$$

Thus

$$\|u_n - u_m\| \leq D(A(\frac{1}{n})),$$

which implies that the sequence $\{u_n\}$ is a Cauchy sequence and it converges, that is, $u_k \longrightarrow u$ in K. From Theorem 19.1 and Theorem 15.2, we know that the set $A(\epsilon)$ is a closed set. Thus

$$u \in \cup_{\epsilon>0} A(\epsilon),$$

so that u is solution of the equilibrium problem (18.1). From the second condition of well-posedness, we see that the solution of the equilibrium problem (18.1) is unique. □

Theorem 19.4. *Let $F(.,.)$ be pseudomnotone and hemicontinuous. If $A(\epsilon) \neq 0, \forall \epsilon > 0$. and $A(\epsilon)$ is bounded for some ϵ_0, then the equilibrium problem (18.1) has at least one solution.*

Proof. Let $u_n \in A(1/n)$. Then $A(1/n) \subset A(\epsilon)$, for n large enough. Thus for some subsequence $u_n \longrightarrow u \in K$, we have

$$\begin{aligned} F(v, u_n) &\leq \frac{1}{n}\|v - u_n\|, \quad \text{for all} \quad v \in K \\ &\leq \frac{1}{n}\{\|v\| + c\}, \quad \text{for all} \quad v \in K. \end{aligned}$$

Taking the limit as $n \longrightarrow \infty$, we have

$$F(v, u) \leq 0,$$

which implies that $u \in B(0) = A(0)$, by Theorem 19.1. This shows that $u \in A(0)$, from which it follows that the equilibrium problem (14.1) has at least one solution. □

Remark 19.1. I. *If the equilibrium problem (18.1) has a unique solution, then it is clear that $A(\epsilon) \neq 0, \forall \epsilon > 0$ and $\cap_{\epsilon > 0} A(\epsilon) = \{u_0\}$.*

II. *It is known that [77, 78]61,62] if the variational inequality (2.1) has a unique solution, then it is not well-posed.*

III. *From Theorem 19.3, we conclude that the unique solution of the equilibrium problem (18.1) can be computed by using the ϵ-equilibrium problem, that is, find $u_\epsilon \in K$ such that*

$$F(u_\epsilon, v) \geq -\epsilon\|v - u_\epsilon\|, \quad \forall v \in K.$$

20. Bivariational inequalities

In this section, we consider the biconvex variational inequalities, which are closely related to the biconvex functions. Noor and Noor [32, 33, 35] introduced and studied some new classes of biconvex sets and biconvex functions. These new concepts may be viewed as refinements of the known concepts. They have shown that the biconvex functions enjoy some nice properties, which convex have. We mainly review these developments and include the necessary information for the convenience of the interested readers. It is shown that the optimality conditions of the differentiable biconvex functions can be characterized by a class of variational inequalities, which is called bidirectional inequality. Due to the inherent structure of these bidirectional inequalities, the projection-type methods and their invariant forms can not be used for solving the bidirectional inequalities. To overcome this drawback, one usually uses the auxiliary principle technique. This technique deals with finding a suitable auxiliary problem and proving that the solution of the auxiliary problem is the solution of the original problem by using the fixed point approach. We use this technique to suggest and analyze several methods for solving bidirectional inequalities and variational inequalities. One can show that a substantial number of numerical methods can be obtained as special cases from this technique. Some iterative methods are suggested for solving bidirectional inequalities using the auxiliary principle technique. Convergence criteria is also discussed using the pseudo monotonicity, which is a weaker condition then monotonicity. It is expected that the ideas and techniques of this paper may stimulate further research in this field.

Let K_β be a closed biconvex set in H. Let $F : K_\beta \to R$ be a continuous function and let $\beta(. - .) : K_\beta \times K_\beta \to R$ be an arbitrary continuous bifunction.
We now recall the known concepts and basic results, which are mainly due to Noor et al. [32, 33, 35].

Definition 20.1. *The set K_β in H is said to be biconvex set with respect to an arbitrary bifunction $\beta(\cdot - \cdot)$, if*

$$u + \lambda\beta(v - u) \in K_\beta, \qquad \forall u, v \in K_\beta, \lambda \in [0, 1].$$

The biconvex set K_β is also called β-connected set. Note that the biconvex set with $\beta(v,u) = v - u$ is a convex set K, but the converse is not true. For example, the set $K_\beta = R - (-\frac{1}{2}, \frac{1}{2})$ is an biconvex set with respect to η, where

$$\beta(v-u) = \begin{cases} v-u, & \text{for} \quad v>0, u>0 \quad \text{or} \quad v<0, u<0 \\ u-v, & \text{for} \quad v<0, u>0 \quad \text{or} \quad v<0, u<0. \end{cases} \tag{20.1}$$

It is clear that K_β is not a convex set.

From now onward K_β is a nonempty closed biconvex set in H with respect to the bifunction $\beta(\cdot - \cdot)$, unless otherwise specified.

We now introduce some new concepts of biconvex functions and their variants forms, which is the main motivation of this paper.

Definition 20.2. *The function F on the biconvex set K_β is said to be a biconvex with respect to the bifunction $\beta(\cdot - \cdot)$, if*

$$F(u + \lambda\beta(v-u)) \leq (1-\lambda)F(u) + \lambda F(v), \forall u, v \in K_\beta, \lambda \in [0,1]. \tag{20.2}$$

The function F is said to be biconcave, if and only if, $-F$ is biconvex function. Consequently, we have a new concept.

Definition 20.3. *A function F is said to be affine involving an arbitrary bifunction $\beta(\cdot - \cdot)$, if*

$$F(u + \lambda\beta(v-u)) = (1-\lambda)F(u) + \lambda F(v), \forall u, v \in K_\beta, \lambda \in [0,1]. \tag{20.3}$$

Note that every convex function is a biconvex, but the converse is not true.

If $\beta(v-u) = v-u$, then the biconvex function becomes a convex function, that is,

$$F(u + \lambda(v-u)) \leq (1-\lambda)F(u) + \lambda F(v), \quad \forall u, v \in K, \lambda \in [0,1].$$

For the properties of the convex functions in variational inequalities and equilibrium problems, see Noor [**?**, 11, 29–31].

Definition 20.4. *The function F on the biconvex set K_β is said to be quasi biconvex with respect to the bifunction $\beta(\cdot - \cdot)$, if*

$$F(u + \lambda\beta(v-u)) \leq \max\{F(u), F(v)\}, \quad \forall u, v \in K_\beta, \lambda \in [0,1].$$

Definition 20.5. *The function F on the biconvex set K_β is said to be* log*-biconvex with respect to the bifunction $\beta(\cdot - \cdot)$, if*

$$F(u + \lambda\beta(v-u)) \leq (F(u))^{1-\lambda}(F(v))^{\lambda}, \quad \forall u, v \in K_\beta, \lambda \in [0,1].$$

where $F(\cdot) > 0$.

We can rewrite the Definition 20.5 in the following equivalent form

Definition 20.6. *The function F on the biconvex set K_β is said to be* log*-biconvex with respect to the bifunction $\beta(\cdot - \cdot)$, if*

$$\log F(u + \lambda\beta(v-u)) \leq (1-\lambda)\log F(u) + \lambda \log F(v), \quad \forall u, v \in K_\beta, \lambda \in [0,1].$$

where $F(\cdot) > 0$.

This equivalent definition can be used to discus the properties of the differentiable log-biconvex functions.

From the above definitions, we have

$$\begin{aligned} F(u+\lambda\beta(v-u)) &\leq (F(u))^{1-\lambda}(F(v))^{\lambda} \\ &\leq (1-\lambda)F(u)+\lambda F(v) \\ &\leq \max\{F(u),F(v)\}. \end{aligned}$$

This shows that every log-biconvex function is biconvex function and every biconvex function is a quasi-biconvex function. However, the converse is not true.

For $\lambda = 1$, Definition **??** and **??** reduce to the following condition.

Condition A.

$$F(u+\beta(v-u)) \leq F(v), \quad \forall v,u \in K_{\beta},$$

which is called the Condition A.

We now define the biconvex functions on the interval $K_{\beta} = I_{\beta} = [a, a+\beta(b-a)]$.

Definition 20.7. *Let $I = [a, a+\beta(b-a)]$. Then F is biconvex function, if and only if,*

$$\begin{vmatrix} 1 & 1 & 1 \\ a & x & a+\beta(b-a) \\ F(a) & F(x) & F(b) \end{vmatrix} \geq 0; \quad a \leq x \leq a+\beta(b-a).$$

One can easily show that the following are equivalent:

(1) F is a biconvex function.
(2) $F(x) \leq F(a) + \frac{F(b)-F(a)}{\beta(b-a)}(x-a)$.
(3) $\frac{F(x)-F(a)}{x-a} \leq \frac{F(b)-F(a)}{\beta(b-a)}$.
(4) $\frac{F(a)}{(\beta(b-a))(a-x)} + \frac{F(x)}{(x-a-\beta(b-a))(a-x)} + \frac{F(b)}{\beta(b-a)(x-b)} \leq 0$,

where $x = a+\lambda\beta(b-a) \in [a, a+\beta(b-a)]$.

We now consider some basic properties of biconvex functions and their variant forms.

Theorem 20.1. *Let F be a strictly biconvex function. Then any local minimum of F is a global minimum.*

Proof. Let the biconvex function F have a local minimum at $u \in K_{\beta}$. Assume the contrary, that is, $F(v) < F(u)$ for some $v \in K_{\beta}$. Since F is a biconvex function, so

$$F(u+\lambda\beta(v-u)) < \lambda F(v) + (1-\lambda)F(u), \quad \text{for} \quad 0 < \lambda < 1.$$

Thus

$$F(u+\lambda\beta(v-u)) - F(u) < \lambda[F(v)-F(u)] < 0,$$

from which it follows that

$$F(u+\lambda\beta(v-u)) < F(u),$$

for arbitrary small $\lambda > 0$, contradicting the local minimum. □

Theorem 20.2. *If the function F on the convex set K_{β} is biconvex, then the level set*

$$L_{\alpha} = \{u \in K : F(u) \leq \alpha, \quad \alpha \in R\}$$

is a biconvex set.

Proof. Let $u, v \in L_\alpha$. Then $F(u) \leq \alpha$ and $F(v) \leq \alpha$.
Now, $\forall \lambda \in (0,1), \quad w = u + \lambda\beta(v-u) \in K_\beta$, since K_β is a biconvex set. Thus, by the biconvexity of F, we have

$$\begin{aligned} F(u+\lambda\beta(v-u)) &\leq (1-\lambda)F(u) + \lambda F(b) \\ &\leq (1-t)\alpha + t\alpha = \alpha, \end{aligned}$$

from which it follows that $u + t\beta(v-u) \in L_\alpha$ Hence L_α is a biconvex set. □

Theorem 20.3. *A positive function F is a biconvex, if and only if*

$$epi(F) = \{(u,\alpha) : u \in K_\beta : F(u) \leq \alpha, \alpha \in R\}$$

is a biconvex set.

Proof. Assume that$F$ is biconvex function. Let $(u,\alpha), \quad (v,\beta) \in epi(F)$. Then it follows that $F(u) \leq \alpha$ and $F(v) \leq \beta$. Thus, $\forall \lambda \in [0,1], \quad u, v \in K_\beta$, we have

$$\begin{aligned} F(u+\lambda\beta(v-u)) &\leq (1-\lambda)F(u) + \lambda F(v) \\ &\leq (1-t)\alpha + t\beta, \end{aligned}$$

which implies that

$$(u+\lambda\beta(v-u), (1-\lambda)\alpha + \lambda\beta) \in epi(F).$$

Thus $epi(F)$ is a biconvex set. Conversely, let $epi(F)$ be a biconvex set. Let $u, v \in K_\beta$. Then $(u, F(u)) \in epi(F)$ and $(v, F(v)) \in epi(F)$. Since $epi(F)$ is a biconvex set, we must have

$$(u+\lambda\beta(v-u), (1-\lambda)F(u) + \lambda F(v)) \in epi(F),$$

which implies that

$$F(u+\lambda\beta(v-u)) \leq (1-\lambda)F(u) + \lambda F(v).$$

This shows that F is a biconvex function. □

Theorem 20.4. *A positive function F is quasi biconvex, if and only if, the level set*

$$L_\alpha = \{u \in K_\beta, \alpha \in R : F(u) \leq \alpha\}$$

is a biconvex set.

Proof. Let $u, v \in L_\alpha$. Then $u, v \in K_\beta$ and $\max(F(u), F(v)) \leq \alpha$.
Now for $\lambda \in (0,1), w = u + \lambda\beta(v-u) \in K_\beta$, We have to prove that $u + \lambda\beta(v-u) \in L_\alpha$. By the quasi biconvexity of F, we have

$$F(u+\lambda\beta(v-u)) \leq \max(F(u), F(v)) \leq \alpha,$$

which implies that $u + \lambda\beta(v-u) \in L_\alpha$, showing that the level set L_α is indeed a biconvex set.

Conversely, assume that L_α is a biconvex set. Then $\forall u, v \in L_\alpha, \lambda \in [0,1]$,

$$u + \lambda\beta(v-u) \in L_\alpha.$$

Let $u, v \in L_\alpha$ for

$$\alpha = max(F(u), F(v) \quad \text{and} \quad F(v) \leq F(u).$$

From the definition of the level set L_α, it follows that

$$F(u+\lambda\beta(v-u)) \leq \max(F(u), F(v)) \leq \alpha.$$

Thus F is a quasi biconvex function. This completes the proof. □

Theorem 20.5. *Let F be a biconvex function.. Let $\mu = \inf_{u \in K_\beta} F(u)$. Then the set $E = \{u \in K_\beta : F(u) = \mu\}$ is a biconvex set of K_β. If F is strictly biconvex, then E is a singleton.*

Proof. Let $u, v \in E$. For $0 < \lambda < 1$, let $w = u + \lambda\beta(v - u)$. Since F is a biconvex function, so

$$F(w) = F(u + \lambda\beta(v - u)) \leq (1 - \lambda)F(u) + \lambda F(v) = \lambda\mu + (1 - \lambda)\mu = \mu,$$

which implies that $w \in E$ and hence E is a biconvex set. For the second part, assume to the contrary that $F(u) = F(v) = \mu$. Since K_β is a biconvex set, then for $0 < \lambda < 1, u + \lambda\beta(v - u) \in K_\beta$. Further, since F is strictly biconvex,

$$\begin{aligned} F(u + \lambda\beta(v - u)) &< (1 - \lambda)F(u) + \lambda F(v) \\ &= (1 - t)\mu + t\mu = \mu. \end{aligned}$$

This contradicts the fact that $\mu = \inf_{u \in K_\beta} F(u)$ and hence the result follows. □

Theorem 20.6. *If F is a biconvex function such that $F(v) < F(u), \forall u, v \in K_\beta$, then F is a strictly quasi biconvex function.*

Proof. By the biconvexity of the function F, $\forall u, v \in K_\beta, \lambda \in [0, 1]$, we have

$$F(u + \lambda\beta(v - u)) \leq (1 - \lambda)F(u) + \lambda F(v) < F(u),$$

since $F(v) < F(u)$, which shows that the function F is strictly quasi biconvex. □

We now derive some properties of the differentiable log-biconvex functions.

