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# About the Orders Induced by Implications Satisfying the Law of Importation

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In this paper, an order induced by implications on a bounded lattice under some more lenient conditions than the ones given former studies is defined and some of its properties are discussed. By giving an order based on uninorms on a bounded lattice, the relationships between such generated orders are investigated.

*Keywords*: Implication; partial order; bounded lattice; law of importation.

## 1. Introduction

A fuzzy implication generalizing the classical implication to fuzzy logic plays a significant role in many applications, viz., approximate reasoning, fuzzy control, fuzzy image processing, etc. (see Refs. 1–6). To generalize any logical operators on the unit interval to complete lattices has been extensively studied by several authors.[7–14]

The great quantity of applications has lead to a systematically of implications also from the theoretical point of view. In these theoretical papers, many properties of fuzzy implications have been extensively studied by several authors along the time. One of these properties is the so-called law of importation,

$$I(T(x,y),z) = I(x,I(y,z)), \quad \text{for all} \quad x,y,z \in [0,1]\,,$$

where $T$ is a t-norm or a conjunctive uninorm and $I$ is a fuzzy implication. This property has been extensively studied in Refs. 3, 4 for many kinds of implications derived from t-norms and t-conorms. Moreover, an extension of this property involving uninorms instead of t-norms has been also studied in Refs. 15, 16.

Recently, the order generating problem from logical operators has been hot topic for many researchers.[17–25] In Ref. 21 a partial order on a bounded lattice $L$, called as $T$-partial order, given by for any $x, y \in L$,

$$x \preceq_T y \Leftrightarrow T(\ell, y) = x \quad \text{for some} \quad \ell \in L\,,$$

has been introduced.

In Ref. 25, it has been given a partial order induced by implications satisfying the exchange principle (EP) and the contrapositive symmetry (CP) w.r.t. the strong natural negation $N_I$, as follows: For any $x, y \in L$

$$y \preceq_I x \Leftrightarrow \exists\, \ell \in L \quad \text{such that} \quad I(\ell, x) = y\,,$$

where $I$ is an implication on a bounded lattice $L$.

In Ref. 20, a relation denoted by $\sqsubseteq_I$, has been defined by means of an implication $I$ on a bounded lattice $L$ given as follows: For any $x, y \in L$

$$x \sqsubseteq_I y \Leftrightarrow \exists\, \ell \in L \text{ such that } I(\ell, x) = y\,,$$

and in the same study, it has been shown that the relation $\sqsubseteq_I$ is a partial order if $I$ satisfies the law of importation to a t-norm $T$ and the neutrality principle (NP). The conditions required for $\sqsubseteq_I$ to define a partial order are different from the ones required for $\preceq_I$ given in Ref. 25.

In this paper, we define a partial order $\preceq_I$ induced by implications on a bounded lattice satisfying the law of importation to a conjunctive uninorm $U$ with a neutral element $e$ $(LI_U)$ and the neutrality principle w.r.t. $e$ $(NP_e)$ and discuss some of its properties. By this way, we have a chance to study on wider classes of implications imposing the order $\preceq_I$. Also, we give a partial order denoted by $\leq_U$ induced by uninorms on a bounded lattice and determine some relationships between the order $\preceq_I$ induced by implications derived from uninorms satisfying $(LI_U)$ and $(NP_e)$ and the order $\leq_U$ induced by uninorms. The paper is organized as follows: We shortly recall some basic notions in Section 2. In Section 3, we give two relations induced by uninorms and implications on a bounded lattice, denoted by $\leq_U$ and $\preceq_I$, respectively. We show that $\leq_U$ is a partial order. Also, we prove that $\preceq_I$ is a partial order on a bounded lattice when $I$ satisfies the conditions $(LI_U)$ and $(NP_e)$. We determine a relationship between the order $\preceq_I$ and the order on the lattice. Giving example, we show that the order $\leq_U$ is different from the order $\preceq_U$ introduced in Ref. 18 and clearly it extends the $T$-partial order to more general form. Moreover, we obtain that the order $\preceq_I$ induced by implications derived from uninorms is independent from the order $\leq_U$ induced by uninorms. Also, we show that any implication $I$ satisfying the required conditions to define a partial order is increasing in the second place w.r.t. the order $\preceq_I$. We obtain that the relation induced by $\phi$-conjugate of an implication satisfying the required conditions to define an order is also an order and we determine a relationship between the orders induced by an implication and its $\phi$-conjugate. Also, we present there exists a relationship between the algebraic structures obtained from the orders induced by implications and their $\phi$-conjugates. In Section 4, we investigate the relationships between the order $\preceq_I$ induced by $(U, N)$, $QL$, $D$, $f$-generated implications satisfying the law of importation to a conjunctive uninorm $U$ with $e$ and the neutrality principle wr.t. $e$ and the order $\leq_U$ induced by $U$. Moreover, we determine some relationships between the algebraic structures obtained from these orders. Also, we show that the order based on $g$-generated implications satisfying the law of importation to a t-norm coincides with the natural order.

## 2. Notations, Definitions and a Review of Previous Results

**Definition 1.**[12,21] An operation $T$ ($S$) on a bounded lattice $L$ is called a triangular norm (triangular conorm) if it is commutative, associative, increasing with respect to the both variables and has a neutral element 1 (0).

**Definition 2.**[11] Let $(L, \leq, 0, 1)$ be a bounded lattice. An operation $U: L^2 \to L$ is called a uninorm on $L$, if it is commutative, associative, increasing with respect to the both variables and has a neutral element $e \in L$.

It is clear that the function $U$ becomes a t-norm when $e = 1$ and a t-conorm when $e = 0$. For any uninorm we have $U(0,1) \in \{0, 1\}$, and a uninorm $U$ is said conjunctive when $U(1,0) = 0$ and disjunctive $U(1,0) = 1$.

In this study, the notation $\mathcal{U}(e)$ will be used for the set of all uninorms on $L$ with a neutral element $e \in L$.

**Definition 3.**[26] A uninorm $U$ with neutral element $e \in (0,1)$ is said to be in $\mathcal{U}_{\min}$ when it is given by

$$U(x,y) = \begin{cases} e \cdot T\left(\dfrac{x}{e}, \dfrac{y}{e}\right) & (x,y) \in [0,e]^2\,, \\[2ex] e + (1-e) \cdot S\left(\dfrac{x-e}{1-e}, \dfrac{y-e}{1-e}\right) & (x,y) \in [e,1]^2\,, \\[2ex] \min(x,y) & \text{otherwise,} \end{cases}$$

and is said to be in $\mathcal{U}_{\max}$ when it is given by

$$U(x,y) = \begin{cases} e \cdot T\left(\dfrac{x}{e}, \dfrac{y}{e}\right) & (x,y) \in [0,e]^2\,, \\[2ex] e + (1-e) \cdot S\left(\dfrac{x-e}{1-e}, \dfrac{y-e}{1-e}\right) & (x,y) \in [e,1]^2, \\[2ex] \max(x,y) & \text{otherwise.} \end{cases}$$

In both expressions $T$ denotes a t-norm and $S$ denotes a t-conorm.

**Definition 4.**[1,12,25] Let $(L, \leq, 0, 1)$ be a bounded lattice. A decreasing function $N: L \to L$ is called a negation if $N(0) = 1$ and $N(1) = 0$. A negation $N$ on $L$ is called strong if it is an involution, i.e., $N(N(x)) = x$, for all $x \in L$.

**Definition 5.**[15] Let $U$ be a uninorm on a bounded lattice $L$ and $N$ be a strong negation on $L$. The uninorm $U_N$ defined by

$$U_N(x,y) = N(U(N(x), N(y))) \quad \text{for any } x, y \in L$$

is called the $N$-dual of $U$. In this sense, the $N$-dual t-conorm of a t-norm $T$ defined by

$$S(x,y) = N(T(N(x),N(y))) \quad \text{for any } x,y \in L\,.$$

**Definition 6.**[1] A function $I\colon L^2 \to L$ on a bounded lattice $(L,\le,0,1)$ is called an implication if it satisfies the following conditions:

**(I1)** $I$ is a decreasing operation on the first variable, that is, for every $a,b \in L$ with $a \le b$, $I(b,y) \le I(a,y)$ for all $y \in L$.
**(I2)** $I$ is an increasing operation on the second variable, that is, for every $a,b \in L$ with $a \le b$, $I(x,a) \le I(x,b)$ for all $x \in L$.
**(I3)** $I(0,0) = I(1,1) = 1$ and $I(1,0) = 0$.