To derive the main results, we need the following assumptions regarding the bifunction $\beta(\cdot - \cdot)$.

Condition M. The bifunction $\beta(, -,)$ said to satisfy the assumptions, if

$$\begin{aligned} (i). \quad \beta(\gamma\beta(v - u)) &= \gamma\beta(v - u), \forall u, v \in K_\beta, \quad \gamma \in R^n. \\ (ii). \quad \beta(v - u - \gamma\beta(v - u)) &= (1 - \gamma)\beta(v - u), \forall u, v \in K_\beta. \end{aligned}$$

Remark 20.1. *Let $\beta(\cdot - \cdot) : K_\beta \times K_\beta \to H$ satisfy the assumption*

$$\beta(v - u) = \beta(v - z) + \beta(z - u), \forall u, v, z \in K_\beta.$$

One can easily show that $\beta(v - u) = 0 \quad \forall u, v \in K_\beta$. Consequently

$$\beta(v - u) = 0 \quad \Leftrightarrow \quad v = u, \forall u, v \in K_\beta.$$

Also

$$\beta(v - u) + \beta(u - v) = 0, \quad \forall u, v \in K_\beta.$$

This implies that the bifunction $\beta(. - .)$ is skew symmetric.

Theorem 20.7. *Let F be a differentiable function on the biconvex set K_β and let the condition M hold. Then the function F is* log*-biconvex function, if and only if,*

$$\log F(v) - \log F(u) \geq \langle \frac{F'(u)}{F(u)}, \beta(v - u)\rangle, \quad \forall v, u \in K_\beta. \tag{20.4}$$

Proof. Let F be a log-biconvex function. Then

$$\log F(u + \lambda\beta(v - u)) \leq (1 - \lambda)\log F(u) + \lambda \log F(v), \qquad \forall u, v \in K_\beta,$$

which can be written as

$$\log F(v) - \log F(u) \geq \{\frac{\log F(u + \lambda\beta(v - u)) - \log F(u)}{\lambda}\}.$$

Taking the limit in the above inequality as $\lambda \to 0$, we have

$$\log F(v) - \log F(u) \geq \langle \frac{F'(u)}{F(u)}, \beta(v - u)\rangle,$$

which is (20.4), the required result.

Conversely, let (20.4) hold. Then $\forall u, v \in K_\beta, \lambda \in [0,1]$, $v_\lambda = u + \lambda\beta(v-u) \in K_\beta$ and using the condition M, we have

$$\begin{aligned} \log F(v) - \log F(v_\lambda) &\geq \langle \frac{F'(v_\lambda)}{F(v_\lambda)}, \beta(v - v_\lambda))\rangle \\ &= (1-\lambda)\langle \frac{F'(v_\lambda)}{F(v_\lambda)}, \beta(v-u)\rangle. \end{aligned} \tag{20.5}$$

In a similar way, we have

$$\begin{aligned} \log F(u) - \log F(v_\lambda) &\geq \langle \frac{F'(v_\lambda)}{F(v_\lambda)}, \beta(u - v_\lambda)\rangle \\ &= -\lambda\langle \frac{F'(v_\lambda)}{F(v_\lambda)}, \beta(v-u)\rangle. \end{aligned} \tag{20.6}$$

Multiplying (20.5) by λ and (20.6) by $(1-\lambda)$ and adding the resultant, we have

$$\log F(u + \lambda\beta(v-u)) \leq (1-\lambda)\log F(u) + \lambda \log F(v),$$

showing that F is a log-biconvex function. □

Remark 20.2. *From (23.2), we have*

$$F(v) \geq F(u)exp\{\langle \frac{F'(u)}{F(u)}, \beta(v-u)\rangle\}, \quad u, v \in K_\beta.$$

Changing the role of u and v in the above inequality, we also have

$$F(u) \geq F(v)exp\{\langle \frac{F'(v)}{F(v)}, \beta(u-v)\rangle\}, \quad u, v \in K_\beta.$$

Thus, we can obtain the following inequality

$$\begin{aligned} F(u) + F(v) &\geq F(v)exp\{\langle \frac{F'(v)}{F(v)}, \beta(u-v)\rangle\}, \\ &\quad + F(u)exp\{\langle \frac{F'(u)}{F(u)}, \beta(v-u)\rangle\} \quad u, v \in K_\beta. \end{aligned}$$

Theorem 20.8. *Let F be a differentiable function on the biconvex set K_β and Condition M hold. Then the function F is* log*-biconvex function, if and only if,*

$$\langle \frac{F'(u)}{F(u)}, \beta(v-u)\rangle + \langle \frac{F'(v)}{F(v)}, \beta(u-v)\rangle \leq 0, \quad \forall v, u \in K_\beta. \tag{20.7}$$

Proof. Let F be a differentiable function on the biconvex set K_β. Then, from Theorem **??**, it follows that

$$\log F(v) - \log F(u) \geq \langle \frac{F'(u)}{F(u)}, \beta(v-u)\rangle, \quad \forall v, u \in K_\beta. \tag{20.8}$$

Changing the role of u and v in (20.8), we have

$$\log F(u) - \log F(v) \geq \langle \frac{F'(v)}{F(v)}, \beta(v-u)\rangle, \quad \forall v, u \in K_\beta. \tag{20.9}$$

Adding (20.8) and (20.9), we have

$$\langle \frac{F'(u)}{F(u)}, \beta(v-u)\rangle + \langle \frac{F'(v)}{F(v)}, \beta(u-v)\rangle \leq 0, \quad \forall v, u \in K_\beta,$$

which is the required (20.7).
Since K_β is a biconvex set, so, $\forall u, v \in K_\beta, \quad \lambda \in [0,1]$, $v_\lambda = u + \lambda\beta(v-u) \in K_\beta$. Conversely, from (20.7), we have

$$\begin{aligned}\langle \frac{F'(v_\lambda)}{F(v_\lambda)}, \beta(u - v_\lambda)\rangle &\leq \langle \frac{F'(u)}{F(u)}, \beta(u - v_\lambda)\rangle \\ &= -\lambda\langle \frac{F'(u)}{F(u)}, \beta(v-u)\rangle, \end{aligned} \tag{20.10}$$

which implies that

$$\langle \frac{F'(v_\lambda)}{F(v_\lambda)}, \beta(v-u)\rangle \geq \langle \frac{F'(u)}{F(u)}, \beta(v-u)\rangle. \tag{20.11}$$

Consider the auxiliary function

$$\xi(\lambda) = \log F(u + \lambda(v-u)),$$

from which, we have

$$\xi(1) = \log F(v), \quad \xi(0) = \log F(u).$$

Then, from (20.11), we have

$$\xi'(\lambda) = \langle \frac{F'(v_\lambda)}{F(v_\lambda)}, \beta(v-u)\rangle \geq \langle \frac{F'(u)}{F(u)}, \beta(v-u)\rangle. \tag{20.12}$$

Integrating (20.12) between 0 and 1, we have

$$\xi(1) - \xi(0) = \int_0^1 \xi'(t)dt \geq \langle \frac{F'(u)}{F(u)}, \beta(v-u)\rangle.$$

Thus it follows that

$$\log F(v) - \log F(u) \geq \langle \frac{F'(u)}{F(u)}, \beta(v-u)\rangle,$$

which is the required (20.4). □

Definition 20.8. *An operator $T : K_\beta \to H$ is said to be:*

(1) *β-monotone, iff,*
$$\langle Tu, \beta(v-u)\rangle + \langle Tv, \beta(u-v)\rangle \leq 0, \quad \forall u, v \in K_\beta.$$
(2) *β-pseudomonotone, iff,*
$$\langle Tu, \beta(v-u)\rangle \geq 0 \Rightarrow -\langle Tv, \beta(u-v)\rangle \geq 0, \quad \forall u, v \in K_\beta.$$
(3) *relaxed β-pseudomonotone, iff,*
$$\langle Tu, \beta(v-u)\rangle \geq 0 \Rightarrow -\langle Tv, \beta(u-v)\rangle \geq 0, \quad \forall u, v \in K_\beta.$$
(4) *strictly β-monotone, iff,*
$$\langle Tu, \beta(v-u)\rangle + \langle Tv, \beta(u-v)\rangle < 0, \quad \forall u, v \in K_\beta.$$
(5) *β-pseudomonotone, iff,*
$$\langle Tu, \beta(v-u)\rangle \geq 0 \Rightarrow \langle Tv, \eta(u,v)\rangle \leq 0, \quad \forall u, v \in K_\beta.$$
(6) *quasi β-monotone, iff,*
$$\langle Tu, \beta(v-u)\rangle > 0 \Rightarrow \langle Tv, \beta(u-v)\rangle \leq 0, \quad \forall u, v \in K_\beta.$$
(7) *strictly β-pseudomonotone, iff,*
$$\langle Tu, \beta(v-u)\rangle \geq 0 \Rightarrow \langle Tv, \beta(u-v)\rangle < 0, \quad \forall u, v \in K_\beta.$$

Definition 20.9. *A differentiable function F on the biconvex set K_η is said to be pseudo β-biconvex function, iff,*

$$\langle\frac{F'(u)}{F(u)},\beta(v-u)\rangle \geq 0 \Rightarrow F(v)-F(u)\geq 0, \qquad \forall u,v\in K_\beta.$$

Definition 20.10. *A differentiable function F on K_β is said to be quasi-biconvex function, iff,*

$$\begin{aligned} & F(v)\leq F(u) \\ & \Rightarrow \\ & \langle\frac{F'(u)}{F(u)},\beta(v-u)\rangle \leq 0, \quad \forall u,v\in K_\beta. \end{aligned}$$

Definition 20.11. *The function F on the set K_β is said to be pseudo-biconvex, iff,*

$$\langle\frac{F'(u)}{F(u)},\beta(v-u)\rangle \geq 0 \Rightarrow F(v)\geq F(u), \qquad \forall u,v\in K_\beta.$$

Definition 20.12. *The differentiable function F on the K_β is said to be quasi-biconvex function, iff,*

$$F(v)\leq F(u) \Rightarrow \langle\frac{F'(u)}{F(u)},\beta(v-u)\rangle \leq 0, \qquad \forall u,v\in K_\beta.$$

We remark that the concepts introduced in this paper represent significant improvement of the previously known ones. All these new concepts may play important and fundamental part in the mathematical programming and optimization.

Theorem 20.9. *Let F be a differentiable function on the biconvex set K_β in H and let the condition M hold. Then the function F is a biconvex function, if and only if, F is a biconvex function.*

Proof. Let F be a biconvex function on the biconvex set K_β. Then

$$F(u+\lambda\beta(v-u)) \leq (1-\lambda)F(u)+\lambda F(v) \quad \forall u,v\in K_\beta, \lambda\in[0,1].$$

which can be written as

$$F(v)-F(u)\geq \{\frac{F(u+\lambda\beta(v-u))-F(u)}{\lambda}\}.$$

Taking the limit in the above inequality as $\lambda\to 0$, we have

$$F(v)-F(u)\geq \langle F'(u),\beta(v-u))\rangle.$$

This shows that F is a biconvex function.

Conversely, let F be a biconvex function on the biconvex set K_β. Then, $\forall u,v\in K_\beta, \lambda\in[0,1]$, $v_t = u+\lambda\beta(v-u)\in K_\beta$ and using the condition M, we have

$$\begin{aligned} & F(v)-F(u+\lambda\beta(v-u)) \\ \geq\ & \langle F'(u+\lambda\beta(v-u)),\beta(v-u+\lambda\beta(v-u))\rangle \\ =\ & (1-\lambda)F'(u+\lambda\beta(v-u)),\beta(v-u)\rangle. \end{aligned} \tag{20.13}$$

In a similar way, we have

$$\begin{aligned} & F(u)-F(u+\lambda\beta(v-u)) \\ \geq\ & \langle F'(u+\lambda\beta(v-u)),\beta(u,u+\lambda\beta(v-u))\rangle \\ =\ & -\lambda F'(u+\lambda\beta(v-u)),\beta(v-u)\rangle. \end{aligned} \tag{20.14}$$

Multiplying (20.13) by λ and (20.14) by $(1-\lambda)$ and adding the resultant, we have

$$F(u+\lambda\beta(v-u)) \leq (1-\lambda)F(u)+\lambda F(v),$$

showing that F is a biconvex function. □

Theorem 20.10. *Let F be a differentiable biconvex function on the biconvex set K_β. If F is a biconvex function, then*

$$\langle F'(u), \beta(v-u))\rangle + \langle F'(v), \beta(u-v)\rangle \le 0, \forall u, v \in K_\beta. \tag{20.15}$$

Proof. Let F be a biconvex function on the biconvex set K_β. Then

$$F(v) - F(u) \ge \langle F'(u), \beta(v-u))\rangle, \quad \forall u, v \in K_\beta. \tag{20.16}$$

Changing the role of u and v in (20.16), we have

$$F(u) - F(v) \ge \langle F'(v), \beta(u-v)\rangle, \quad \forall u, v \in K_\beta. \tag{20.17}$$

Adding (20.16) and (20.17), we have

$$\langle F'(u), \beta(v-u))\rangle + \langle F'(v), \beta(u-v)\rangle \le 0, \forall u, v \in K_\beta, \tag{20.18}$$

which shows that $F'(.)$ is a β-monotone operator. □

Theorem 20.11. *If the differential $F'(.)$ is a β-monotone, then*

$$F(v) - F(u) \ge \langle F'(u), \beta(v-u)\rangle.$$

Proof. Let $F'(.)$ be a β-monotone. From (20.18), we have

$$\langle F'(v), \beta(u-v)\rangle \ge \langle F'(u), \beta(v-u))\rangle. \tag{20.19}$$

Since K_β is an biconvex set, $\forall u, v \in K_\beta$, $\lambda \in [0,1]$ $v_t = u + \lambda\beta(v-u) \in K_\beta$. Taking $v = v_\lambda$ in (20.19) and using Condition M, we have

$$\begin{aligned} \langle F'(v_\lambda), \beta(u - u - \lambda\beta(v-u))\rangle &\le \langle F'(u), \eta(u + \lambda\beta(v-u) - u))\rangle \\ &\quad + \|\beta(u - u - \lambda\beta(v-u)\|^p\} \\ &= -\lambda\langle F'(u), \beta(v-u)\rangle, \end{aligned}$$

which implies that

$$\langle F'(v_\lambda), \beta(v-u)\rangle \ge \langle F'(u), \beta(v-u)\rangle. \tag{20.20}$$

Let $\xi(\lambda) = F(u + \lambda\beta(v-u))$. Then, from (20.20), we have

$$\begin{aligned} \xi'(\lambda) &= \langle F'(u + \lambda\beta(v-u)), \beta(v-u)\rangle \\ &\ge \langle F'(u), \beta(v-u)\rangle. \end{aligned} \tag{20.21}$$

Integrating (20.21) between 0 and 1, we have

$$\xi(1) - \xi(0) \ge \langle F'(u), \beta(v-u)\rangle,$$

that is,

$$F(u + \lambda\beta(v-u)) - F(u) \ge \langle F'(u), \beta(v-u)\rangle.$$

By using Condition A, we have

$$F(v) - F(u) \ge \langle F'(u), \beta(v-u)\rangle.$$

the required result. □

We now give a necessary condition for β-pseudo-biconvex function.

Theorem 20.12. *Let $F'(.)$ be a relaxed β-pseudomonotone operator and Conditions A and M hold. Then F is a β-pseudo-biconvex function.*

Proof. Let F' be a relaxed β-pseudomonotone. Then, $\forall u, v \in K_\beta$,

$$\langle F'(u), \beta(v-u)\rangle \geq 0,$$

implies that

$$-\langle F'(v), \beta(u-v)\rangle \geq 0. \tag{20.22}$$

Since K is an biconvex set, $\forall u, v \in K_\eta$, $\lambda \in [0,1]$, $v_\lambda = u + \lambda\beta(v-u) \in K_\beta$. Taking $v = v_\lambda$ in (20.22) and using condition Condition M, we have

$$-\langle F'(u+\lambda\beta(v-u)), \beta(u-v)\rangle \geq 0. \tag{20.23}$$

Let

$$\xi(\lambda) = F(u+\lambda\beta(v-u)), \quad \forall u, v \in K_\beta, \lambda \in [0,1].$$

Then, using (20.23), we have

$$\xi'(\lambda) = \langle F'(u+\lambda\beta(v-u)), \beta(u-v)\rangle \geq 0.$$

Integrating the above relation between 0 to 1, we have

$$\xi(1) - \xi(0) \geq 0,$$

that is,

$$F(u+\lambda\beta(v-u)) - F(u) \geq 0,$$

which implies, using Condition A,

$$F(v) - F(u) \geq 0,$$

showing that F is a β-pseudo-biconvex function. □

Definition 20.13. *The function F is said to be sharply pseudo biconvex, if*

$$\langle F'(u), \beta(v-u)\rangle \geq 0$$
$$\Rightarrow$$
$$F(v) \geq F(v+\lambda\beta(v-u)), \forall u, v \in K_\beta, \lambda \in [0,1].$$

Theorem 20.13. *Let F be a sharply pseudo biconvex function on K_β. Then*

$$-\langle F'(v), \beta(v-u)\rangle \geq 0, \quad \forall u, v \in K_\beta.$$

Proof. Let F be a sharply pseudo biconvex function on K_β. Then

$$F(v) \geq F(v+\lambda\beta(v-u)), \quad \forall u, v \in K_\beta, \lambda \in [0,1],$$

from which we have

$$\frac{F(v+\lambda\beta(v-u)) - F(v)}{\lambda} \leq 0.$$

Taking limit in the above-mentioned inequality, as $\lambda \to 0$, we have

$$-\langle F'(v), \beta(v-u)\rangle \geq 0,$$

the required result. □

Definition 20.14. *A function F is said to be a pseudo biconvex function with respect to strictly positive bifunction $W(.,.)$, if*

$$F(v) < F(u)$$
$$\Rightarrow$$
$$F(u+\lambda\beta(v-u)) < F(u) + \lambda(\lambda-1)W(v,u), \forall u, v \in K_\beta, \lambda \in [0,1].$$

Theorem 20.14. *If the function F is a biconvex function such that $F(v) < F(u)$, then the function F is pseudo biconvex.*

Proof. Since $F(v) < F(u)$ and F is biconvex function, then
$\forall u, v \in K_\eta, \quad \lambda \in [0,1]$, we have

$$\begin{aligned} F(u+\lambda\beta(v-u)) &\leq F(u)+\lambda(F(v)-F(u)) \\ &< F(u)+\lambda(1-\lambda)(F(v)-F(u)) \\ &= F(u)+\lambda(\lambda-1)(F(u)-F(v)) \\ &< F(u)+\lambda(\lambda-1)W(u,v), \end{aligned}$$

where $W(u,v) = F(u) - F(v) > 0$. This shows that the function F is a pseudo biconvex. □

We now consider the bidirectional inequalities and suggest some iterative methods by using the auxiliary principle techniques involving the Bregman distance functions.

For the readers convenience, we recall some basic properties of the Bregman [4] convex functions. For strongly convex function F, we define the Bregman distance function as

$$B(v,u) = F(v) - F(u) - \langle F'(u), v-u\rangle \geq \alpha\|v-u\|^2, \forall u, v \in K. \tag{20.24}$$

It is important to emphasize that various types of function F gives different Bregman distance. We give the following important examples of some practical important types of function F and their corresponding Bregman distance, see [23]

Examples

(1) If $f(v) = \|v\|^2$, then $B(v,u) = \|v-u\|^2$, which is the squared Euclidean distance (SE).

(2) If $f(v) = \sum_{i=1}^n a_i \log(v_i)$, which is known as Shannon entropy, then its corresponding Bregman distance is given as

$$B(v,u) = \sum^n i = 1\left(v_i \log(\frac{v_i}{u_i}) + u_i - v_i\right),$$

This distance is called KullbackLeibler distance (KL) and as become a very important tool in several areas of applied mathematics such as machine learning.

(3) If $f(v) = -\sum_{i=1}^n \log(v_i)$, which is called Burg entropy, then its corresponding Bregman distance is given as

$$B(v,u) = \sum_{i=1}^n \left(\log(\frac{v_i}{u_i}) + \frac{v_i}{u_i} - 1\right).$$

This is called ItakuraSaito distance (IS),which is very important in the information theory, data analysis and machine learning.

Remark 20.3. *It is a challenging problem to explore the applications of Bregman distance functions for other types of noncongeal functions as biconvex, k-convex functions and harmonic functions.*

We now discuss the optimality conditions for the differentiable biconvex functions.

Theorem 20.15. *Let F be a differentiable biconvex function with modulus $\mu > 0$. If $u \in K_\beta$ is the minimum of the function F, if and only if, $u \in K_\beta$ satisfies the*

$$\langle F'(u), \beta(v-u)\rangle \geq 0, \quad \forall v \in K_\beta. \tag{20.25}$$

Proof. Let $u \in K_\beta$ be a minimum of the function F. Then

$$F(u) \leq F(v), \forall v \in K_\beta. \tag{20.26}$$

Since K_β is a biconvex set, so, $\forall u, v \in K_\beta, \quad \lambda \in [0,1]$,

$$v_\lambda = u + \lambda\beta(v-u) \in K_\beta.$$

Taking $v = v_\lambda$ in (20.26), we have

$$0 \leq \lim_{\lambda \to 0}\{\frac{F(u+\lambda\beta((v-u)) - F(u)}{\lambda}\} = \langle F'(u), \beta(v-u)\rangle, \tag{20.27}$$

which is the inequality (20.25).
Since F is differentiable biconvex function, so

$$F(u+\lambda\beta(v-u)) \leq F(u) + \lambda(F(v) - F(u)), \forall u, v \in K_\beta,$$

from which, using (20.25), we have

$$\begin{aligned} F(v) - F(u) &\geq \lim_{\lambda \to 0}\{\frac{F(u+\lambda\beta(v-u)) - F(u)}{\lambda}\} \\ &= \langle F'(u), \beta(v,u)\rangle \geq 0, \end{aligned}$$

from which, we have

$$F(u) \leq F(v), \forall v \in K_\beta. \tag{20.28}$$

which implies that $u \in K_\beta$ is the minimum of the biconvex functions.

□

Remark: We would like to mention that, if $u \in K_\beta$ satisfies the inequality

$$\langle F'(u), \beta(v-u)\rangle \geq 0, \quad \forall u, v \in K_\beta, \tag{20.29}$$

then $u \in K_\beta$ is the minimum of the differentiable biconvex function F.
The inequality of the type (20.29) is called the bidirectional inequality and appears to new one.

It is worth mentioning that inequalities of the type (20.29) may not arise as a minimization of the biconvex functions. This motivated us to consider a more general bidirectional inequality of which (20.29) is a special case.

For a given operator T, bifunction $\beta(. - .)$, consider the problem of finding $u \in K_\beta$, such that

$$\langle Tu, \beta(v-u)\rangle \geq 0, \forall v \in K_\beta, \tag{20.30}$$

which is called bidirectional inequality.

It is worth mentioning that for suitable and appropriate choice of the operators, biconvex sets and spaces, one can obtain a wide class of variational inequalities and optimization problems. This shows that the bidirectional inequalities are quite flexible and unified ones.

Due to the inherent nonlinearity, the projection method and its variant form can not be used to suggest the iterative methods for solving these bidirectional inequalities. To overcome these drawback, one may use the auxiliary principle technique of Glowinski et al. [6] as developed by Noor [**?**, 29–31] Noor et al. [34, 36–38] to suggest and analyze some iterative methods for solving the bidirectional-like inequalities(20.30). This technique does not involve the concept of the projection, which is the main advantage of this technique. We again use the auxiliary principle technique coupled with Bergman functions. These applications are based on the type of convex functions associated with the Bregman distance. We now suggest and analyze some iterative methods for bidirectional inequalities (20.30) using the auxiliary principle technique coupled with Bregman distance functions.

For a given $u \in K_\beta$ satisfying the bidirectional inequality (20.30), we consider the auxiliary problem of finding a $w \in K$ such that

$$\langle \rho Tw, \beta(v-w) + \langle E'(w) - E'(u), \beta(v-w)\rangle \geq 0, \quad \forall v \in K_\beta, \tag{20.31}$$

where $\rho > 0$ is a constant and $E'(u)$ is the differential of a strongly biconvex function $E(u)$ at $u \in K_\beta$. Since $E(u)$ is a strongly biconvex function, this implies that its differential E' is strongly β-monotone. Consequently it follows that the problem (20.30) has an unique solution.

Remark 3.1: The function $B(w,u) = E(w) - E(u) - \langle E'(u), \beta(w,u)\rangle$ associated with the biconvex function $E(u)$ is called the generalized Bregman function. By the strongly boiconvexity of the function $E(u)$, the Bregman function $B(.,.)$ is nonnegative and $B(w,u) = 0$, if and only if $u = w, \forall u, w \in K_\beta$. For the applications of the Bregman function in solving variational inequalities and complementarity problems, see [24, 32–38].

We note that, if $w = u$, then clearly w is solution of the bidirectional inequality (20.30). This observation enables us to suggest and analyze the following iterative method for solving (20.30).

Algorithm 20.1. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho T u_{n+1}, \beta(v - u_{n+1})\rangle + \langle E'(u_{n+1}) - E'(u_n), \beta(v - u_{n+1})\rangle \geq 0, \quad \forall v \in K_\beta, \tag{20.32}$$

where $\rho > 0$ is a constant. Algorithm 20.1 is called the proximal method for solving bidirectional inequalities (20.30). In passing we remark that the proximal point method was suggested in the context of convex programming problems as a regularization technique.

If $\beta(v - u) = v - u$, then Algorithm 20.1 collapses to:

Algorithm 20.2. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho T(u_{n+1}), v - u_{n+1}\rangle + \langle E'(u_{n+1}) - E'(u), v - u_{n+1}\rangle \geq 0, \quad \forall v \in K,$$

for solving the variational inequality.

For suitable and appropriate choice of the operators and the spaces, one can obtain a number of known and new algorithms for solving variational inequalities and related problems.

Theorem 20.16. *Let the bifunction T be pseudomonotone. Let E be differentiable higher order strongly biconvex function with module $\nu > 0$ and Condition M hold. If $\rho\mu \leq \nu$, then the approximate solution u_{n+1} obtained from Algorithm 20.1 converges to a solution $u \in K$ satisfying the bidirectional inequality(20.30).*

Proof. Let $u \in K$ be a solution of bidirectional inequality(20.30). Then

$$\langle Tu, \beta(v - u)\rangle \geq 0, \quad \forall v \in K_\beta,$$

implies that

$$-\langle Tv, \beta(u - v))\rangle \geq 0, \quad \forall v \in K_\beta, \tag{20.33}$$

since T is β-pseudomonotone.