**Definition 7.**[1,15,27] An implication $I$ on a bounded lattice $L$ is said to satisfy the law of importation to a conjunctive uninorm $U$ with a neutral element $e$ if for all $x,y,z \in L$

$$I(x,I(y,z)) = I(U(x,y),z) \qquad (LI_U)$$

holds.

An implication $I$ is said to satisfy the left neutrality principle w.r.t. $e$ if for all $y \in L$

$$I(e,y) = y \qquad (NP_e)$$

holds.

If $I$ is an implication on $L$ with $I(1,e) = 0$ for some $e \in L \setminus \{1\}$, then the function $N_I^e\colon L \to L$ given by

$$N_I^e(x) = I(x,e), \quad \text{for any } x \in L$$

is called the natural negation of $I$ with respect to $e$.

The most usual ways to define implication functions from uninorms are as in Definition 8.

**Definition 8.**[15]

(i) $(U,N)$-implications are obtained from a disjunctive uninorm $U_d$ and strong negation $N$ as follows:

$$I_{U_d,N}(x,y) = U_d(N(x),y) \quad \text{for all } x,y \in [0,1]\,.$$

(ii) $RU$-implications are obtained from a uninorm $U$ such that $U(x,0) = 0$ for all $x < 1$ as follows:

$$I_U(x,y) = \sup\{z \in [0,1] |\quad U(x,z) \le y\} \quad \text{for all } x,y \in [0,1]\,.$$

Note that for $RU$-implications, the condition $U(x,0) = 0$ for all $x < 1$ is a necessary and sufficient condition to obtain an implication (see Ref. 28).

(iii) $QL$-operators are obtained from a disjunctive uninorm $U_d$, conjunctive uninorm $U_c$ and strong negation $N$ as follows:

$$I_{QL}(x, y) = U_d(N(x), U_c(x, y)) \quad \text{for all } x, y \in [0, 1] .$$

By Ref. 29, it is proven that the following necessary condition for $I_{QL}$ to be an implication: $U_d$ must be a t-conorm such that $U_d(x, N(x)) = 1$ for all $x \in [0, 1]$. In this case, $I_{QL}$ is called as $QL$-implication.

(iv) $D$-operators are obtained from a disjunctive uninorm $U_d$, conjunctive uninorm $U_c$ and strong negation $N$ as follows:

$$I_D(x, y) = U_d(U_c(N(x), N(y)), y) \quad \text{for all } x, y \in [0, 1] .$$

Note that, in order for $I_D$ to be an implication, it is proven in Ref. 29 that necessarily $U_d$ must be a t-conorm satisfying the condition $U_d(x, N(x)) = 1$ for all $x \in [0, 1]$. In this case, $I_D$ is called as $D$-implication.

The methods given in Definition 8 can be used similarly to define implication functions derived from uninorms on a bounded lattice.

**Definition 9.**[1] Let $f\colon [0, 1] \to [0, \infty]$ be a strictly decreasing and continuous function with $f(1) = 0$. The function $I_f\colon [0, 1]^2 \to [0, 1]$ defined by

$$I_f(x, y) = f^{-1}(x \cdot f(y)), \quad x, y \in [0, 1] ,$$

with the understanding $0.\infty = 0$, is called an $f$-generated implication.

**Definition 10.**[1] Let $g\colon [0, 1] \to [0, \infty]$ be a strictly increasing and continuous function with $g(0) = 0$. The function $I_g\colon [0, 1]^2 \to [0, 1]$ defined by

$$I_g(x, y) = g^{(-1)}\left(\frac{1}{x} \cdot g(y)\right), \quad x, y \in [0, 1] , \tag{1}$$

with the understanding $\frac{1}{0} = \infty$, is called a $g$-generated implication, where the function $g^{(-1)}$ is the pseudo-inverse of $g$ given by

$$g^{(-1)}(x) = \begin{cases} g^{-1}(x) & x \in [0, g(1)] , \\ 1 & x \in [g(1), \infty] . \end{cases}$$

Notice that the formula (1) can also be written in the following form without explicitly using the pseudo-inverse of $g$:

$$I(x, y) = g^{-1}\left(\min\left(\frac{1}{x} \cdot g(y), g(1)\right)\right), \quad x, y \in [0, 1] .$$

**Definition 11.**[1] If $I$ is an implication on the unit interval $[0, 1]$ and $\phi\colon [0, 1] \to [0, 1]$ an order-preserving bijection, then the operation $I_\phi\colon [0, 1]^2 \to [0, 1]$ given by

$$I_\phi(x, y) = \phi^{-1}(I(\phi(x), \phi(y)))$$

is also an implication. Such an implication is called $\phi$-conjugate of $I$.

The $\phi$-conjugate of an implication (t-norm, t-conorm, uninorm) on a bounded lattice is defined as similar to Definition 11.

**Definition 12.**[21] Let $L$ be a bounded lattice, $T$ be a t-norm on $L$. The order defined as the following is called a $T-$ *partial order* (triangular order) for the t-norm $T$:

$$x \preceq_T y \Leftrightarrow T(\ell, y) = x \quad \text{for some} \quad \ell \in L\,.$$

**Definition 13.**[18] Let $L$ be a bounded lattice, $S$ be a t-conorm on $L$. The order defined as the following is called an $S-$ *partial order* for t-conorm $S$:

$$x \preceq_S y \Leftrightarrow S(\ell, x) = y \quad \text{for some} \quad \ell \in L\,.$$

**Definition 14.**[18] Let $(L, \leq, 0, 1)$ be a bounded lattice and $U \in \mathcal{U}(e)$. The order defined as the following is called a $U$-*partial order* for $U$: For every $x, y \in L$

$$x \preceq_U y \Leftrightarrow \begin{cases} \text{if} \quad x, y \in [0, e] \quad \text{and there exists} \quad k \in [0, e] \\ \text{such that} \quad U(k, y) = x \quad \text{or,} \\ \text{if} \quad x, y \in [e, 1] \quad \text{and there exists} \quad \ell \in [e, 1] \\ \text{such that} \quad U(x, \ell) = y \quad \text{or,} \\ \text{if} \quad (x, y) \in L^* \quad \text{and} \quad x \leq y\,, \end{cases} \tag{2}$$

where $I_e = \{x \in L \mid x \| e\}$ and $L^* = [0, e] \times [e, 1] \cup [0, e] \times I_e \cup [e, 1] \times [0, e] \cup [e, 1] \times I_e \cup I_e \times [0, e] \cup I_e \times [e, 1] \cup I_e \times I_e$.

Here, note that the notation $x || y$ denotes that $x$ and $y$ are incomparable.

**Definition 15.**[20] Let $I\colon L \times L \to L$ be an implication on a bounded lattice $L$. For $x, y \in L$ we say that

$$x \sqsubseteq_I y \Leftrightarrow \exists\, \ell \in L \quad \text{such that} \quad I(\ell, x) = y\,.$$

**Theorem 1.**[20] *Let $(L, \leq, 0, 1)$ be a bounded lattice and $\otimes$ be a t-norm and $I$ an implication on $L$, respectively. For all $a, b, c \in L$, let $I$ satisfy the following:*

$$I(a \otimes b, c) = I(a, I(b, c)) \quad \textit{and} \quad I(1, b) = b\,.$$

*Then,*

(i) $\sqsubseteq_I$ *is an order on $L$.*
(ii) *Further, $a \sqsubseteq b$ implies that $a \leq b$.*

## 3. The Orders Induced by Implications and Uninorms

In this section, it is introduced a new order which is different from the one given in Ref. 18 by means of uninorms. Giving an order induced by implications satisfying the law of importation with a conjunctive uninorm $U$ with neutral element $e$ $(LI_U)$ and the neutrality principle w.r.t. $e$ $(NP_e)$, it is discussed some properties of these orders. Determining a connection between the orders induced by implications and their $\phi$-conjugates, it is presented a relationship between the algebraic structures obtained from the orders induced by them.

**Proposition 1.** *Let $(L, \leq, 0, 1)$ be a bounded lattice and $U \in \mathcal{U}(e)$. Then, the relation $\leq_U$ defined as, for any $x, y \in L$,*

$$x \leq_U y \Leftrightarrow \quad \exists \ell^* \leq e \quad \textit{such that} \quad U(\ell^*, y) = x\,. \tag{3}$$

*is a partial order.*

**Proof.** If we take as $\ell^* = e \leq e$, then we obtain that for any $x \in L$, $x \leq_U x$ since $U(e, x) = x$. Thus, $\leq_U$ satisfies the reflexivity.