Taking $v = u$ in (20.32) and $v = u_{n+1}$ in (20.33), we have

$$\langle \rho T(u_{n+1}), \beta(u, u - n + 1)\rangle + \langle E'(u_{n+1}) - E'_k(u_n, \beta(u - u_{n+1})\rangle \geq 0. \tag{20.34}$$

and

$$-\langle Tu_{n+1}, \beta(u - u_{n+1})\rangle \geq 0. \tag{20.35}$$

We now consider the Bregman distance function

$$B(u,w) = E(u) - E(w) - \langle E'(w, \beta(u - w)\rangle \geq \nu\|\beta(v - u)\|^2, \tag{20.36}$$

using higher order strongly biconvexity of E.
Now combining (20.34),(20.35) and (20.36), we have

$$\begin{aligned} B(u,u_n)-B(u,u_{n+1}) &= E(u_{n+1})-E(u_n)-\langle E'(u_n),\beta(u-u_n)\rangle \\ &\quad +\langle E'(u_{n+1}),\beta(u-u_{n+1})\rangle \\ &= E(u_{n+1})-E(u_n)-\langle E'(u_n)-E'(u_{n+1},\beta(u-u_{n+1})\rangle \\ &\quad -\langle E'(u_n,u_{n+1}-u_n\rangle \\ &\geq \nu\|\beta(u_{n+1}-u_n)\|^2+\langle E'(u_{n+1})-E'(u_n),\beta(u-u_{n+1})\rangle \\ &\geq \nu\|\beta(u_{n+1}-u_n)\|^2-\rho\langle T(u_{n+1}),\beta(u-u_{n+1})\rangle \\ &\quad -\rho\mu\|\beta(u-u_{n+1})\|^2 \\ &\geq (\nu-\rho\mu)\|\beta(u_{n+1}-u_n)\|^2. \end{aligned}$$

If $u_{n+1} = u_n$, then clearly u_n is a solution of the problem(20.30). Otherwise, it follows that $B(u,u_n) - B(u,u_{n+1})$ is nonnegative and we must have

$$\lim_{n\to\infty} \|\beta(u_{n+1}-u_n)\| = 0.$$

from which, we have

$$\lim_{n\to\infty} \|u_{n+1}-u_n\| = 0.$$

It follows that the sequence $\{u_n\}$ is bounded. Let $\bar{u}$ be a cluster point of the subsequence $\{u_{n_i}\}$, and let $\{u_{n_i}\}$ be a subsequence converging toward $\bar{u}$. Now using the technique of Zhu and Marcotte [24], it can be shown that the entire sequence $\{u_n\}$ converges to the cluster point $\bar{u}$ satisfying the bidirectional inequality(20.30).

□

It is well-known that to implement the proximal point methods, one has to find the approximate solution implicitly, which is itself a difficult problem. To overcome this drawback, we now consider another method for solving the bidirectional inequality(20.30) using the auxiliary principle technique.

For a given $u \in K_\beta$, find $w \in K_\beta$ such that

$$\langle \rho T(u,\beta(v-w)\rangle + \langle E'(w)-E',\beta(v-w)\rangle \geq 0, \quad \forall v \in K_\beta, \tag{20.37}$$

where $E'(u)$ is the differential of a biconvex function $E(u)$ at $u \in K_\beta$. Problem (20.37) has a unique solution, since E is strongly biconvex function. Note that problems (20.37) and (20.32) are quite different problems.
It is clear that for $w = u$, w is a solution of (20.30). This fact allows us to suggest and analyze another iterative method for solving the bidirectional inequality (20.30).

Algorithm 20.3. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho Tu_n,\beta(v-u_{n+1})\rangle + \langle E'(u_{n+1})-E'(u_n),\beta(v-u_{n+1})\rangle \geq 0, \quad \forall v \in K_\beta, \tag{20.38}$$

for solving the bidirectional inequality (20.30).
If $\beta(v,u)) = v-u$, Algorithm 20.3 collapses to:

Algorithm 20.4. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\rho\langle Tu_n,\beta(v-u_{n+1})\rangle + \langle E'(u_{n+1})-E'(u_n),\beta(v-u_{n+1})\rangle \geq 0, \quad \forall v \in K_\beta,$$

for solving the bidirectional inequalities and appears to be a new one.

We now again use the auxiliary principle to suggest some more iterative methods for solving bidirectional inequalities.
For a given $u \in K_\beta$ satisfying (20.30), find $w \in K_\beta$ such that

$$\langle \rho T(w, \beta(v-w)) \rangle + \langle w - u + \alpha(u-u), v - w \rangle \geq 0, \quad \forall v \in K_\beta, \tag{20.39}$$

which is the auxiliary bidirectional inequality. We note that, if $w = u$, w is a solution of (20.30). This fact allows us to suggest and analyze another iterative method for solving the bidirectional inequality (20.30).

Algorithm 20.5. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\rho\langle Tu_{n+1}, \beta(v - u_{n+1})\rangle + \langle u_{n+1} - u_n) + \alpha(u_n - u_{n-1}), v - u_{n+1}\rangle \geq 0, \quad \forall v \in K_\beta, \tag{20.40}$$

where α is a constant. Algorithm 20.5 is called the inertial proximal method for solving the bidirectional inequalities (20.30). For $\alpha = 0$, Algorithm 20.5 becomes:

Algorithm 20.6. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\rho\langle Tu_{n+1}, \beta(v - u_{n+1})\rangle + \langle u_{n+1} - u_n), v - u_{n+1}\rangle \geq 0, \quad \forall v \in K_\beta,$$

which is called the proximal method for solving the bidirectional inequalities (20.30).
If $\beta. - .) = v - u$, then the biconvex set K_β becomes the convex set K. Consequently Algorithm 20.5 reduces to:

Algorithm 20.7. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\rho\langle Tu_{n+1}, v - u_{n+1}\rangle + \langle u_{n+1} - u_n) + \alpha(u_n - u_{n-1}), v - u_{n+1}\rangle \geq 0, \quad \forall v \in K.$$

Algorithm 20.7 is known as the inertial proximal method for solving variational inequalities.
We now consider the convergence analysis of Algorithm 20.5.

Theorem 20.17. *Let $\bar{u} \in K_\beta$ be a solution of (20.30) and let u_{n+1} be the approximate solution obtained from Algorithm 20.5. If the $T : H \longrightarrow R$ is pseudo β-monotone, then*

$$\begin{aligned} \|u_{n+1} - \bar{u}\|^2 \quad \leq \quad & \|u_n - \bar{u}\|^2 - \|u_{n+1} - u_n - \alpha_n(u_n - u_{n-1})\|^2 \\ & + \alpha_n\{\|u_n - \bar{u}\|^2 - \|u_{n-1} - \bar{u}\|^2 + 2\|u_n - u_{n-1}\|^2\}. \end{aligned} \tag{20.41}$$

Proof. Let $\bar{u} \in K_\beta$ be a solution of (20.30). Then

$$\langle Tu, \beta(v-u)\rangle \geq 0, \quad \forall v \in K_\beta,$$

implies that

$$-\langle Tv, \beta(\bar{u} - v)) \geq 0, \tag{20.42}$$

since T is pseudo β-monotone.

Taking $v = u_{n+1}$ in (20.42), we have

$$\langle Tu_{n+1}, \beta(\bar{u} - u_{n+1})\rangle \geq 0. \tag{20.43}$$

Now taking $v = \bar{u}$ in (20.38), we obtain

$$\langle \rho Tu_{n+1}, \beta(\bar{u} - u_{n+1})\rangle \quad + \quad \langle u_{n+1} - u_n - \alpha_n(u_n - u_{n-1}), \bar{u} - u_{n+1}\rangle \geq 0. \tag{20.44}$$

From (20.43) and (20.44), we have

$$\langle u_{n+1} - u_n - \alpha_n(u_n - u_{n-1}) \quad , \quad \bar{u} - u_{n+1}\rangle \geq -\langle \rho Tu_{n+1}, \beta(\bar{u} - u_{n+1})\rangle \geq 0, \tag{20.45}$$

One can write (20.44) in the form

$$\langle u_{n+1} - u_n, \bar{u} - u_{n+1}\rangle \geq \alpha_n\langle u_n - u_{n-1}, \bar{u} - u_n + u_n - u_{n+1}\rangle. \tag{20.46}$$

Using the inequality $2\langle u, v\rangle = \|u+v\|^2 - \|u\|^2 - \|v\|^2, \forall u, v \in H$ and rearranging the terms in (20.46), one can easily obtain the required result. □

Theorem 20.18. *Let H be a finite dimensional space. Let u_{n+1} be the approximate solution obtained from Algorithm 20.5 and $\bar{u} \in K_\beta$ be a solution of (20.30). If there exists $\alpha \in (0,1)$ such that $0 \leq \alpha_n \leq \alpha, \quad \forall n \in N$ and*

$$\sum_{n=1}^{\infty} \alpha_n \|u_n - u_{n-1}\|^2 \leq \infty,$$

then $\lim_{n \longrightarrow \infty} u_n = \bar{u}$.

Proof. Let $\bar{u} \in K_\beta$ be a solution of (20.30). First we consider the case $\alpha_n = 0$. In this case, we see from (20.39) that the sequence $\{\|\bar{u} - u_n\|\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. Also from (20.39), we have

$$\sum_{n=0}^{\infty} \|u_{n+1} - u_n\|^2 \leq \|u_0 - \bar{u}\|^2,$$

which implies that

$$\lim_{n \longrightarrow \infty} \|u_{n+1} - u_n\| = 0. \tag{20.47}$$

Let $\hat{u}$ be the cluster point of $\{u_n\}$ and the subsequence $\{u_{n_j}\}$ of the sequence $\{u_n\}$ converge to $\hat{u} \in H$. Replacing u_n by u_{n_j} in (20.38) and taking the limit $n_j \longrightarrow \infty$ and using (20.47), we have

$$\langle T\hat{u}, \beta(v - \hat{u})\rangle \geq 0, \quad \forall v \in K_\beta,$$

which implies that $\hat{u}$ solves the bihemivariational inequality problem (20.30) and

$$\|u_{n+1} - u_n\|^2 \leq \|u_n - \bar{u}\|^2.$$

Thus it follows from the above inequality that the sequence $\{u_n\}$ has exactly one cluster point $\hat{u}$ and $\lim_{n \longrightarrow \infty} u_n = \hat{u}$.

Now we consider the case $\alpha_n > 0$. From (20.38), we have

$$\begin{aligned}\sum_{n+1}^{\infty} \|u_{n+1} - u_n \quad - \quad & \alpha_n(u_n - u_{n-1})\|^2 \leq \|u_0 - \bar{u}\|^2 \\ & + \sum_{n=1}^{\infty} \{\alpha \|u_n - \bar{u}\|^2 + 2\|u_n - u_{n-1}\|^2\} \leq \infty,\end{aligned}$$

which implies that

$$\lim_{n \longrightarrow \infty} \|u_{n+1} - u_n - \alpha_n(u_n - u_{n-1})\|^2 = 0.$$

Repeating the above arguments as in the case $\alpha_n = 0$, one can easily show that

$$\lim_{n \to \infty} u_n = \hat{u},$$

the required result. □

For a given $u \in K_\beta$ satisfying the bidirectional inequality (20.30), consider the auxiliary problem of finding $w \in K_\beta$ such that

$$\langle \rho T u, \beta(v - u)\rangle + \langle w - u, v - w\rangle \geq 0, \quad \forall v \in K_\beta, \tag{20.48}$$

where $\rho > 0$ is a constant. Problem (20.48) is known as the auxiliary bidirectional inequality. We note that if $w = u$, then clearly w is a solution of the problem (20.30). This observation enables us to suggest and analyze the following iterative method for solving the problem(20.30).

Algorithm 20.8. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho T w_n, \beta(v - w_n)\rangle + \langle u_{n+1} - w_n, v - u_{n+1}\rangle \geq 0, \forall v \in K_\beta$$
$$\langle \nu T u_n, \beta(v - u_n)\rangle + \langle w_n - u_n, v - w_n\rangle \geq 0, \quad \forall v \in K_\beta,$$

where $\rho > 0$ and $\nu > 0$ are constants. Algorithm 3.8 is two-step predictor-corrector method for solving the bidirectional inequalities (20.30).

Remark 20.4. *For suitable and appropriate choice of the operators and the spaces, one can obtain various known and new algorithms for solving bidirectional inequality (20.30) and related optimization problems. Convergence analysis of these new algorithms can be considered and investigated using the above techniques and ideas. It is an interesting problem from both analytically and numerically point of views.*

Conclusion

In this section, we have introduced and studied some new classes of biconvex functions. These concepts are more general and unifying ones. Several new properties of these strongly biconvex functions are discussed and their relations with previously known results are highlighted. It is shown that the optimality conditions of the differentiable biconvex functions can be characterised by a class of bidirectional inequalities. This result is used to introduce a more general class of bidirectional inequalities (20.30). Auxiliary principle techniques is used to suggest and analyze some iterative methods for solving the bidirectional inequalities. Convergence analysis of the proposed methods is condition using the pseudo monotonicity which is a weaker condition than monotonicity. Our method of proofs is very simple as compared with other techniques. It is itself an interesting problem to develop some efficient numerical methods for solving bidirectional inequalities along with applications in pure and applied sciences. Despite the current activities in these fields, much clearly remains to be done in these fields. It is expected that the ideas and techniques of this paper may be starting point for future research activities.

21. Bifunction variational inequalities

Much attention has been given to study the bifunction variational inequality, which can be viewed as a useful and significant extension of the variational inequalities. Crespi et al [1-4], Fang and Hu [5], Lalitha and Mehra [6] and Noor [7] have studied various aspects of the bifunction variational inequalities. There are a substantial number of numerical methods including projection technique and its variant forms, Wiener-Hopf equations, auxiliary principle, dynamical systems and resolvent equations methods for solving variational inequalities. However, it is known that projection, Wiener-Hopf equations, proximal and resolvent equations techniques cannot be extended and generalized to suggest and analyze similar iterative methods for solving bifunction variational inequalities. This fact has motivated to use the auxiliary principle technique, which is mainly due to mainly due to Glowinski, Lions and Tremolieres [8], to suggest and analyze an implicit iterative method for solving the bifunction variational inequalities. We also study the convergence of this new method under the pseudomonotonicity, which is a weaker condition than monotonicity. Our method of proof is very simple.

For a given continuous bifunction function $T(.,.) : K \times K \longrightarrow H$, consider the problem of finding $u \in K$, such that

$$T(u, v - u) \geq 0, \quad \forall v \in K, \tag{21.1}$$

which is called an *bifunction variational inequality.* A number of problems arising in various branches of pure and applied sciences can be studied via the bifunction variational inequalities. It has been shown [7] that the minimum of a directionally differentiable convex function on a convex set can be characterized by the bifunction variational inequality (21.1). In a similar way, one can show that the minimum of a Lipschitz continuous noncongeal satisfies is a solution of the bifunction variational inequality (21.1).

(I). If $T(u, v-u) = \langle G(u), v-u\rangle$, where G is a nonlinear operator, then (21.1) is equivalent to finding $u \in K$, such that

$$\langle G(u), v-u\rangle \geq 0, \quad \forall v \in K, \tag{21.2}$$

which is known as the classical variational inequality (2.1) introduced by Stampacchia [9].
Definition 2.1. The bifunction function $T(.,.): K \times K \longrightarrow H$ is said to be pseudomonotone, iff

$$T(u, v-u) \geq 0 \quad \Longrightarrow -T(v, u-v) \geq 0, \quad \forall u, v \in K.$$

21.1. **Iterative methods.**

In this section, we consider an iterative method for solving (21.1) by using the technique of the auxiliary principle technique involving the Bregman distance function.

For a given $u \in K$ satisfying (21.1), we consider the auxiliary bifunction variational inequality problem of finding $w \in K$, such that

$$\rho T(w, v-w) + \langle E'(w) - E'(u), v-w\rangle \geq 0, \quad \forall v \in K, \tag{21.3}$$

where $\rho > 0$ is a constant and $E'(u)$ is the differential of a strongly convex function E at $u \in K$. From the strongly convexity of the differentiable function $E(u)$, it follows that problem (21.3) has a unique solution. It is clear that if $w = u$, then w is a solution of problem (21.1). This observation enables to suggest and analyze the following iterative method for solving (21.1).

Algorithm 21.1. *For a given $u_0 \in H$, calculate the approximate solution u_{n+1} by the iterative scheme*

$$\rho T(u_{n+1}, v-u_{n+1}) + \langle E'(u_{n+1}) - E'(u_n), v-u_{n+1}\rangle \geq 0, \quad \forall v \in K, \tag{21.4}$$

where $\rho > 0$ is a constant.

Note that, if $T(u, v-u) = \langle G(u), v-u\rangle$, then Algorithm 21.1) reduces to the following iterative scheme for solving (22.1).

Algorithm 21.2. *For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho G(Tu_{n+1}) + E'(u_{n+1}) - E'(u_n), v-u_{n+1}\rangle \geq 0, \quad \forall v \in K,$$

where $\rho > 0$ is a constant.

For appropriate and suitable choice of the bifunction and the spaces, one can obtain a number of iterative methods for solving the bifunction variational inequalities and related optimization problems.

We now study the convergence criteria of Algorithm 21.3 and this is the main motivation of next result.

Theorem 21.1. *Let the function $T(.,.)$ be pseudomonotone and let $E(u)$ be strongly convex function with modulus $\beta > 0$. Then the approximate solution u_{n+1} obtained from Algorithm 21.3 converges to a solution $u \in K$ of the problem (21.1)*

Proof. Let $u \in K$ be a solution of (21.1). Then, using the pseudomonotonicity of $T(.,.)$, we have

$$-T(v, u-v) \quad \geq \quad 0, \quad \forall v \in K. \tag{21.5}$$

Taking $v = u_{n+1}$ in (21.5) and $v = u$ in (21.4), we have

$$-T(u_{n+1}, u - u_{n+1}) \geq 0 \quad (21.6)$$

$$\rho T(u_{n+1}, u - u_{n+1}) + \langle E'(u_{n+1}) - E'(u_n), u - u_{n+1}\rangle \geq 0. \quad (21.7)$$

Now we consider the generalized Bergman function as

$$B(u, z) = E(u) - E(z) - \langle E'(z), u - z\rangle \geq \beta\|u - z\|^2, \quad (21.8)$$

where we have used the fact that the function $E(u)$ is strongly convex.

Combining (21.6), (21.7) and (21.8), we have

$$\begin{aligned} B(u, u_n) - B(u, u_{n+1}) &= E(u_{n+1}) - E(u_n) - \langle E'(u_n), u - u_n\rangle \\ &\quad +\langle E'(u_{n+1}), u - u_{n+1}\rangle \\ &= E(u_{n+1}) - E(u_n) - \langle E'(u_n) - E'(u_{n+1}), u - u_{n+1}\rangle \\ &\quad -\langle E'(u_n), u_{n+1} - u_n\rangle \\ &\geq \beta\|u_{n+1} - u_n\|^2 + \langle E'(u_{n+1}) - E'(u_n), u - u_{n+1}\rangle \\ &\geq \beta\|u_{n+1} - u_n\|^2 - \rho T(u_{n+1}, u - u_{n+1}) \\ &\geq \beta\|u_{n+1} - u_n\|^2. \end{aligned}$$

If $u_{n+1} = u_n$, then clearly u_n is a solution of (1). Otherwise, for $\beta > 0$, the sequences $B(u, u_n) - B(u, u_{n+1})$ is nonnegative and we must have

$$\lim_{n\to\infty}(\|u_{n+1} - u_n\|) = 0.$$

It follows that the sequence $\{u_n\}$ is bounded. Let $\bar{u}$ be a cluster point of the subsequence $\{u_{n_i}\}$, and let $\{u_{n_i}\}$ be a subsequence converging toward $\bar{u}$. Now using the technique of Zhu and Marcotte [13], it can be shown that the entire sequence $\{u_n\}$ converges to the cluster point $\bar{u}$ satisfying (21.1). □

We now consider another iterative method for solving the bifunction variational inequality (21.1). For a given $u \in K$ satisfying (21.1), we consider the auxiliary bifunction variational inequality problem of finding $w \in K$, such that

$$\rho T(u, v - w) + \langle E'(w) - E'(u), v - w\rangle \geq 0, \quad \forall v \in K, \quad (21.9)$$

where $\rho > 0$ is a constant and $E'(u)$ is the differential of a strongly convex function E at $u \in K$. From the strongly convexity of the differentiable function $E(u)$, it follows that problem (21.9) has a unique solution. It is clear that if $w = u$, then w is a solution of problem (21.1). This observation enables to suggest and analyze the following iterative method for solving (21.1).

Algorithm 21.3. *For a given $u_0 \in H$, calculate the approximate solution u_{n+1} by the iterative scheme*

$$\rho T(u_n, v - u_{n+1}) + \langle E'(u_{n+1}) - E'(u_n), v - u_{n+1}\rangle \geq 0, \quad \forall v \in K, \quad (21.10)$$

which is called the explicit method for solving the bifunction variational inequality (21.1).

We gain use the auxiliary principle technique without Bregman distance function to suggest some iterative methods for solving (21.1).

For a given $u \in K$ satisfying the bifunction variational inequality (20.2), consider the auxiliary problem of finding $w \in K$ such that

$$\langle \rho T(u, v - u\rangle + \langle w - u, v - w\rangle \geq 0, \quad \forall v \in K, \quad (21.11)$$

where $\rho > 0$ is a constant. Problem (21.11) is known as the auxiliary bifunction variational inequality. We note that if $w = u$, then clearly w is a solution of the problem (21.1). This observation enables us to suggest and analyze the following iterative method for solving the problem(21.1).

Algorithm 21.4. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho T(w_n, v - w_n\rangle + \langle u_{n+1} - w_n, v - u_{n+1}\rangle \geq 0, \forall v \in K$$
$$\langle \nu T(u_n, v - u_n\rangle + \langle w_n - u_n, v - w_n\rangle \geq 0, \quad \forall v \in K,$$

where $\rho > 0$ and $\nu > 0$ are constants. Algorithm 21.4 is two-step predictor-corrector method for solving the bifunction variational inequalities (21.1).
In a similar way, we suggest an implicit iterative method.

For a given $u \in K_\beta$ satisfying the bifunction variational inequality (21.1), consider the auxiliary problem of finding $w \in K$ such that

$$\langle \rho T(w, v - w\rangle + \langle w - u, v - w\rangle \geq 0, \quad \forall v \in K, \tag{21.12}$$

where $\rho > 0$ is a constant. Problem (21.12) is known as the auxiliary bidirectional inequality. We note that if $w = u$, then clearly w is a solution of the problem (21.1). This observation enables us to suggest and analyze the following iterative method for solving the problem(21.1).

Algorithm 21.5. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho T(u_{n+1}, v - u_{n+1}\rangle + \langle u_{n+1} - u_n, v - u_{n+1}\rangle \geq 0, \quad \forall v \in K,$$

which is known as implicit method.

Remark 21.1. *For appropriate and suitable choice of the operator, convex set and parameters, one can obtain a wide class of iterative methods for solving bifunction variational inequalities and related optimization problems. Using the idea of this section, one can consider bifunction variational-like inequalities and bifunction bidirectional inequalities. One can explore the applications of these inequalities in various fields of pure and applied sciences.*

22. Bifunction hemivariational inequalities

Variational inequalities have been generalized and extended in several directions. A useful and important generalization of variational inequalities is a class of variational inequalities known as hemivariational inequalities involving the nonlinear Lipschitz continuous functions. Hemivariational inequalities are connected with noncongeal and possibly nonsmooth energy functions. It has been shown that the hemivariational inequalities have important applications in structural analysis and noncongeal optimization. In particular, it has been shown that if a nonsmooth and noncongeal superpotential of a structure is quasidifferentiable, then these problems can be studied by the hemivariational inequalities. In passing we remark that, if the nonlinear locally Lipschitz function is a convex function, then hemivariational inequalities coincide with the mildly nonlinear variational inequalities introduced and studied by Noor [16] in 1975. However, numerical techniques considered for solving mildly nonlinear variational inequalities cannot be extended for the hemivariational inequalities due to the presence of nonlinear and nonlinear differentiable functionals.

In recent years, it have been shown that the minimum of directionally differentiable convex function on a convex set can be characterized by a class of variational inequalities, which is called the bifunction variational inequality. For the formulation, applications, numerical methods and other aspects of the bifunction variational inequalities, see [2-5,7,30,32] and the references therein. It is clear that the bifunction variational inequalities and the hemivariational inequalities are two remarkably different and useful generalizations of the classical variational inequality in several aspects. It is natural to study these different problems in a unified framework. This fact motivated us to introduce and study a new class of variational inequalities, which is called the *bifunction hemivariational inequalities.* It is shown that the bifunction hemivariational inequalities unify and combine these two type of bifunction variational inequalities and hemivariational inequalities.

Due to the nature of the bifunction hemivariational inequalities projection and resolvent methods can not be applied for solving bifunction hemivariational inequalities. To overcome these difficulties, one usually uses the auxiliary principle technique. The main idea involving this technique is to first consider an auxiliary problem and then to show that the solution of the auxiliary problem is the solution of the original problem by using the fixed-point approach. In this section, we show that the auxiliary principle technique can be used to suggest some iterative schemes for solving the bifunction hemivariational inequalities. We prove that the convergence of these methods require either pseudomonotonicity or partially relaxed strongly monotonicity. These are weaker conditions than monotonicity. As a special cases, we obtain iterative schemes for solving hemivariational inequalities and related optimization problems. The comparison of these methods with other methods is a subject of future research.

First of all, we recall the following concepts and results from nonsmooth analysis, see Clarke et al [1].

Definition 22.1. *Let f be locally Lipschitz continuous at a given point $x \in H$ and v be any other vector in H. The Clarke's generalized directional derivative of f at x in the direction v, denoted by $f^0(x; v)$, is defined as*

$$f^0(x; v) = \lim_{t \to 0^+} \sup_{h \to 0} \frac{f(x + h + tv) - f(x + h)}{t}.$$

The generalized gradient of f at x, denoted $\partial f(x)$, is defined to be subdifferential of the function $f^0(x; v)$ at 0. That is

$$\partial f(x) = \{w \in H : \langle w, v\rangle \leq f^0(x; v), \quad \forall v \in H.\}$$

Lemma 22.1. *Let f be a locally Lipschitz continuous at a given point $x \in H$ with a constant L. Then*
(i). $\partial f(x)$ is a none-empty weakly compact subset of H and $\|\xi\| \leq L$ for each $\xi \in \partial f(x)$.
(ii). For every $v \in H$, $f^0(x; v) = max\{\langle \xi, v\rangle : \xi \in \partial f(x)\}$.
(iii). The function $v \longrightarrow f^0(x; v)$ is finite, positively homogeneous, subadditive, convex and continuous.
(iv). $f^0(x; -v) = (-f)^0(x; v)$.
(v). $f^0(x; v)$ is upper semicontinuous as a function of $(x; v)$.
(vi). $\forall x \in H$, there exists a constant $\alpha > 0$ such that

$$|f^0(x; v)| \leq \alpha\|v\|, \quad \forall v \in H.$$

For a given nonlinear bifunction $T(., .)$, consider the problem of finding $u \in K$ such that

$$T(u, v - u) + \int_\Omega f^0(u; v - u) d\Omega \geq 0, \quad \forall v \in K, \tag{22.1}$$

which is called the bifunction hemivariational inequality. Here $f^0(u; v-u) := f^0(u, v-u)$ denotes the generalized directional derivative of Lipschitz continuous function $f(x, .)$ and Ω is an open bounded subset of R^n.

If $f(.; .) = 0$, then bifunction hemivariational inequality problem (20.4) reduces to finding $u \in K$ such that

$$T(u, v - u) \geq 0, \quad \forall v \in K, \tag{22.2}$$

which is called the bifunction variational inequality. For the formulation, numerical methods and other aspects of the bifunction variational inequalities, see [2-5,7,30,32].

If f $T(u, v-u) = \langle Au, v-u\rangle$, where A is a nonlinear operator, then the bifunction hemivariational inequality (22.1) is equivalent to finding $u \in K$ such that

$$\langle Au, v-u\rangle + \int_\Omega f^0(u; v-u)d\Omega \geq 0, \quad \forall v \in K, \tag{22.3}$$

which is known as the *hemivariational inequality* introduced and studied by Panagiotopoulos [39,40] in order to formulate variational principles associated with energy functions which are neither convex nor smooth. It is has been shown that the technique of hemivariational inequalities is very efficient to describe the behaviour of complex structure arising in engineering and industrial sciences, see [6,14,15, 28,29,33, 39,40] and the references therein.

If $f(.,.)$ is a smooth and convex function, then problem (22.3) reduces to finding $u \in K$ such that

$$\langle Au, v-u\rangle + \langle f'(u), v-u\rangle \geq 0, \quad \forall v \in K, \tag{22.4}$$

where $f'(u)$ is the differential of a convex function f at u. Problem (22.4) is known as the mildly nonlinear variational inequality which was introduced and studied by Noor [16] in 1975. For recent applications, extensions, numerical methods and other aspects of mildly nonlinear variational inequalities, see [16-39].