Now, let $x \leq_U y$ and $y \leq_U x$ for any $x, y \in L$. Then, there exist two elements $\ell_1, \ell_2 \leq e$ such that

$$U(\ell_1, y) = x \quad \text{and} \quad U(\ell_2, x) = y\,,$$

whence we have that $x = U(\ell_1, y) \leq U(e, y) = y$ and $y = U(\ell_2, x) \leq U(e, x) = x$. Thus, $x = y$, which shows that the antisymmetry holds.

Let $x \leq_U y$ and $y \leq_U z$ for any $x, y, z \in L$. Then, there exist two elements $\ell_1, \ell_2 \leq e$ such that

$$U(\ell_1, y) = x \quad \text{and} \quad U(\ell_2, z) = y\,.$$

Since $x = U(\ell_1, y) = U(\ell_1, U(\ell_2, z)) = U(U(\ell_1, \ell_2), z)$ and $U(\ell_1, \ell_2) \leq e$, we obtain that $x \leq_U z$. Thus, the transitivity holds. □

Theorem 1 shows that the relation $\sqsubseteq_I$ given in Definition 15 is an order on a bounded lattice $L$ when the implication $I$ with $I(1, y) = y$ for any $y \in L$ satisfies the law of importation to a t-norm $T$. Taking a conjunctive uninorm $U$ instead of $T$ in Definition 15, we will define the following relation $\preceq_I$.

**Definition 16.** Let $(L, \leq, 0, 1)$ be a bounded lattice, $U \in \mathcal{U}(e)$ a conjunctive uninorm and $I$ an implication satisfying $(LI_U)$. Define the following relation: for $x, y \in L$

$$x \preceq_I y \Leftrightarrow \quad \exists \ell \leq e \quad \text{such that} \quad I(\ell, x) = y\,, \tag{4}$$

**Proposition 2.** *Let $(L, \leq, 0, 1)$ be a bounded lattice, $U \in \mathcal{U}(e)$ a conjunctive uninorm and $I$ an implication satisfying $(LI_U)$. If $I$ satisfies the neutrality principle w.r.t. $e$ $(NP_e)$, then the relation $\preceq_I$ given by (4) is a partial order on $L$.*

**Proof.** For every $x \in L$, if we choose $\ell = e \leq e$, we have that $x \preceq_I x$ since $I(\ell, x) = I(e, x) = x$ by $(NP_e)$. Thus, the relation $\preceq_I$ satisfies the reflexivity.

Let $x \preceq_I y$ and $y \preceq_I x$ for any $x, y \in L$. Then, there exist some elements $\ell_1, \ell_2 \leq e$ such that

$$I(\ell_1, x) = y \quad \text{and} \quad I(\ell_2, y) = x\,.$$

By (I1) and $(NP_e)$, since

$$x = I(e, x) \leq I(\ell_1, x) = y \quad \text{and} \quad y = I(e, y) \leq I(\ell_2, y) = x\,,$$

we have that $x = y$, which shows that the relation $\preceq_I$ satisfies the antisymmetry.

Let $x \preceq_I y$ and $y \preceq_I z$ for any elements $x, y, z \in L$. Then, there exist some elements $\ell_1, \ell_2 \leq e$ such that

$$I(\ell_1, x) = y \quad \text{and} \quad I(\ell_2, y) = z\,.$$

By the property $(LI_U)$, we have that

$$z = I(\ell_2, y) = I(\ell_2, I(\ell_1, x)) = I(U(\ell_2, \ell_1), x)\,.$$

Also, it is clear that $U(\ell_1, \ell_2) \leq U(e, e) = e$ since $\ell_1, \ell_2 \leq e$. Then, it is obtained that $x \preceq_I z$. Thus, the transitivity holds. □

**Proposition 3.** *Let $(L, \leq, 0, 1)$ be a bounded lattice, $U \in \mathcal{U}(e)$ a conjunctive uninorm and $I$ an implication satisfying $(LI_U)$ and $(NP_e)$. If $x \preceq_I y$ for any $x, y \in L$, then $x \leq y$.*

**Proof.** Let $x \preceq_I y$ for any $x, y \in L$. Then, there exists an element $\ell \leq e$ such that

$$I(\ell, x) = y\,.$$

By $(NP_e)$ and (I1), we have that $x = I(e, x) \leq I(\ell, x) = y$, that is, $x \leq y$. □

It is obvious that the similar relation to Proposition 3 is true for the order $\leq_U$ given in (3).

The converse of Proposition 3 may not be satisfied. Before then, let us look at the following Proposition 4 and Remark 1.

**Proposition 4.** [16] *Let $U = \langle T, S, e \rangle$ be a uninorm in $\mathcal{U}_{\min}$ with $T$ and $S$ left continuous and $I_U$, its residual implication. Then, $I_U$ always satisfies the law of importation with the same $U$.*

**Remark 1.** Let $U = \langle T, S, e \rangle$ be a uninorm in $\mathcal{U}_{\min}$ with $T$ and $S$ left continuous. Recall that in this case its residual implication is given by Refs. 28 and 30.

$$I_U(x, y) = \begin{cases} e \cdot I_T\left(\dfrac{x}{e}, \dfrac{y}{e}\right) & x, y \in [0, e) \quad \text{and} \quad x > y\,, \\[2ex] e + (1-e) \cdot I_S\left(\dfrac{x-e}{1-e}, \dfrac{y-e}{1-e}\right) & x, y \in [e, 1] \quad \text{and} \quad x \leq y\,, \\[2ex] e & x, y \in [e, 1] \quad \text{and} \quad x > y\,, \\[2ex] I_{GD}(x, y) & \text{otherwise,} \end{cases}$$

where $I_S$ stands for the residuation operator derived from the t-conorm $S$ which is given by

$$I_S(x, y) = \sup\{z \in [0, 1] | \quad S(x, z) \leq y\}\,.$$

By Proposition 4, it is clear that $I_U$ satisfies $(LI_U)$. Also, by Proposition 5.4.2 in Ref. 1, $I_U(e, y) = y$ for any $y \in [0, 1]$. Thus, by Proposition 2, $([0, 1], \preceq_{I_U})$ is a partially ordered set.

Now, we show that the converse of Proposition 3 may not be true. Let us look at the following example.

**Example 1.** Let us consider $U_P = \langle T_P, S_P, 0.5\rangle \in \mathcal{U}_{\min}$. Then, the *RU*-implication obtained from $U_P$ is given by Ref. 1:

$$I_{U_P}(x, y) = \begin{cases} \dfrac{y}{2x} & x, y \in [0, 0.5) \quad \text{and} \quad x > y\,, \\ 0.5 + \dfrac{y - x}{2.(1 - x)} & x, y \in [0.5, 1] \quad \text{and} \quad x \leq y\,, \\ 0.5 & x, y \in [0.5, 1] \quad \text{and} \quad x > y\,, \\ I_{GD}(x, y) & \text{otherwise,} \end{cases}$$

By Remark 1, it is clear that $([0, 1], \preceq_{I_{U_P}})$ is a partially ordered set.

Although $\frac{1}{4} \leq \frac{1}{2}$, the case $\frac{1}{4} \preceq_{I_{U_P}} \frac{1}{2}$ does not hold. Indeed, suppose that $\frac{1}{4} \preceq_{I_{U_P}} \frac{1}{2}$. Then, there exists an element $\ell \leq \frac{1}{2}$ such that

$$I_{U_P}\left(\ell, \frac{1}{4}\right) = \frac{1}{2}$$

holds. If $\ell = \frac{1}{2}$, then we would have that $\frac{1}{2} = I_{U_P}(\ell, \frac{1}{4}) = I_{GD}(\frac{1}{2}, \frac{1}{4}) = \frac{1}{4}$, a contradiction. Thus, it must be $\ell < \frac{1}{2}$. If $\ell \leq \frac{1}{4}$, then we would have a contradiction since $\frac{1}{2} = I_{U_P}(\ell, \frac{1}{4}) = I_{GD}(\ell, \frac{1}{4}) = 1$. Thus, it must be $\ell \in (\frac{1}{4}, \frac{1}{2})$. In this case, since

$$\frac{1}{2} = I_{U_P}\left(\ell, \frac{1}{4}\right) = \frac{\frac{1}{4}}{2.\ell}\,,$$

it is obtained that $\ell = \frac{1}{4}$, which contradicts that $\ell \in (\frac{1}{4}, \frac{1}{2})$. Then, we have that $\frac{1}{4} \not\preceq_{I_{U_P}} \frac{1}{2}$.

**Remark 2.** Note that, when $e = 1$, the orders $\leq_U$ and $\preceq_I$ given respectively in (3) and (4) coincide with the $T$-partial order $\preceq_T$ in Ref. 21 and the order $\sqsubseteq_I$ induced by implications on a bounded lattice,[20] respectively. Then, the orders $\preceq_T$ and $\sqsubseteq_I$ are extended to more general forms.

It is obvious that the orders $\leq_U$ given in (3) and $\preceq_U$ given in Ref. 18 coincide on $[0, e]$ but in general the orders are different. We give an example illustrating the difference of two orders.

**Example 2.** Consider the lattice $L = \{0, a, b, c, d, e, 1\}$ with $0 < a < b < c < d < e < 1$ and take the function $U \colon L^2 \to L$ as in Table 1.

By Remark 7 in Ref. 18 it is clear that $U$ is a uninorm on $L$ with neutral element $c$. Since $b < c < e$, by the definition of $\preceq_U$, it is obvious that $b \preceq_U e$. Although, since there doesn't exist an element $\ell \leq c$ such that $U(\ell, e) = b$, it is not possible

Table 1. The uninorm $U$ on $L$.

| $U$ | 0 | $a$ | $b$ | $c$ | $d$ | $e$ | 1 |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a$ | 0 | $a$ | $a$ | $a$ | $d$ | $d$ | 1 |
| $b$ | 0 | $a$ | $b$ | $b$ | $d$ | $e$ | 1 |
| $c$ | 0 | $a$ | $b$ | $c$ | $d$ | $e$ | 1 |
| $d$ | 0 | $d$ | $d$ | $d$ | 1 | 1 | 1 |
| $e$ | 0 | $d$ | $e$ | $e$ | 1 | 1 | 1 |
| 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |

the case $b \leq_U e$. On the other hand, since $U(a, e) = d$, it is clear that $d \leq_U e$. But, since there doesn't exist an element $\ell \geq c$ such that

$$U(\ell, d) = e\,,$$

we have that $d \npreceq_U e$. This shows that the orders $\leq_U$ and $\preceq_U$ are independent.

Let $(L, \leq, 0, 1)$ be a bounded lattice, $U \in \mathcal{U}(e)$ a conjunctive uninorm and $I$ an implication satisfying $(LI_U)$ and $(NP_e)$. The partial orders induced by $I$ and $U$ are independent. The following example illustrates this independency.

**Example 3.** Let us consider the uninorm $U = \langle T^{nM}, S_M, \frac{1}{2}\rangle \in \mathcal{U}_{\min}$ and take the $RU$-implication $I_U$ obtained from $U$. Since

$$I_U\left(\frac{1}{4}, \frac{1}{8}\right) = \frac{1}{2} I_{FD}\left(\frac{1}{2}, \frac{1}{4}\right) = \frac{1}{2}\max\left(1 - \frac{1}{2}, \frac{1}{4}\right)$$
$$= \frac{1}{2}\max\left(\frac{1}{2}, \frac{1}{4}\right) = \frac{1}{4},$$

if we choose $\ell := \frac{1}{4} \leq \frac{1}{2}$, we have that

$$I_U\left(\ell, \frac{1}{8}\right) = \frac{1}{4}\,,$$

whence $\frac{1}{8} \preceq_{I_U} \frac{1}{4}$. Although, let us show that $\frac{1}{8} \nleq_U \frac{1}{4}$. Suppose that $\frac{1}{8} \leq_U \frac{1}{4}$. Then, there exists an element $\ell \leq \frac{1}{2}$ such that

$$U\left(\ell, \frac{1}{4}\right) = \frac{1}{8}\,.$$

Since $\ell, \frac{1}{4} \in [0, \frac{1}{2}]$, we have that $\frac{1}{2}T^{nM}(2\ell, \frac{1}{2}) = \frac{1}{8}$, whence $T^{nM}(2\ell, \frac{1}{2}) = \frac{1}{4}$. If $2\ell + \frac{1}{2} \leq 1$, then it would be $\frac{1}{4} = T^{nM}(2\ell, \frac{1}{2}) = 0$, a contradiction. Then, $2\ell + \frac{1}{2} > 1$. In this case, since

$$\frac{1}{4} = T^{nM}\left(2\ell, \frac{1}{2}\right) = \min\left(2\ell, \frac{1}{2}\right),$$

we have that $\ell = \frac{1}{8}$ which contradicts that $2\ell + \frac{1}{2} > 1$. This shows that there doesn't exist any element $\ell \leq \frac{1}{2}$ such that $U(\ell, \frac{1}{4}) = \frac{1}{8}$, that is, $\frac{1}{8} \nleq_U \frac{1}{4}$.

On the other side, since

$$U\left(\frac{1}{8},\frac{1}{2}\right) = \frac{1}{2}T^{nM}\left(\frac{1}{4},1\right) = \frac{1}{8} \text{ and } \ell^* = \frac{1}{8} \leq \frac{1}{2},$$

we have that $\frac{1}{8} \leq_U \frac{1}{2}$. Although, $\frac{1}{8} \not\preceq_{I_U} \frac{1}{2}$. Indeed, let $\frac{1}{8} \preceq_{I_U} \frac{1}{2}$. Then, there exists an element $\ell \leq \frac{1}{2}$ such that

$$I_U\left(\ell,\frac{1}{8}\right) = \frac{1}{2}.$$

If $\ell \leq \frac{1}{8}$, since $\frac{1}{2} = I_U(\ell,\frac{1}{8}) = I_{GD}(\ell,\frac{1}{8}) = 1$, we would have a contradiction. If $\ell = \frac{1}{2}$, we would obtain a contradiction again since $\frac{1}{2} = I_U(\ell,\frac{1}{8}) = I_U(\frac{1}{2},\frac{1}{8}) = \frac{1}{8}$. So, it must be $\frac{1}{8} < \ell < \frac{1}{2}$. Then, since $2\ell > \frac{1}{4}$, we have the following equalities,

$$\begin{aligned}\frac{1}{2} = I_U\left(\ell,\frac{1}{8}\right) &= \frac{1}{2}I_{FD}\left(2\ell,\frac{1}{4}\right)\\ &= \frac{1}{2}\max\left(1-2\ell,\frac{1}{4}\right).\end{aligned}$$

Then, it follows $\ell = 0$ from $\max(1-2\ell,\frac{1}{4}) = 1$. This contradicts that $\ell > \frac{1}{8}$. Then, there doesn't exist any element $\ell \leq \frac{1}{2}$ satisfying that $I_U(\ell,\frac{1}{8}) = \frac{1}{2}$, that is, $\frac{1}{8} \not\preceq_{I_U} \frac{1}{2}$.

**Remark 3.**

(i) Let $U \in \mathcal{U}(e)$. If $U$ is conjunctive, since $U(0,x) = 0$ for any $x \in L$ it is clear that $0 \leq_U x$. Thus, 0 is the least element of $(L,\leq_U)$. If $U$ is not a conjunctive uninorm, the partial order $(L,\leq_U)$ needs not to have the bottom and top elements. For example, if we consider the uninorm $U = \langle T^{nM}, S_M, \frac{1}{2}\rangle \in \mathcal{U}_{\max}$, it can be easily seen that $([0,1],\leq_U)$ has not the bottom and top elements.

(ii) Let $I$ be an implication on a bounded lattice $L$ and $\preceq_I$ as in (3) define an order on $L$. Since $I(0,x) = 1$ for any $x \in L$, it is clear that $x \preceq_I 1$. Thus, $(L,\preceq_I)$ has the top element 1.

(iii) It need not to exist the bottom element of $(L,\preceq_I)$. We look at the following example.

**Example 4.** Consider the $RU$-implication $I_{U_P}$ given in Example 1. Suppose that an element $a \in [0,1]$ is the bottom element on $([0,1],\preceq_{I_{U_P}})$. Then, for any $x \in [0,1]$, $a \preceq_{I_{U_P}} x$. By Proposition 3, we have that for any $x \in [0,1]$, $a \leq x$, that is, $a = 0$. Although, 0 is not the bottom element of $([0,1],\preceq_{I_{U_P}})$. Indeed, suppose that $0 \preceq_{I_{U_P}} x$ for any $x \in [0,1]$. Then, it must be $0 \preceq_{I_{U_P}} 0.5$. Then, there exists an element $\ell \leq 0.5$ such that

$$I_{U_P}(\ell,0) = 0.5$$

holds. If $\ell = 0$, it would be $0.5 = I_{U_P}(\ell,0) = I_{U_P}(0,0) = 1$, a contradiction. If $\ell = 0.5$, since $0.5 = I_{U_P}(\ell,0) = I_{GD}(0.5,0) = 0$, which is a contradiction again.