From the above discussion, it is obvious that the bifunction hemivariational inequalities are more general than bifunction variational inequalities, hemivariational inequalities and mildly nonlinear variational inequalities and include these problems as special cases.

Definition 22.2. *The bifunction $T(.,.)$ is said to be*

(a) monotone, iff,

$$T(u, v-u) + T(v, u-v) \leq 0, \quad \forall u, v \in K.$$

(b) pseudomonotone with respect to $\int_\Omega f^0(u; v-u)d\Omega$, iff,

$$T(u, v-u) + \int_\Omega f^0(u; v-u)d\Omega \geq 0$$
$$\Longrightarrow$$
$$-T(v, u-v) - \int_\Omega f^0(v; v-u)d\Omega \geq 0, \quad \forall u, v \in K.$$

(c) partially relaxed strongly monotone, if there exists a constant $\gamma > 0$ such that

$$T(u, u-v) + T(v, z-v) \leq \gamma\|z-u\|^2, \quad \forall u, v, z \in K.$$

Note that for $z = u$ partially relaxed strongly monotonicity reduces to monotonicity. This shows that partially relaxed strongly monotonicity implies monotonicity, but the converse is not true.

Definition 22.3. *The function $\int_\Omega f^0(u; v-u)d\Omega$ is said to be partially relaxed strongly monotone, if there exists a constant $\alpha > 0$ such that*

$$\int_\Omega f^0(u; v-u)d\Omega + \int_\Omega f^0(z; u-v)d\Omega \leq \alpha\|z-v\|^2, \quad \forall u, v, z \in H.$$

Note that for $z = v$, partially relaxed strongly monotonicity reduces to monotonicity, that is,

$$\int_\Omega f^0(u; v-u)d\Omega + \int_\Omega f^0(v; u-v)d\Omega \leq 0, \quad \forall u, v \in H.$$

We now suggest and analyze some iterative methods for bifunction hemivariational inequalities (22.1) using the auxiliary principle technique.

For a given $u \in K$, we consider the auxiliary problem of finding a $w \in K$ such that

$$\rho T(w, v-w) + \langle E'(w) - E'(u), v-w\rangle + \rho \int_{\Omega} f^0(w; v-w) d\Omega \geq 0, \quad \forall v \in K, \tag{22.5}$$

where $\rho > 0$ is a constant and $E'(u)$ is the differential of a strongly convex function $E(u)$ at $u \in K$. Since $E(u)$ is a strongly convex function, this implies that its differential E' is strongly monotone. Consequently it follows that the problem (22.1) has an unique solution.

Remark 22.1. *The function $B(w,u) = E(w) - E(u) - \langle E'(u), w-u\rangle$ associated with the convex function $E(u)$ is called the generalized Bregman function. By the strongly convexity of the function $E(u)$, the Bregman function $B(.,.)$ is nonnegative and $B(w,u) = 0$, if and only if $u = w, \forall u, w \in K$. For the applications of the Bregman function in solving variational inequalities and complementarity problems, see [26,27].*

We note that, if $w = u$, then clearly w is solution of the bifunction hemivariational inequalities (20.4). This observation enables us to suggest and analyze the following iterative method for solving (20.4).

Algorithm 22.1. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho T(u_{n+1}, v-u_{n+1}) + \langle E'(u_{n+1}) - E'(u_n), v - u_{n+1})\rangle + \rho \int_{\Omega} f^0(u_{n+1}; v-u_{n+1}) d\Omega \geq 0, \quad \forall v \in K, \tag{22.6}$$

where $\rho > 0$ is a constant. Algorithm 22.1 is called the proximal method for solving bifunction hemivariational inequalities (22.1). In passing we remark that the proximal point method was suggested by Martinet [12] in the context of convex programming problems as a regularization technique.

If $f(x,u) = 0$, then Algorithm 22.1 collapses to:

Algorithm 22.2. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho T(u_{n+1}, v-u_{n+1}) + \langle E'(u_{n+1}) - E'(u_n), v - u_{n+1})\rangle \geq 0, \quad \forall v \in K,$$

for solving the bifunction variational inequality (22.2). For the convergence analysis of Algorithm 22.2; see Noor [32].

For suitable and appropriate choice of the operators and the spaces, one can obtain a number of known and new algorithms for solving variational inequalities and related problems.

Theorem 22.1. *Let the bifunction $T(.,.)$ be pseudomonotone with respect to $\int_{\Omega} f^0(u; v-u) d\Omega$. If E be differentiable strongly convex function with module $\beta > 0$, then the approximate solution u_{n+1} obtained from Algorithm* **??** *converges to a solution $u \in K$ satisfying the bifunction hemivariational inequality (22.1).*

Proof. Let $u \in K$ be a solution of (22.1). Then

$$T(u, v-u) + \int_{\Omega} f^0(u; v-u) d\Omega \geq 0, \quad \forall v \in K,$$

implies that

$$-T(v, u-v) - \int_{\Omega} f^0(u; v-u) d\Omega \geq 0, \quad \forall v \in K, \tag{22.7}$$

since $T(.,.)$ is pseudomonotone with respect to $\int_{\Omega} f^0(u; v-u) d\Omega$.

Taking $v = u$ in (**??**) and $v = u_{n+1}$ in (22.7), we have

$$\rho T(u_{n+1}, u - u_{n+1}) \quad + \quad \langle E'(u_{n+1}) - E'(u_n), u - u_{n+1}\rangle$$
$$\geq -\rho \int_\Omega f^0(u_{n+1}; u - u_{n+1}) d\Omega. \qquad (22.8)$$

and

$$-T(u_{n+1}, u - u_{n+1}) - \int_\Omega f^0(u_{n+1}; u_{n+1} - u) d\Omega \geq 0. \qquad (22.9)$$

We now consider the Bregman function

$$B(u, w) = E(u) - E(w) - \langle E'(w), u - w\rangle \geq \beta \|u - w\|^2, \quad \text{using strongly convexity of E.} \quad (22.10)$$

Now combining (22.8),(22.9) and (22.10), we have

$$\begin{aligned} B(u, u_n) - B(u, u_{n+1}) &= E(u_{n+1}) - E(u_n) - \langle E'(u_n), u - u_n\rangle \\ &\quad +\langle E'(u_{n+1}), u - u_{n+1}\rangle \\ &= E(u_{n+1}) - E(u_n) - \langle E'(u_n) - E'(u_{n+1}), u - u_{n+1}\rangle \\ &\quad -\langle E'(u_n), u_{n+1} - u_n\rangle \\ &\geq \beta\|u_{n+1} - u_n\|^2 + \langle E'(u_{n+1}) - E'(u_n), u - u_{n+1}\rangle \\ &\geq \beta\|u_{n+1} - u_n\|^2 - \rho T(u_{n+1}, u - u_{n+1}) \\ &\quad -\rho \int_\Omega f^0(u_{n+1}; u - u_{n+1}) d\Omega \\ &\geq \beta\|u_{n+1} - u_n\|^2. \end{aligned}$$

If $u_{n+1} = u_n$, then clearly u_n is a solution of the bifunction hemivariational inequality (22.1). Otherwise, it follows that $B(u, u_n) - B(u, u_{n+1})$ is nonnegative and we must have

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0.$$

It follows that the sequence $\{u_n\}$ is bounded. Let $\bar{u}$ be a cluster point of the subsequence $\{u_{n_i}\}$, and let $\{u_{n_i}\}$ be a subsequence converging toward $\bar{u}$. Now using the technique of Zhu and Marcotte [43], it can be shown that the entire sequence $\{u_n\}$ converges to the cluster point $\bar{u}$ satisfying the bifunction hemivariational inequalities (22.1). □

It is well-known that to implement the proximal point methods, one has to find the approximate solution implicitly, which is itself a difficult problem. To overcome this drawback, we now consider another method for solving the bifunction hemivariational inequality (22.1) using the auxiliary principle technique.

For a given $u \in K$, find $w \in K$ such that

$$\rho T(u, v - w) \quad + \quad \langle E'(w) - E'(u), v - w\rangle$$
$$+\rho \int_\Omega f^0(u; v - w) d\Omega \geq 0, \quad \forall v \in K, \qquad (22.11)$$

where $E'(u)$ is the differential of a strongly convex function $E(u)$ at $u \in K$. Problem (22.11) has a unique solution, since E is strongly convex function. Note that problems (22.6) and (22.11) are quite different problems. It is clear that for $w = u$, w is a solution of (22.1). This fact allows us to suggest and analyze another iterative method for solving the bifunction hemivariational inequalities (22.1).

Algorithm 22.3. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\rho T(u_n, v - u_{n+1}) \quad + \quad \langle E'(u_{n+1}) - E'(u_n), v - u_{n+1}\rangle$$
$$\geq -\rho \int_\Omega (u_n; v - u_{n+1}) d\Omega, \quad \forall v \in K, \qquad (22.12)$$

for solving the bifunction hemivariational inequalities (22.1).

If $T(u, v-u) = \langle Au, v-u\rangle$, Algorithm 22.3 collapses to:

Algorithm 22.4. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative schemes*

$$\begin{aligned}\rho\langle Au_n, v-u_{n+1}\rangle \quad &+ \quad \langle E'(u_{n+1}) - E'(u_n), v-u_{n+1}\rangle \\ &\geq -\rho\int_\Omega (u_n; v-u_{n+1})d\Omega, \quad \forall v \in K,\end{aligned}$$

for solving the hemivariational inequalities (22.3) and appears to be a new one. For suitable and appropriate choice of the operators and the spaces, one can obtain various known and new algorithms for solving bifunction hemivariational inequalities and related optimization problems.

We now consider the convergence analysis of Algorithm 22.3 using essentially the technique of Theorem 22.1 For the sake of completeness and to convey an idea of the technique, we sketch the main points.

Theorem 22.2. *Let $T(.,.)$ and $\int_\Omega f^0(u; v-u)d\Omega$ be partially relaxed strongly monotone with constants $\gamma > 0$ and $\alpha > 0$ respectively. If E is strongly convex function with modulus $\beta > 0$ and $0 < \rho < \beta/(\alpha+\gamma)$, then the approximate solution u_{n+1} obtained from Algorithm 22.3 converges to a solution of bifunction hemivariational inequality (22.1).*

Proof. Let $u \in K$ be a solution of (22.1). Setting $v = u_{n+1}$ in (20.4) and $v = u$ in (22.12), we have

$$T(u, u_{n+1}-u) + \int_\Omega f^0(u; u_{n+1}-u)d\Omega \geq 0. \tag{22.13}$$

and

$$\begin{aligned}\rho T(u_n, u-u_{n+1}) \quad &+ \quad \langle E'(u_{n+1}) - E'(u_n), u-u_{n+1}\rangle \\ &\geq -\rho\int_\Omega f^0(u_n; u-u_{n+1})d\Omega. \end{aligned}\tag{22.14}$$

As in Theorem 22.1 and from (22.13) and (22.14), we have

$$\begin{aligned}B(u,u_n) - B(u,u_{n+1}) &= E(u_{n+1}) - E(u_n) - \langle E'(u_n), u-u_n\rangle \\ &\quad +\langle E'(u_{n+1}), u-u_{n+1}\rangle \\ &= E(u_{n+1}) - E(u_n) - \langle E'(u_n) - E'(u_{n+1}), u-u_{n+1}\rangle \\ &\quad -\langle E'(u_n), u_{n+1}-u_n\rangle \\ &\geq \beta\|u_{n+1}-u_n\|^2 + \langle E'(u_{n+1}) - E'(u_n), u-u_{n+1}\rangle \\ &\geq \beta\|u_{n+1}-u_n\|^2 - \rho\{T(u, u_{n+1}-u) \\ &\quad +T(u_n, u-u_{n+1})\} - \rho\{\int_\Omega f^0(u_n; u-u_{n+1})d\Omega \\ &\quad + \int_\Omega f^0(u; u_{n+1}-u)d\Omega\} \\ &\geq \beta\|u_{n+1}-u_n\|^2 - \rho(\alpha+\gamma)\|u_{n+1}-u_n\|^2 \\ &= \{\beta - \rho(\alpha+\gamma)\}\|u_{n+1}-u_n\|^2,\end{aligned}$$

where we have used the fact that $T(.,.)$ and $\int_\Omega f^0(u;\eta(v,u))d\Omega$ are partially relaxed strongly monotone with constants $\alpha > 0$ and $\gamma > 0$ respectively.

If $u_{n+1} = u_n$, then clearly u_n is a solution of the bifunction hemivariational inequality (22.1).Otherwise, for $0 < \rho < \frac{\beta}{\alpha+\gamma}$, it follows that $B(u,u_n) - B(u,u_{n+1})$ is nonnegative and we must have

$$\lim_{n\to\infty} \|u_{n+1}-u_n\| = 0.$$

Now using the technique of Zhu and Marcotte (Ref. 43), it can be shown that the entire sequence $\{u_n\}$ converges to the cluster point u satisfying the bifunction hemivariational inequality (22.1). □

Conclusion. In this section, we have introduced and considered a new class of variational inequalities, which is called the bifunction hemivariational inequality. Several iterative methods for solving the bifunction hemivariational inequalities are suggested and analyzed by using the auxiliary principle technique in conjunction with Bregman function. Some special cases are also considered. We expect that the ideas and techniques of this paper will motivate and inspire the interested readers to explore the applications of bifunction hemivariational inequalities in various fields of mathematical and engineering sciences.