Thus, it must be $\ell \in (0, 0.5)$. In this case, we have that

$$0.5 = I_{U_P}(\ell, 0) = \frac{0}{2.\ell} = 0\,,$$

a contradiction. Thus, 0 can not be the bottom element of $([0,1], \preceq_{I_{U_P}})$.

Although, it can not be said that the bottom element of $(L, \preceq_I)$ for any implication $I$ satisfying the $(LI_U)$ and $(NP_e)$ does not exist. The following is an example for a bounded partially ordered set $(L, \preceq_I)$.

**Example 5.** Let us consider a disjunctive uninorm $U_P$ from the class $\mathcal{U}_{\max}$ generated by the triple $\langle T_P, S_P, \frac{1}{2}\rangle$. Then, the corresponding $(U,N)$-implication obtained from $U_P$ and the classical negation $N_C$ is as follows:[1]

$$I(x,y) = \begin{cases} 2y - 2x & \max(1-x, y) \leq \dfrac{1}{2}\,, \\[2ex] 1 - 2x + 2xy & \max(1-x, y) > \dfrac{1}{2}\,, \\[1ex] I_{KD}(x,y) & \text{otherwise.} \end{cases}$$

It is clear that $([0,1], \preceq_I)$ is a partially ordered set since $I$ satisfies the law of importation to the $N_C$-dual of $U_P$ and $I(\frac{1}{2}, y) = y$ for any $y \in [0,1]$. Now, let us show that for any $x \in [0,1]$, $0 \preceq_I x$. If $x = 0$, then it is clear that $0 \preceq_I 0 = x$. Take $\ell = \frac{1-x}{2}$ for any $x \in (0,1]$. It is obvious that $\ell < \frac{1}{2}$. Since $\max(1-\ell, 0) = 1 - \ell > \frac{1}{2}$, we obtain that

$$I(\ell, 0) = I\left(\frac{1-x}{2}, 0\right) = 1 - 2\left(\frac{1-x}{2}\right) = x\,,$$

whence $0 \preceq_I x$. Thus, 0 is the bottom element of the partially ordered set $([0,1], \preceq_I)$.

**Proposition 5.** *Let $U = \langle T, S, e\rangle \in \mathcal{U}_{\min}$ with neutral element $e \in (0,1)$, where $T$ and $S$ are left-continuous and let $I_U$ be the corresponding RU-implication. Let $n_{I_T}(x) = I_T(x, 0)$ be the natural negation of the residual implication $I_T$. If $n_{I_T}$ is surjective, then for any $y \in [0, e)$, $0 \preceq_{I_U} y$.*

**Proof.** For $y = 0$, the claim is obvious. Let $y \in (0, e)$. Then, $0 < \frac{y}{e} < 1$. By the surjectivity of $n_{I_T}$, there exists an element $\ell \in [0,1]$ such that

$$n_{I_T}(\ell) = I_T(\ell, 0) = \frac{y}{e}\,.$$

For $\ell \in [0,1]$, it is clear that $e \cdot \ell \in [0, e]$. Say $\ell' := e \cdot \ell$. If $\ell' = e$, then it would be $\ell = 1$, whence we have that $\frac{y}{e} = I_T(1, 0) = 0$, a contradiction. Also, it is clear that $\ell' \neq 0$ from $e \neq 0$. Then, it must be $\ell' \in (0, e)$. Thus,

$$I_U(\ell', 0) = e \cdot I_T\left(\frac{\ell'}{e}, 0\right) = e \cdot I_T(\ell, 0) = e \cdot \frac{y}{e} = y\,,$$

whence we have that $0 \preceq_{I_U} y$. □

By the following proposition, we show that any implication $I$ satisfying the required conditions to define a partial order is increasing in the second variable w.r.t. the order $\preceq_I$.

**Proposition 6.** *Let $(L, \leq, 0, 1)$ be a bounded lattice, $U \in \mathcal{U}(e)$ a conjunctive uninorm and $I: L^2 \to L$ an implication satisfying $(LI_U)$ and $(NP_e)$. Then, for any $x, y \in L$*

$$x \preceq_I y \text{ implies that } I(z, x) \preceq_I I(z, y) \quad \text{for all } z \in L\,.$$

**Proof.** Let $x \preceq_I y$ for any $x, y \in L$. Then, there exists an element $\ell \leq e$ such that

$$I(\ell, x) = y\,.$$

By $(LI_U)$, we have the following equalities,

$$\begin{aligned} I(z, y) = I(z, I(\ell, x)) &= I(U(z, \ell), x) \\ &= I(U(\ell, z), x) \\ &= I(\ell, I(z, x))\,. \end{aligned}$$

Then, it follows $I(z, x) \preceq_I I(z, y)$ from $I(z, y) = I(\ell, I(z, x))$ and $\ell \leq e$. □

An implication $I$ need not be decreasing w.r.t. $\preceq_I$ in the first variable. Let us consider the following example.

**Example 6.** Let us consider a disjunctive uninorm $U_M$ from the class $\mathcal{U}_{\max}$ generated from the triple $\langle T_M, S_M, \frac{1}{2}\rangle$. Then, the corresponding $(U, N)$-implication obtained from $U_M$ and the classical negation $N_C$ is as follows:[1]

$$I(x, y) = \begin{cases} \min(1 - x, y) & \max(1 - x, y) \leq \dfrac{1}{2}\,, \\ I_{KD}(x, y) & \text{otherwise.} \end{cases}$$

It is clear that $([0, 1], \preceq_I)$ is a partially ordered set by Proposition 2 in the present paper and Theorem 4 in Ref. 15. Obviously, $\frac{1}{2} \preceq_I \frac{3}{4}$ since $I(\frac{1}{4}, \frac{1}{2}) = \max(1 - \frac{1}{4}, \frac{1}{2}) = \frac{3}{4}$. Suppose that $I$ is decreasing w.r.t. $\preceq_I$ in the first variable. Thus, it must be

$$\frac{1}{4} = I\left(\frac{3}{4}, \frac{1}{2}\right) \preceq_I I\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}\,.$$

Then, there exists an element $\ell \leq \frac{1}{2}$ such that

$$I\left(\ell, \frac{1}{4}\right) = \frac{1}{2}\,.$$

If $\ell = \frac{1}{2}$, it would be $\frac{1}{2} = I(\ell, \frac{1}{4}) = \frac{1}{4}$, a contradiction. Then, it must be $\ell < \frac{1}{2}$. Also, since $\max(1 - \ell, \frac{1}{4}) > \frac{1}{2}$, we have that

$$\frac{1}{2} = I\left(\ell, \frac{1}{4}\right) = I_{KD}\left(\ell, \frac{1}{4}\right) = 1 - \ell\,,$$

whence $\ell = \frac{1}{2}$, a contradiction. Thus, $\frac{1}{4} \not\preceq_I \frac{1}{2}$, which shows that $I$ is not decreasing w.r.t. $\preceq_I$ in the first variable.

**Proposition 7.** *Let $(L, \leq, 0, 1)$ be a bounded lattice, $U \in \mathcal{U}(e)$ a conjunctive uninorm and $I\colon L^2 \to L$ an implication satisfying $(LI_U)$ and $(NP_e)$. If $\phi$ is an order-preserving bijection on L such that $\phi(e) = e$, then $(L, \preceq_{I_\phi})$ is also a partially ordered set.*

**Proof.** It can be easily seen that $I_\phi$ satisfies the law of importation with $U_\phi$. Also, it is clear that $U_\phi$ is a conjunctive uninorm and $I$ satisfies $(NP_e)$. Thus, by Proposition 2, $(L, \preceq_{I_\phi})$ is a partially ordered set. □

The following Proposition presents the relationship between the orders $\preceq_I$ and $\preceq_{I_\phi}$.

**Proposition 8.** *Let $(L, \leq, 0, 1)$ be a bounded lattice, $U \in \mathcal{U}(e)$ a conjunctive uninorm and $I\colon L^2 \to L$ an implication satisfying $(LI_U)$ and $(NP_e)$. Let $\phi\colon L \to L$ is an order-preserving bijection with $\phi(e) = e$. Then, for any $a, b \in L$*

$$a \preceq_{I_\phi} b \quad \textit{iff } \phi(a) \preceq_I \phi(b)\,.$$

**Proof.** Let $a \preceq_{I_\phi} b$ for any $a, b \in L$. Then, there exists an element $\ell \leq e$ such that

$$I_\phi(\ell, a) = b\,,$$

whence $\phi^{-1}(I(\phi(\ell), \phi(a))) = b$, that is, we have that

$$I(\phi(\ell), \phi(a)) = \phi(b)\,.$$

Also, it is clear that $\phi(\ell) \leq \phi(e) = e$. Thus, it follows $\phi(a) \preceq_I \phi(b)$.

Conversely, let $\phi(a) \preceq_I \phi(b)$. Then, there exists an element $\ell \leq e$ such that

$$I(\ell, \phi(a)) = \phi(b)\,.$$

Since $\phi$ is surjective, for $\ell \in L$, there exists an element $\ell^* \in L$ such that $\phi(\ell^*) = \ell$. Since $\phi$ is an order-preserving bijection and $\ell \leq e$, it is clear that $\ell^* \leq e$. Since

$$\phi(b) = I(\ell, \phi(a)) = I(\phi(\ell^*), \phi(a))\,,$$

we have that $I_\phi(\ell^*, a) = \phi^{-1}(I(\phi(\ell^*), \phi(a))) = b$, where $\ell^* \leq e$. This shows that $a \preceq_{I_\phi} b$. □

**Remark 4.** Let $X$ be any subset of $L$. In the whole study, for any binary operation $M$ on $L$, we denote the set of the upper (lower) bounds of $X$ w.r.t. the order $\preceq_M$ by $\overline{X}_{\preceq_M}$ $(\underline{X}_{\preceq_M})$. Also, for any $a, b \in L$, $a \wedge_M b$ $(a \vee_M b)$ denotes the greatest (least) of the lower (upper) bounds w.r.t. $\preceq_M$, if there exists.

**Theorem 2.** *Let $(L, \leq, 0, 1)$ be a bounded lattice, $U \in \mathcal{U}(e)$ a conjunctive uninorm and $I\colon L^2 \to L$ an implication satisfying $(LI_U)$ and $(NP_e)$. Let $\phi\colon L \to L$ is an order-preserving bijection with $\phi(e) = e$. Then,*

*$(L, \preceq_I)$ is a lattice iff $(L, \preceq_{I_\phi})$ is a lattice.*

**Proof.** Let $(L, \preceq_I)$ is a lattice and take any elements $x, y \in L$. Then, $\phi(x), \phi(y) \in L$. Since $(L, \preceq_I)$ is a lattice, there exists the infimum and supremum of $\phi(x)$ and $\phi(y)$ w.r.t. the order $\preceq_I$. Let

$$\phi(x) \wedge_I \phi(y) = k \text{ and } \phi(x) \vee_I \phi(y) = t\,.$$

Since $\phi(x) \wedge_I \phi(y) = k$, it is clear that

$$k \preceq_I \phi(x) \text{ and } k \preceq_I \phi(y)\,.$$

Since $\phi(\phi^{-1}(k)) = k \preceq_I \phi(x)$ and $\phi(\phi^{-1}(k)) = k \preceq_I \phi(y)$, by Proposition 8, we have that $\phi^{-1}(k) \preceq_{I_\phi} x$ and $\phi^{-1}(k) \preceq_{I_\phi} y$. That is, $\phi^{-1}(k) \in \underline{\{x, y\}}_{\preceq_{I_\phi}}$. Let $m \in \underline{\{x, y\}}_{\preceq_{I_\phi}}$ be arbitrary. Then,

$$m \preceq_{I_\phi} x \text{ and } m \preceq_{I_\phi} y\,.$$

By Proposition 8, we have that

$$\phi(m) \preceq_I \phi(x) \text{ and } \phi(m) \preceq_I \phi(y)\,,$$

whence $\phi(m) \in \underline{\{\phi(x), \phi(y)\}}_{\preceq_I}$. Since $\phi(x) \wedge_I \phi(y) = k$, it must be

$$\phi(m) \preceq_I k\,.$$

Since $\phi(m) \preceq_I k = \phi(\phi^{-1}(k))$, by Proposition 8, we obtain that

$$m \preceq_{I_\phi} \phi^{-1}(k)\,.$$

Thus, $\phi^{-1}(k)$ is the greatest upper bound of $\phi(x)$ and $\phi(y)$ w.r.t. $\preceq_{I_\phi}$, that is,

$$x \wedge_{I_\phi} y = \phi^{-1}(k)\,.$$

Similarly, it can be easily shown that

$$x \vee_{I_\phi} y = \phi^{-1}(t)\,.$$

Thus, $(L, \preceq_{I_\phi})$ is also a lattice.

Conversely, let $(L, \preceq_{I_\phi})$ be a lattice. Say $\psi := \phi^{-1} : L \to L$. Then, it is clear that $\psi$ is an order-preserving bijection with $\psi(e) = e$. Since $(I_\phi)_\psi = I_{\phi\circ\psi} = I$, by the first part of the proof, we have that $(L, \preceq_{(I_\phi)_\psi}) = (L, \preceq_I)$ is a lattice. □

## 4. The Relationships Between the Orders $\leq_U$ and $\preceq_I$

In this section, it is determined some relationships between the order $\preceq_I$ induced by the $(U, N)$, $QL$, $D$ and $f$-generated implications satisfying the law of importation to a conjunctive uninorm $U$ with $e$ and the neutrality principle w.r.t. $e$ and the order $\leq_U$ induced by $U$. It is studied on some relationships between the algebraic structures obtained from these orders. Also, the order based on $g$-generated implications satisfying the law of importation to a t-norm is characterized.

**Proposition 9.** *Let $(L, \leq, 0, 1)$ be a bounded lattice, $U \in \mathcal{U}(e)$ a conjunctive uninorm and $I{:}L^2 \to L$ an implication satisfying $(LI_U)$ and $(NP_e)$. Then, for any $x, y \in L$,*

$$x \leq_U y \text{ implies that } I(y,z) \preceq_I I(x,z) \quad \text{for all } z \in L\,.$$

**Proof.** Let $x \leq_U y$. Then, there exists an element $\ell \leq e$ such that

$$U(\ell, y) = x\,.$$

Since $I(\ell, I(y,z)) = I(U(\ell,y),z) = I(x,z)$ and $\ell \leq e$, we have that

$$I(y,z) \preceq_I I(x,z)\,. \qquad \square$$

**Corollary 1.** *Let $(L, \leq, 0, 1)$ be a bounded lattice, $U \in \mathcal{U}(e)$ a conjunctive uninorm with $e \in [0, 1)$ and $I{:}L^2 \to L$ an implication with $I(1, e) = 0$ satisfying $(LI_U)$ and $(NP_e)$. Then, for any $x, y \in L$,*

$$x \leq_U y \text{ implies that } N_I^e(y) \preceq_I N_I^e(x)\,.$$

**Theorem 3.**[16] *Let $U_d$ be a disjunctive uninorm, $N$ a strong negation and $I_{U_d,N}$ its associated $(U,N)$-implication. Then, $I_{U_d,N}$ satisfies $(LI_{U_c})$ for a conjunctive uninorm $U_c$ if and only if $U_c$, defined by*

$$U_c(x,y) = N(U_d(N(x), N(y)))\,,$$

*is the $N$-dual of $U_d$.*

Obviously, Theorem 3 is also true on a bounded lattice $L$. That is, if $L$ is a bounded lattice, $U_d$ is a disjunctive uninorm on $L$, $N\colon L \to L$ is a strong negation and $I_{U_d,N}$ is its associated $(U,N)$-implication on $L$, then the statement of Theorem 3 holds again.

**Proposition 10.** *Let $(L, \leq, 0, 1)$ be a bounded lattice, $U_d \in \mathcal{U}(e)$ a disjunctive uninorm and $N$ a strong negation on $L$. Let $I_{U_d,N}$ be the associated $(U,N)$-implication satisfying the law of importation $(LI_{U_c})$ with a conjunctive uninorm $U_c$. Then, for any $x, y \in L$*

$$x \preceq_{I_{U_d,N}} y \text{ iff } N(y) \leq_{U_c} N(x)\,.$$

**Proof.** If the associated $(U,N)$-implication $I_{U_d,N}$ satisfies the law of importation with a conjunctive uninorm $U_c$, then it is clear that $U_c$ is the $N$-dual of $U_d$, by Theorem 3. Since $N$ is strong, it is obvious that $N$ is surjective. Then, for $e \in L$, there exists an element $e'$ of $L$ such that

$$N(e') = e\,.$$

Since $U_c$ is the $N$-dual of $U_d$ and $N(e') = e$, we have that

$$\begin{aligned} U_c(e', y) &= N(U_d(N(e'), N(y))) \\ &= N(U_d(e, N(y))) = N(N(y)) = y\,, \end{aligned}$$

which shows that the element $e' \in L$ is the neutral element of $U_c$. Also, since

$$\begin{aligned} I_{U_d,N}(e', y) &= U_d(N(e'), y) \\ &= U_d(e, y) = y\,, \end{aligned}$$

we see that $I_{U_d,N}$ satisfies the neutrality principle with $e'$. Thus, by Proposition 2, we obtain that $(L, \preceq_{I_{U_d,N}})$ is a partially ordered set.