23. Exponential convex functions and variational inequalities

Convex functions and convex sets have played an important and fundamental part in the development of various fields of pure and applied sciences. Convexity theory describes a broad spectrum of very interesting developments involving a link among various fields of mathematics, physics, economics and engineering sciences. Some of these developments have made mutually enriching contacts with other fields. Ideas explaining these concepts led to the developments of new and powerful techniques to solve a wide class of linear and nonlinear problems. The development of convexity theory can be viewed as the simultaneous pursuit of two different lines of research. On the one hand, it reveals the fundamental facts on the qualitative behaviour of solutions (regarding its existence, uniqueness and regularity) to important classes of problems; on the other hand, it also enables us to develop highly efficient and powerful new numerical methods to solve nonlinear problems. In recent years, various extensions and generalizations of convex functions and convex sets have been considered and studied using innovative ideas and techniques. It is known that more accurate and inequalities can be obtained using the logarithmically convex functions than the convex functions. Closely related to the log-convex functions, we have the concept of exponentially convex(concave) functions, the origin of exponentially convex functions can be traced back to Bernstein [**?**]. Avriel [**?**] introduced and studied the concept of r-convex functions. For further properties of the r-convex functions, see Zhao et al [22] and the references therein. which have important applications in information theory, big data analysis, machine learning and statistic.
Motivated and inspired by the ongoing research in this interesting, applicable and dynamic field, we again consider the concept of exponentially convex functions. We discuss the basic properties of the exponentially convex functions. It is has been shown that the exponentially convex(concave) have nice properties which convex functions enjoy. Several new concepts have been introduced and investigated. We show that the local minimum of the exponentially convex functions is the global minimum. The optimal conditions of the differentiable exponentially convex functions can be characterized by a class of variational inequalities, which is itself an interesting outcome of our main results. The difference (sum) of the exponentially convex function and exponentially affine convex function is again a exponentially convex function. The ideas and techniques of this paper may be starting point for further research in these areas.

We now consider the concept of the exponentially convex function, which is mainly due to Noor and Noor [33, 35] and Rashid et al [**?**, **?**] as:

Definition 23.1. *[5] A function F is said to be exponentially convex function, if*

$$e^{F((1-t)u+tv)} \leq (1-t)e^{F(u)} + te^{F(v)}, \quad \forall u, v \in K, \quad t \in [0,1].$$

We remark that Definition 23.3 can be rewritten in the following equivalent way, which is due to Antczak [5].

Definition 23.2. *A function F is said to be exponentially convex function, if*

$$F((1-t)u+tv) \leq \log[(1-t)e^{F(u)} + te^{F(v)}], \quad \forall u, v \in K, \quad t \in [0,1]. \tag{23.1}$$

A function is called the exponentially concave function f, if $-f$ is exponentially convex function. It is obvious that two concepts are equivalent. This equivalent have been used to discuss various aspects of the exponentially convex functions. It is worth mentioning that one can also deduce the concept of exponentially convex functions from r-convex functions, which were considered by Avriel [**?**] and Bernstein [**?**].

For the applications of the exponentially convex functions in the mathematical programming and information theory, see Antczak [5],Alirezaei and Mathar [4] and Pal et al [39].. For the applications of the exponentially concave function in the communication and information theory, we have the following example.

Example [4]: The error function

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt,$$

becomes an exponentially concave function in the form $erf(\sqrt{x}), \quad x \geq 0$, which describes the bit/symbol error probability of communication systems depending on the square root of the underlying signal-to-noise ratio. This shows that the exponentially concave functions can play important part in communication theory and information theory.

Definition 23.3. *[5] A function F is said to be exponentially affine convex function, if*

$$e^{F((1-t)u+tv)} = (1-t)e^{F(u)} + te^{F(v)}, \quad \forall u, v \in K, \quad t \in [0,1].$$

Definition 23.4. *The function F on the convex set K is said to be exponentially quasi convex, if*

$$e^{F(u+t(v-u))} \leq \max\{e^{F(u)}, e^{F(v)}\}, \quad \forall u, v \in K, t \in [0,1].$$

From the above definitions, we have

$$\begin{aligned} e^{F(u+t(v-u))} &\leq (1-t)e^{F(u)} + te^{F(v)}) \\ &\leq \max\{e^{F(u)}, e^{F(v)}\}. \end{aligned}$$

This shows that every exponentially convex function is a exponentially quasi-convex function. However, the converse is not true.

Let $K = I = [a, b]$ be the interval. We now define the exponentially convex functions on I.

Definition 23.5. *Let $I = [a, b]$. Then F is exponentially convex function, if and only if,*

$$\begin{vmatrix} 1 & 1 & 1 \\ a & x & b \\ e^{F(a)} & e^{F(x)} & e^{F(b)} \end{vmatrix} \geq 0; \quad a \leq x \leq b.$$

One can easily show that the following are equivalent:

(1) F is exponentially convex function.
(2) $e^{F(x)} \leq e^{F(a)} + \frac{e^{F(b)} - e^{F(a)}}{b-a}(x-a)$.
(3) $\frac{e^{F(x)} - e^{F(a)}}{x-a} \leq \frac{e^{F(b)} - e^{F(a)}}{b-a}$.
(4) $(b-x)e^{F(a)} + (a-b)e^{F(x)} + (x-a)e^{F(b)}) \geq 0$.
(5) $\frac{F(a)}{(b-a)(a-x)} + \frac{e^{F(x)}}{(x-b)(a-x)} + \frac{e^{F(b}}{(b-a)(x-b)} \leq 0$,

where $x = (1-t)a + tb \in [0,1]$.

We now consider some basic properties of exponentially convex functions.

Theorem 23.1. *Let F be a strictly exponentially convex function. Then any local minimum of F is a global minimum.*

Proof. Let the exponentially convex function F have a local minimum at $u \in K$. Assume the contrary, that is, $F(v) < F(u)$ for some $v \in K$. Since F is exponentially convex, so

$$e^{F(u+t(v-u))} < te^{F(v)} + (1-t)e^{F(u)}, \quad \text{for} \quad 0 < t < 1.$$

Thus

$$e^{F(u+t(v-u))} - e^{F(u)} < t[e^{F(v)} - e^{F(u)}] < 0,$$

from which it follows that

$$e^{F(u+t(v-u))} < e^{F(u)},$$

for arbitrary small $t > 0$, contradicting the local minimum. □

Theorem 23.2. *If the function F on the convex set K is exponentially convex, then the level set $L_\alpha = \{u \in K : e^{F(u)} \leq \alpha, \quad \alpha \in R\}$ is a convex set.*

Proof. Let $u, v \in L_\alpha$. Then $e^{F(u)} \leq \alpha$ and $e^{F(v)} \leq \alpha$. Now, $\forall t \in (0,1), \quad w = v + t(u-v) \in K$, since K is a convex set. Thus, by the exponentially convexity of F, we have

$$\begin{aligned} Fe^{(v+t(u-v))} &\leq (1-t)e^{F(v)} + te^{F(u)} \\ &\leq (1-t)\alpha + t\alpha = \alpha, \end{aligned}$$

from which it follows that $v + t(u-v) \in L_\alpha$ Hence L_α is convex set. □

Theorem 23.3. *The function F is exponentially convex, if and only if*

$$epi(F) = \{(u,\alpha) : u \in K : e^{F(u)} \leq \alpha, \alpha \in R\}$$

is a convex set.

Proof. Assume that$F$ is exponentially convex. Let $(u,\alpha), \quad (v,\beta) \in epi(F)$. Then it follows that $e^{F(u)} \leq \alpha$ and $e^{F(v)} \leq \beta$. Thus, $\forall t \in [0,1], \quad u, v \in K$, we have

$$\begin{aligned} e^{F(u+t(v-u))} &\leq (1-t)e^{F(u)} + te^{F(v)} \\ &\leq (1-t)\alpha + t\beta, \end{aligned}$$

which implies that

$$(u + t(v-u), (1-t)\alpha + t\beta) \in epi(F).$$

Thus $epi(F)$ is a convex set. Conversely, let $epi(F)$ be a convex set. Let $u, v \in K$. Then $(u, e^{F(u)}) \in epi(F)$ and $(v, e^{F(v)}) \in epi(F)$. Since $epi(F)$ is a convex set, we must have

$$(u + t(v-u), (1-t)e^{F(u)} + te^{F(v)}) \in epi(F),$$

which implies that

$$e^{F(u+t(v-u))} \leq (1-t)e^{F(u)} + te^{F(u)}.$$

This shows that F is exponentially convex function. □

Theorem 23.4. *The function F is exponentially quasi convex, if and only if, the level set $L_\alpha = \{u \in K, \alpha \in R : e^{F(u)} \leq \alpha\}$ is a convex set.*

Proof. Let $u, v \in L_\alpha$. Then $u, v \in K$ and $\max(e^{F(u)}, e^{F(v)}) \leq \alpha$. Now for $t \in (0,1), w = u + t(v-u) \in K$, We have to prove that $u + t(v-u) \in L_\alpha$. By the exponentially quasi convexity of F, we have

$$e^{F(u+t(v-u))} \leq \max\left(e^{F(u)}, e^{F(v)}\right) \leq \alpha,$$

which implies that $u + t(v-u) \in L_\alpha$, showing that the level set L_α is indeed a convex set.

Conversely, assume that L_α is a convex set. Then for any $u, v \in L_\alpha, t \in [0,1]$, $u + t(v-u) \in L_\alpha$. Let $u, v \in L_\alpha$ for

$$\alpha = max(e^{F(u)}, e^{F(v)}) \quad \text{and} \quad e^{F(v)} \leq e^{F(u)}.$$

Then from the definition of the level set L_α, it follows that

$$e^{F(u+t\ (v,u))} \leq \max(e^{F(u)}, e^{F(v)}) \leq \alpha.$$

Thus F is an exponentially quasi convex function. This completes the proof. □

Theorem 23.5. *Let F be an exponentially convex function.. Let $\mu = \inf_{u\in K} F(u)$. Then the set $E = \{u \in K : e^{F(u)} = \mu\}$ is a convex set of K. If F is strictly exponentially , then E is a singleton.*

Proof. Let $u, v \ \in E$. For $0 < t < 1$, let $w = u + t(v-u)$. Since F is a exponentially convex function, then

$$\begin{aligned} F(w) = e^{F(u+t(v-u))} &\leq (1-t)e^{F(u)} + te^{F(v)} \\ &= t\mu + (1-t)\mu = \mu, \end{aligned}$$

which implies that to $w \in E$. and hence E is a convex set. For the second part, assume to the contrary that $F(u) = F(v) = \mu$. Since K is a convex set, then for $0 < t < 1, u + t(v-u) \in K$. Further, since F is strictly exponentially convex,

$$\begin{aligned} e^{F(u+t(v-u))} &< (1-t)e^{F(u)} + te^{F(v)} \\ &= (1-t)\mu + t\mu = \mu. \end{aligned}$$

This contradicts the fact that $\mu = \inf_{u\in K} F(u)$ and hence the result follows. □

Theorem 23.6. *If F is an exponentially convex function such that $e^{F(v)} < e^{F(u)}, \forall u, v \in K$, then F is a strictly exponentially quasi convex function.*

Proof. By the exponentially convexity of the function F, $\forall u, v \in K, t \in [0,1]$, we have

$$e^{F(u+t(v-u))} \leq (1-t)e^{F(u)} + te^{F(v)} < e^F(u),$$

since $e^{F(v)} < e^{F(u)}$, which shows that the function F is strictly exponentially quasi convex. □

We now derive some properties of the differentiable exponentially convex functions.

Theorem 23.7. *Let F be a differentiable function on the convex set K. Then the function F is exponentially convex function, if and only if,*

$$e^{F(v)} - e^{F(u)} \geq \langle e^{F(u)}F'(u), v-u\rangle, \quad \forall v, u \in K. \tag{23.2}$$

Proof. Let F be a exponentially convex function. Then

$$e^{F(u+t(v-u))} \leq (1-t)e^{F(u)} + te^{F(v)}, \qquad \forall u, v \in K,$$

which can be written as

$$e^{F(v)} - e^{F(u)} \geq \{\frac{e^{F(u+t(v-u))} - e^{F(u)}}{t}\}.$$

Taking the limit in the above inequality as $t \to 0$, we have

$$e^{F(v)} - e^{F(u)} \geq \langle e^{F(u)}F'(u), v-u)\rangle,$$

which is (23.2), the required result.

Conversely, let (23.2) hold. Then
$\forall u, v \in K, t \in [0,1]$, $v_t = u + t(v-u) \in K$, we have

$$\begin{aligned} e^{F(v)} - e^{F(v_t)} &\geq \langle e^{F(v_t)} F'(v_t), v - v_t)\rangle \\ &= (1-t)\langle e^{F(v_t)} F'(v_t), v-u\rangle. \end{aligned} \tag{23.3}$$

In a similar way, we have

$$\begin{aligned} e^{F(u)} - e^{F(v_t)} &\geq \langle e^{F(v_t)} F'(v_t), u - v_t)\rangle \\ &= -t\langle e^{F(v_t)} F'(v_t), v-u\rangle. \end{aligned} \tag{23.4}$$

Multiplying (23.3) by t and (23.4) by $(1-t)$ and adding the resultant, we have

$$e^{F(u+t(v-u))} \leq (1-t)e^{F(u)} + te^{F(v)},$$

showing that F is a exponentially convex function. □

Remark 23.1. *From* (23.2)*, we have*

$$e^{F(v)-F(u)} - 1 \geq \langle F'(u), v-u\rangle, \quad \forall v, u \in K,$$

which can be written as

$$F(v) - F(u) \geq \log\{1 + \langle F'(u), v-u\rangle\} \quad \forall v, u \in K, \tag{23.5}$$

Changing the role of u *and* v *in* (23.5)*, we also have*

$$F(u) - F(v) \geq \log\{1 + \langle F'(v), u-v\rangle\} \quad \forall v, u \in K, \tag{23.6}$$

Adding (23.5) *and* (23.6)*, we have*

$$\langle F'(u) - F'(v), u-v\rangle \geq (\langle F'(u)u-v\rangle)(\langle F'(v), u-v\rangle)$$

which express the monotonicity of the differential $F'(.)$ *of the exponentially convex function.*

Theorem 23.7 enables us to introduce the concept of the exponentially monotone operators, which appears to be new ones.

Definition 23.6. *The differential* $F'(.)$ *is said to be exponentially monotone, if*

$$\langle e^{F(u)} F'(u) - e^{F(v)} F'(v), u-v\rangle \geq 0, \quad \forall u, v \in H.$$

Definition 23.7. *The differential* $F'(.)$ *is said to be exponentially pseudo-monotone, if*

$$\langle e^{F(u)} F'(u), v-u\rangle \geq 0, \quad \Rightarrow \langle e^{F(v)} F'(v), v-u\rangle \geq 0, \quad \forall u, v \in H.$$

From these definitions, it follows that exponentially monotonicity implies exponentially pseudo-monotonicity, but the converse is not true.