Since $I_{U_d,N}(1, e) = 0$, $N^e_{I_{U_d,N}}$ is a negation and $N^e_{I_{U_d,N}} = N$. Let $N(y) \leq_{U_c} N(x)$ for any $x, y \in L$. By Corollary 1 and the strongness of $N$, we have immediately that $x \preceq_{I_{U_d,N}} y$.

Conversely, let $x \preceq_{I_{U_d,N}} y$ for any $x, y \in L$. Then, there exists an element $\ell \leq e'$ such that

$$I_{U_d,N}(\ell, x) = y\,,$$

whence $U_d(N(\ell), x) = y$. Since $U_c$ is the $N$-dual of $U_d$, we have that

$$\begin{aligned} U_c(\ell, N(x)) &= N(U_d(N(\ell), N(N(x)))) \\ &= N(U_d(N(\ell), x)) = N(y)\,. \end{aligned}$$

Thus, we obtain that $N(y) \leq_{U_c} N(x)$. □

**Theorem 4.** *Let $(L, \leq, 0, 1)$ be a bounded lattice, $U_d \in \mathcal{U}(e)$ a disjunctive uninorm and $N$ a strong negation on $L$. Let $I_{U_d,N}$ be the associated $(U,N)$-implication satisfying the law of importation $(LI_{U_c})$ with a conjunctive uninorm $U_c$. Then, $(L, \preceq_{I_{U_d,N}})$ is a meet (join) semi-lattice iff $(L, \leq_{U_c})$ is a join (meet) semi-lattice.*

**Proof.** Let $(L, \preceq_{I_{U_d,N}})$ is a meet semi-lattice and $x, y \in L$. Then, $N(x), N(y) \in L$. Since $(L, \preceq_{I_{U_d,N}})$ is a meet semi-lattice, there exists the infimum of $N(x)$ and $N(y)$ w.r.t. $\preceq_{I_{U_d,N}}$. Let $N(x) \wedge_{I_{U_d,N}} N(y) = k$. Then,

$$k \preceq_{I_{U_d,N}} N(x) \text{ and } k \preceq_{I_{U_d,N}} N(y)\,.$$

Since $N$ is surjective, for $k \in L$, there exists an element $k^*$ such that

$$N(k^*) = k\,.$$

Thus, we have that

$$N(k^*) \preceq_{I_{U_d,N}} N(x) \text{ and } N(k^*) \preceq_{I_{U_d,N}} N(y)\,.$$

By Proposition 10, it is clear that

$$x \leq_{U_c} k^* = N(k) \text{ and } y \leq_{U_c} k^* = N(k)\,.$$

That is, $N(k) = k^* \in \overline{\{x,y\}}_{\leq_{U_c}}$. Let $m \in \overline{\{x,y\}}_{\leq_{U_c}}$ be arbitrary. Then,

$$x \leq_{U_c} m \text{ and } y \leq_{U_c} m\,.$$

By Proposition 10, we have that

$$N(m) \preceq_{I_{U_d,N}} N(x) \text{ and } N(m) \preceq_{I_{U_d,N}} N(y)\,,$$

i.e., $N(m) \in \underline{\{N(x), N(y)\}}_{\preceq_{I_{U_d,N}}}$. Since $N(x) \wedge_{I_{U_d,N}} N(y) = k$, it is clear that

$$N(m) \preceq_{I_{U_d,N}} k\,.$$

By Proposition 10, we have that

$$N(k) \leq_{U_c} N(N(m)) = m\,,$$

which shows that $N(k)$ is the least upper bound of the elements $x$ and $y$ w.r.t. $\leq_{U_c}$. That is,

$$x \vee_{U_c} y = N(k)\,.$$

Thus, $(L, \leq_{U_c})$ is a join semi-lattice.

Conversely, it can be easily shown that if $(L, \leq_{U_c})$ is a join semi-lattice, then $(L, \preceq_{I_{U_d,N}})$ is a meet semi-lattice in a similar way.

Also, to prove that $(L, \preceq_{I_{U_d,N}})$ is a join semi-lattice iff $(L, \leq_{U_c})$ is a meet semi-lattice is similar. □

**Corollary 2.** *Let $(L, \leq, 0, 1)$ be a bounded lattice, $U_d \in \mathcal{U}(e)$ a disjunctive uninorm and $N$ a strong negation on $L$. Let $I_{U_d,N}$ be the associated $(U,N)$-implication satisfying the law of importation $(LI_{U_c})$ with a conjunctive uninorm $U_c$. Then, $(L, \preceq_{I_{U_d,N}})$ is a lattice iff $(L, \leq_{U_c})$ is a lattice.*

**Proposition 11.**[15] *Let $S$ be a t-conorm with $S(x, N(x)) = 1$ for all $x \in [0,1]$, $U$ a conjunctive uninorm and $N$ a strong negation. If the corresponding $QL$-operator satisfies the law of importation with a conjunctive uninorm $U'$, then $U'$ and $U$ are both t-norms.*

It is clear that Proposition 11 is also true on a bounded lattice $L$.

**Proposition 12.** *Let $I_{QL}$ be a $QL$-implication on a bounded lattice $(L, \leq, 0, 1)$ derived from a conjunctive uninorm $U_c$, a disjunctive uninorm $U_d$ and a strong negation $N^*$. If $I_{QL}$ satisfies the law of importation with a conjunctive uninorm $U$, then for any $x, y \in L$*

$$x \preceq_{I_{QL}} y \text{ iff } N^*(y) \leq_U N^*(x)\,.$$

**Proof.** Let $I_{QL}$ be a $QL$-implication derived from a conjunctive uninorm $U_c$, a disjunctive uninorm $U_d$ and a strong negation $N^*$. Then,

$$I_{QL}(x,y) = U_d(N^*(x), U_c(x,y))\,.$$

Since $I_{QL}$ is an implication, by Ref. 29 (see Definition 8 (iii)), $U_d$ must be a t-conorm such that $U_d(x, N^*(x)) = 1$ for all $x \in L$. If $I_{QL}$ satisfies $(LI_U)$ with a conjunctive uninorm $U$, by Proposition 11, we have that $U_c$ and $U$ are both t-norms and the neutral element $e$ of $U$ must be 1. Thus, note that the order $\leq_U$ is coincident with the $T$-partial order. Then,

$$\begin{aligned} I_{QL}(1, y) &= U_d(N^*(1), U_c(1, y)) \\ &= U_d(0, y) = y \end{aligned}$$

holds. Thus, since $I_{QL}$ satisfies $(LI_U)$ with $U$ and the neutrality principle with $e = 1$, by Proposition 2, we obtain that $(L, \preceq_{I_{QL}})$ is a partially ordered set. Moreover, since $I_{QL}$ satisfies $(LI_U)$ and $U$ is a t-norm, then by Proposition 7.3.9 in Ref. 1 which is also true on a bounded lattice, we have that $I_{QL}$ is an $(S, N)$-implication obtained from the strong negation $N^*$ and the t-conorm

$$S(x, y) = U_d(x, U_c(N^*(x), y)),$$

and $U$ is the $N^*$-dual of $S$. That is,

$$I_{QL}(x, y) = S(N^*(x), y). \tag{5}$$

Now, let $x \preceq_{I_{QL}} y$ for any $x, y \in L$. Then, there exists an element $\ell \leq e = 1$ such that

$$I_{QL}(\ell, x) = y.$$

Since $I_{QL}$ is an $(S, N)$-implication given as in (5), we have that

$$y = I_{QL}(\ell, x) = S(N^*(\ell), x).$$

Also, since $U$ is the $N^*$-dual of $S$, we have that

$$N^*(y) = N^*(S(N^*(\ell), N^*(N^*(x)))) = U(\ell, N^*(x)).$$

Thus, we obtain that

$$N^*(y) \leq_U N^*(x).$$

Conversely, let $N^*(y) \leq_U N^*(x)$. Then, there exists an element $\ell \in L$ such that

$$U(\ell, N^*(x)) = N^*(y).$$

Since $U$ is the $N^*$-dual of $S$, then we have that

$$\begin{aligned} N^*(y) = U(\ell, N^*(x)) &= N^*(S(N^*(\ell), N^*(N^*(x)))) \\ &= N^*(S(N^*(\ell), x)), \end{aligned}$$

whence $y = S(N^*(\ell), x) = I_{QL}(\ell, x)$. Since $y = I_{QL}(\ell, x)$ and $\ell \leq e = 1$, we obtain that $x \preceq_{I_{QL}} y$. □

**Theorem 5.** *Let $I_{QL}$ be a QL-implication on a bounded lattice $(L, \leq, 0, 1)$ derived from a conjunctive uninorm $U_c$, a disjunctive uninorm $U_d$ and a strong negation $N^*$. Let $I_{QL}$ satisfy the law of importation with a conjunctive uninorm $U$. Then, $(L, \preceq_{I_{QL}})$ is a meet (join) semi-lattice iff $(L, \leq_U)$ is a join (meet) semi-lattice.*

**Proof.** The proof is similar to the proof of Theorem 4. □

**Proposition 13.**[15] *Let $S$ be a t-conorm satisfying $S(x, N(x)) = 1$ for all $x \in [0,1]$, $U$ a conjunctive uninorm and $N$ a strong negation. If the corresponding $D$-operator $I_D$ satisfies the law of importation with a conjunctive uninorm $U_c$, then $U_c$ must be a t-norm.*

Obviously, Proposition 13 is also true on a bounded lattice.