Theorem 23.8. *Let* F *be differentiable on the convex set* K*. Then* (23.2) *holds, if and only if,* F' *satisfies*

$$\langle e^{F(u)} F'(u) - e^{F(v)} F'(v), u-v\rangle \geq 0, \quad \forall u, v \in K. \tag{23.7}$$

Proof. Let F be a exponentially convex function on the convex set K. Then, from Theorem 3.1, we have

$$e^{F(v)} - e^{F(u)} \geq \langle e^{F(u)} F'(u), v-u\rangle, \quad \forall u, v \in K. \tag{23.8}$$

Changing the role of u and v in (23.8), we have

$$e^{F(u)} - e^{F(v)} \geq \langle e^{F(v)} F'(v), u-v)\rangle, \quad \forall u, v \in K. \tag{23.9}$$

Adding (23.8) and (23.9), we have

$$\langle e^{F(u)} F'(u) - e^{F(v)} F'(v), u-v\rangle \geq 0,$$

which shows that F' is exponentially monotone.

Conversely, from (23.7), we have

$$\langle e^{F(v)}F'(v), u-v\rangle \leq \langle e^{F(u)}F'(u), u-v)\rangle. \tag{23.10}$$

Since K is an convex set, $\forall u, v \in K, \quad t \in [0,1]$ $v_t = u + t(v-u) \in K$.
Taking $v = v_t$ in (23.10), we have

$$\begin{aligned}\langle e^{F(v_t)}F'(v_t), u-v_t\rangle &\leq \langle e^{F(u)}F'(u), u-v_t\rangle \\ &= -t\langle e^{F(u)}F'(u), v-u\rangle,\end{aligned}$$

which implies that

$$\langle e^{F(v_t)}F'(v_t), v-u\rangle \geq \langle e^{F(u)}F'(u), v-u\rangle. \tag{23.11}$$

Consider the auxiliary function

$$g(t) = e^{F(u+t(v-u))},$$

from which, we have

$$g(1) = e^{F(v)}, \quad g(0) = e^{F(u)}.$$

Then, from (23.11), we have

$$g'(t) \quad = \quad \langle e^{F(v_t)}F'(v_t, v-u\rangle \geq \langle e^{F(u)}F'(u), v-u\rangle. \tag{23.12}$$

Integrating (23.12) between 0 and 1, we have

$$g(1) - g(0) = \int_0^1 g'(t)dt \geq \langle e^{F(u)}F'(u), v-u\rangle.$$

Thus it follows that

$$e^{F(v)} - e^{F(u)} \geq \langle e^{F(u)}F'(u), v-u\rangle,$$

which is the required (23.2). □

We now give a necessary condition for exponentially pseudo-convex function.

Theorem 23.9. *Let F' be exponentially pseudomonotone. Then F is a exponentially pseudo-convex function.*

Proof. Let F' be a exponentially pseudomonotone. Then, $\forall u, v \in K$,

$$\langle e^{F(u)}F'(u), v-u\rangle \geq 0.$$

implies that

$$\langle e^{F(v)}F'(v), v-u\rangle \geq 0. \tag{23.13}$$

Since K is an convex set, $\forall u, v \in K$, $t \in [0,1]$, $v_t = u + t(v-u) \in K$.
Taking $v = v_t$ in (23.13), we have

$$\langle e^{F(v_t)}F'(v_t), v-u\rangle \geq 0. \tag{23.14}$$

Consider the auxiliary function

$$g(t) = e^{F(u+t(v-u))} = e^{F(v_t)}, \quad \forall u, v \in K, t \in [0,1],$$

which is differentiable, since F is differentiable function. Then, using (23.14), we have

$$g'(t) = \langle e^{F(v_t)}F'(v_t), v-u)\rangle \geq 0.$$

Integrating the above relation between 0 to 1, we have

$$g(1) - g(0) = \int_0^1 g'(t)dt \geq 0,$$

that is,

$$e^{F(v)} - e^{F(u)} \geq 0,$$

showing that F is a exponentially pseudo-convex function. □

Definition 23.8. *The function F is said to be sharply exponentially pseudo convex, if there exists a constant $\mu > 0$ such that*

$$\begin{aligned} \langle e^{F(u)}F'(u), v-u\rangle &\geq 0 \\ &\Rightarrow \\ F(v) &\geq e^{F(v+t(u-v))}, \quad \forall u, v \in K, t \in [0,1]. \end{aligned}$$

Theorem 23.10. *Let F be a sharply exponentially pseudo convex function on K. Then*

$$\langle e^{F(v)}F'(v), v-u\rangle \geq 0, \quad \forall u, v \in K.$$

Proof. Let F be a sharply exponentially pesudo convex function on K. Then

$$e^{F(v)} \geq e^{F(v+t(u-v))}, \quad \forall u, v \in K, t \in [0,1].$$

from which we have

$$\begin{aligned} 0 &\leq \lim_{t\to 0}\{\frac{e^{F(v+t(u-v))} - e^{F(v)}}{t}\} \\ &= \langle e^{F(v)}F'(v), v-u\rangle, \end{aligned}$$

the required result. □

Definition 23.9. *A function F is said to be a pseudo convex function, if there exists a strictly positive bifunction $W(.,.)$, such that*

$$\begin{aligned} e^{F(v)} &< e^{F(u)} \\ &\Rightarrow \\ e^{F(u+t(v-u))} &< e^{F(u)} + t(t-1)W(v,u), \forall u, v \in K, t \in [0,1]. \end{aligned}$$

Theorem 23.11. *If the function F is exponentially convex function such that $e^{F(v)} < e^{F(u)}$, then the function F is exponentially pseudo convex.*

Proof. Since $e^{F(v)} < e^{F(u)}$ and F is exponentially convex function, then $\forall u, v \in K, \quad t \in [0,1]$, we have

$$\begin{aligned} e^{F(u+t\ l(v,u))} &\leq e^{F(u)} + t(e^{F(v)} - e^{F(u)}) \\ &< e^{F(u)} + t(1-t)(e^{F(v)} - e^{F(u)}) \\ &= e^{F(u)} + t(t-1)(e^{F(u)} - e^{F(v)})) \\ &< e^{F(u)} + t(t-1)W(u,v), \end{aligned}$$

where $W(u,v) = e^{F(u)} - e^{F(v)} > 0$, the required result. This shows that the function F is exponentially convex function. □

We now show that the difference of exponentially convex function and exponentially affine convex function is again an exponentially convex function.

Theorem 23.12. *Let f be a exponentially affine convex function. Then F is a exponentially convex function, if and only if, $g = F - f$ is a exponentially convex function.*

Proof. Let f be exponentially affine convex function. Then

$$e^{f((1-t)u+tv)} = (1-t)e^{f(u)} + te^{f(v)}, \quad \forall u, v \in K, \quad t \in [0,1]. \tag{23.15}$$

From the exponentially convexity of F, we have

$$e^{F((1-t)u+tv)} \leq (1-t)e^{F(u)} + te^{F(v)}, \quad \forall u, v \in K, \quad t \in [0,1]. \tag{23.16}$$

From (23.15) and (23.16), we have

$$e^{F((1-t)u+tv)} - e^{f((1-t)u+tv)} \leq (1-t)(e^{F(u)} - e^{f(u)}) + t(e^{F(v)} - e^{f(v)}), \tag{23.17}$$

from which it follows that

$$\begin{aligned} e^{g((1-t)u+tv)} &= e^{F((1-t)u+tv)} - e^{f((1-t)u+tv)} \\ &\leq (1-t)(e^{F(u)} - e^{f(u)}) + t(e^{F(v)} - e^{f(v)}), \end{aligned}$$

which show that $g = F - f$ is an exponentially convex function.
The inverse implication is obvious. □

23.1. **Exponentially variational inequalities.**

In this section, we introduce a new class of variational inequality, which is called the exponentially variational inequality. First of all, we discuss the optimality condition for the differentiable exponentially convex functions, which is the main motivation of our next result.

Theorem 23.13. *Let F be a differentiable exponentially convex function. Then $u \in K$ is the minimum of the function F, if and only if, $u \in K$ satisfies the inequality*

$$\langle e^{F(u)} F'(u), v - u \rangle \geq 0, \quad \forall u, v \in K. \tag{23.18}$$

Proof. Let $u \in K$ be a minimum of the function F. Then

$$F(u) \leq F(v), \forall v \in K.$$

from which, we have

$$e^{F(u)} \leq e^{F(v)}, \forall v \in K. \tag{23.19}$$

Since K is a convex set, so, $\forall u, v \in K, \quad t \in [0,1]$,

$$v_t = (1-t)u + tv \in K.$$

Taking $v = v_t$ in (23.19), we have

$$0 \leq \lim_{t \to 0} \{ \frac{e^{F(u+t(v-u))} - e^{F(u)}}{t} \} = \langle e^{F(u)} F'(u), v - u \rangle. \tag{23.20}$$

Since F is differentiable exponentially convex function, so

$$e^{F(u+t(v-u))} \leq e^{F(u)} + t(e^{F(v)} - e^{F(u)}, \quad u, v \in K, t \in [0,1],$$

from which, using (23.20), we have

$$e^{F(v)} - e^{F(u)} \geq \lim_{t \to 0} \{ \frac{e^{F(u+t(v-u))} - e^{F(u)}}{t} \} = \langle e^{F(u)} F'(u), v - u \rangle \geq 0,$$

from which, we have

$$F(u) \leq F(v), \quad \forall v \in K.$$

This shows that $u \in K$ is the minimum of the differentiable exponentially convex function, the required result. □

The inequality of the type ((23.18)) is called the exponentially variational inequality and appears to be new one. In many applications, the inequalities of the type ((23.18)) may not arise as the optimality condition of the differentiable exponentially convex functions. This fact motivated us to introduce a more general variational inequality of which the inequality ((23.18)) is a special case.

For a given nonlinear operator T, consider the problem of finding $u \in K$, where K is a convex set in H, such that

$$\langle e^{Tu}, v-u \rangle \geq 0, \quad \forall v \in K, \tag{23.21}$$

which is called the exponentially variational inequality.

Clearly for $e^{Tu} = e^{F(u)}F'(u) = \left(e^{F(u)}\right)'$, the inequality ((23.18)) is special case of inequality ((23.21)).

If K^* is the dual cone of the convex cone, then problem ((23.21)) reduces to finding $u \in K$ such that

$$e^{Tu} \in K^* \quad \text{and} \quad \langle e^{Tu}, u \rangle = 0, \tag{23.22}$$

which is called the exponentially complementarity problem and appears to be a new one.

If $e^{Tu} = \Phi(u)$, then the problem ((23.21)) is equivalent to fining $u \in K$ such that

$$\langle \Phi(u),, v-u \rangle \geq 0, \quad \forall v \in K, \tag{23.23}$$

which is called the classical variational inequalities.

We now define some new concepts

Definition 23.10. *An exponentially operator T is said to:*
(i) exponentially monotone, if

$$\langle e^{Tu} - e^{Tv}, u-v \rangle \geq 0, \quad \forall u, v \in K.$$

(ii). exponentially strongly monotone, if there exists a constant $\eta > 0$ such that

$$\langle e^{Tu} - e^{Tv}, u-v \rangle \geq \eta \|u-v\|^2, \quad \forall u, v \in K.$$

(iii). Lipschitz continuous, if there exists a constant $\beta > 0$ such that

$$\|e^{Tu} - e^{Tv}\| \leq \beta \|u-v\|, \quad \forall u, v \in K.$$

(iv). exponentially pseudo monotone, if

$$\langle e^{Tu}, v-u \rangle \geq 0 \quad \Longrightarrow \quad \langle e^{Tv}, v-u \rangle \geq 0, \quad \forall u, v \in K.$$

Note that every exponentially monotone operator is exponentially pseudo monotone, but the converse is not true.

Theorem 23.14. *Let T be exponentially monotone operator and exponentially hemicontinuous. then $u \in K$ satisfies the inequality (23.21), if only if, $u \in K$ satisfies*

$$\langle e^{Tv}, v-u \rangle \geq 0, \quad \forall v \in K,$$

which is known as Minty exponentially variational inequality.

We suggest and analyze some iterative methods for solving the exponentially variational inequality (23.21) using the auxiliary principle technique.

We use the auxiliary principle technique to suggest some iterative methods for solving (23.21).

For a given $u \in K$ satisfying the exponentially variational inequality (23.21)), consider the auxiliary problem of finding $w \in K$ such that

$$\langle \rho e^{Tw}, v - w\rangle + \langle w - u, v - w\rangle \geq 0, \quad \forall v \in K, \tag{23.24}$$

where $\rho > 0$ is a constant. Problem (23.24) is known as the auxiliary bifunction variational inequality. We note that if $w = u$, then clearly w is a solution of the problem (23.21). This observation enables us to suggest and analyze the following iterative method for solving the problem(23.21).

Algorithm 23.1. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho e^{T(u_{n+1})}, v - u_{n+1})\rangle + \langle u_{n+1} - w_n, v - u_{n+1}\rangle \geq 0, \forall v \in K,$$

where $\rho > 0$ is a constants. Algorithm 23.1 is an implicit method for solving the problem (23.21).

In a similar way, we suggest an explicit iterative method.

For a given $u \in K$ satisfying the exponentially variational inequality (23.21), consider the auxiliary problem of finding $w \in K$ such that

$$\langle \rho e^{Tu}, v - w\rangle + \langle w - u, v - w\rangle \geq 0, \quad \forall v \in K, \tag{23.25}$$

where $\rho > 0$ is a constant. Problem (23.25) is known as the auxiliary bidirectional inequality. We note that if $w = u$, then clearly w is a solution of the problem (23.21). This observation enables us to suggest and analyze the following iterative method for solving the problem(23.21).

Algorithm 23.2. *For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme*

$$\langle \rho e^{T(u_n)}, v - u_{n+1}\rangle + \langle u_{n+1} - u_n, v - u_{n+1}\rangle \geq 0, \quad \forall v \in K,$$

which is known as implicit method.

Using the technique of this paper, one can study the convergence of these iterative methods. One can use the use the projection method, dynamical systems, Wiener-Hopf equations and merit function method to suggest several new methods for solving the exponentially variational inequalities. We have only convey the main idea of the exponentially variational inequalities.

Conclusion

In this section, we have introduced and studied a new class of convex functions, which is called the exponentially convex function. It has been shown that exponentially convex functions enjoy several properties which convex functions have. We have shown that the minimum of the differentiable exponentially convex functions can be characterized by a new class of variational inequalities, which is called the exponential variational inequality. One can explore the applications of the exponentially variational inequalities This may stimulate further research.

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