**Proposition 14.** *Let $U$ be a conjunctive uninorm, $U_d$ a disjunctive uninorm and $N$ a strong negation on $L$. If the corresponding $D$-implication $I_D(x,y) = U_d(U(N(x), N(y)), y)$ satisfies $(LI_{U_c})$ with a conjunctive uninorm $U_c$, then for any $x, y \in L$,*

$$x \preceq_{I_D} y \text{ implies that } x \leq_{U_d} y\,.$$

**Proof.** Let $I_D(x,y) = U_d(U(N(x), N(y)), y)$ be a $D$-implication derived from a conjunctive uninorm $U$, a disjunctive uninorm $U_d$ and a strong negation $N$. By Ref. 29 (see Definition 8(iv)), $U_d$ must be a t-conorm satisfying the condition $U_d(x, N(x)) = 1$ for all $x \in L$. Moreover, if $I_D$ satisfies $(LI_{U_c})$ with a conjunctive uninorm $U_c$, then by Proposition 13, $U_c$ must be a t-norm and $e_c = 1$. Since $U$ is a conjunctive uninorm, it is clear that for any $y \in L$,

$$U(0, y) \leq U(0, 1) = 0\,,$$

whence $U(0, y) = 0$. Then, since $U_d$ is a t-conorm, it follows

$$\begin{aligned} I_D(1, y) &= U_d(U(N(1), N(y)), y) \\ &= U_d(U(0, N(y)), y) = U_d(0, y) = y\,. \end{aligned}$$

Since $I_D$ satisfies $(LI_{U_c})$ with a conjunctive uninorm $U_c$ and the neutrality principle $(NP_{e_c})$ with $e_c = 1$, then $(L, \preceq_{I_D})$ is a partially ordered set by Proposition 2.

Now, let $x \preceq_{I_D} y$ for any $x, y \in L$. Then, there exists an element $\ell \leq e_c = 1$ such that

$$I_D(\ell, x) = y\,.$$

Thus, $U_d(U(N(\ell), N(x)), x) = y$. Say $\ell' := U(N(\ell), N(x)) \in L$. Since $U_d$ is a t-conorm and $U_d(\ell', x) = y$, by the definition of $S$-partial order, we have that $x \leq_{U_d} y$. □

**Proposition 15.** *Let $I_f\colon [0,1]^2 \to [0,1]$ be an $f$-generated implication and $U_c$ a conjunctive uninorm on $[0,1]$. If $I_f$ satisfies the law of importation $(LI_{U_c})$ with $U_c$, then $\preceq_{I_f} = \leq_{U_c}$.*

**Proof.** Let $I_f$ satisfy $(LI_{U_c})$ with $U_c$. Then, for any $x, y, z \in [0,1]$,

$$I_f(U_c(x,y), z) = I_f(x, I_f(y, z))$$

holds. Taking $x = 1$, we obtain that

$$\begin{aligned} f^{-1}(y.f(z)) &= I_f(y,z) = f^{-1}(f(I_f(y,z))) \\ &= f^{-1}(1.f(I_f(y,z))) = I_f(1, I_f(y,z)) \\ &= I_f(U_c(1,y),z) = f^{-1}(U_c(1,y).f(z))\,, \end{aligned}$$

whence $y.f(z) = U_c(1,y).f(z)$. If $z \in (0,1)$, since $f$ is strictly decreasing, $0 < f(z) < \infty$ and hence $U_c(1,y) = y$, that is, $U_c$ is a t-norm. Moreover, it can be shown that $U_c$ is the product t-norm by similar arguments to the ones in the proof of Theorem 7.3.10. in Ref. 1. Since $T_P$ is continuous, it is clear that $\leq_{U_c} = \preceq_{T_P} = \leq$ by Ref. 21. Also, since $I_f(1,y) = f^{-1}(1.f(y)) = y$ for $e_c = 1$, the left neutrality principle $(NP_{e_c})$ holds. By Proposition 2, $([0,1], \preceq_{I_f})$ is a partially ordered set.

Also, it is clear that for any $a, b \in [0,1]$, if $a \preceq_{I_f} b$, then $a \leq b$, whence $a \leq_{U_c} b$. Conversely, let $a \leq_{U_c} b$. Then, $a \leq b$. If $a = 1$, then it is clear that $a \preceq_{I_f} b$ from $b = 1$. Now, let $a \neq 1$. Then, $f(a) \neq 0$. Since $f$ is a strictly decreasing function, we obtain that $f(b) \leq f(a)$. Take $\ell := \frac{f(b)}{f(a)}$. Since

$$I_f(\ell, a) = I_f\left(\frac{f(b)}{f(a)}, a\right) = f^{-1}\left(\frac{f(b)}{f(a)} \cdot f(a)\right) = b\,,$$

we have that $a \preceq_{I_f} b$. □

**Proposition 16.** *Let $I_g\colon [0,1]^2 \to [0,1]$ be a g-generated implication satisfying the law of importation with a t-norm $T$. Then, $\preceq_{I_g} = \leq$.*

**Proof.** Let $I_g\colon [0,1]^2 \to [0,1]$ satisfy the law of importation with a t-norm $T$. It is clear that $I_g$ satisfies the neutrality principle $(NP)$ with 1 by Theorem 3.2.8. in Ref. 1. Thus, by Proposition 2, $([0,1], \preceq_{I_g})$ is a partially ordered set. Also, it is obvious that for any $a, b \in [0,1]$, if $a \preceq_{I_g} b$, then $a \leq b$. Conversely, let $a \leq b$. If $b = 0$, then $a = 0$, whence we have that $a \preceq_{I_g} b$. Let $b \neq 0$. Then, since $g$ is a strictly increasing function, $g(b) \neq 0$ and $g(a) \leq g(b)$. Take $\ell := \frac{g(a)}{g(b)} \leq 1$. Then, since

$$\begin{aligned} I_g(\ell, a) &= g^{-1}\left(\min\left(\frac{1}{\ell} \cdot g(a), g(1)\right)\right) \\ &= g^{-1}\left(\min\left(\frac{g(b)}{g(a)} \cdot g(a), g(1)\right)\right) \\ &= g^{-1}(\min(g(b), g(1))) = b\,, \end{aligned}$$

we have that $a \preceq_{I_g} b$. □

## 5. Concluding Remarks

In this paper, an order, denoted by $\preceq_I$, induced by implications on a bounded lattice satisfying the law of importation to a conjunctive uninorm $U$ with $e$ and the neutrality principle w.r.t. $e$ is defined and some properties are discussed. The required and more lenient conditions than the ones given previous studies[20,25] allows to study on wider classes of implications imposing the order $\preceq_I$. Also, an order, different from the order introduced in Ref. 18 and denoted by $\leq_U$, based on uninorms on a bounded lattice is given and the independency of the orders $\preceq_I$ and $\preceq_U$ are shown. A relationship between the algebraic structures obtained from the orders induced by implications and their $\phi$-conjugates is determined. Some relationships between the order $\preceq_I$ induced by the $(U, N)$, $QL$, $D$ and $f$-generated implications satisfying the law of importation to a conjunctive uninorm $U$ with $e$ and the neutrality principle w.r.t. $e$ and the the $\leq_U$ induced by $U$ are presented. Moreover, some relationships between the algebraic structures obtained from these orders are obtained. It is shown that the order based on $g$-generated implications is coincident with the natural order.

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