Non-fragile $H_\infty$ output feedback control design for continuous-time fuzzy systems

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ABSTRACT

In this paper, we investigate the problem of non-fragile $H_\infty$ fuzzy control design for continuous Takagi–Sugeno (T–S) fuzzy systems with uncertainties, external disturbance and unmeasurable state variables. For the case of controller and observer gain additive variations, we propose a new solution of the fragility problem by developing the non-fragile design schemes ensuring the asymptotic stability and $H_\infty$ performance for the resulting closed loop systems. By considering a fuzzy Lyapunov function and by introducing slack variables, we propose the new sufficient stabilization conditions formulated in LMI constraints which can be easily solved using the convex optimization tools. The effectiveness the proposed results are illustrated through three numerical examples.

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1. Introduction

Over the last two decades, model-based fuzzy control has become a widespread approach to deal with complex nonlinear systems. In this context, (T–S) fuzzy models [1] play an important role due to their capacity to describe a large class of nonlinear systems and the existence of systematic and effective control design tools to complete other nonlinear control techniques [2,3].

Generally speaking, the standard approach for stability and stabilization analysis of (T–S) fuzzy models consists of finding a common quadratic Lyapunov function that satisfies a set of stability sufficient conditions [4–6]. These conditions are frequently formulated in Linear Matrix Inequality (LMI) constraints which can easily be solved using the convex optimization techniques. In this framework, very effective strategies have been suggested to overcome mathematical and numerical difficulties, promoting less-conservative conditions. The piecewise or switched Lyapunov functions are usually considered as an alternative type of Lyapunov functions to overcome the conservatism of the quadratic methods in [7–11]. An interesting alternative consists in selecting fuzzy Lyapunov functions, which are parametrized by the same membership functions as those used to construct the (T–S) fuzzy model. These types of functions, also named basis-dependent Lyapunov functions and non-quadratic Lyapunov functions, have been addressed for continuous fuzzy systems in [12–15] and for discrete-time fuzzy system in [16–18].

In general, an implicit assumption inherent to the fuzzy design techniques is that the controller will be implemented exactly. However, in practical situations, it has been shown that without considering the relatively small uncertainties in controller implementation, the robust controller design could even make the closed-loop system unstable. Such controllers are often named “fragile”. Therefore, it is important that any control system ensures that the closed loop system maintains the stability and performance level when the controller gains change in the predefined admissible range. Many studies have investigated the non-fragile controller design problem [19–23]. Recently, the non-fragile control issue is considered for some practical systems. In [24], the non-fragile $H_\infty$ control for vehicle active suspension systems is investigated. Two aspects of vibration control for structural systems based on resilient controller are concerned in [25].

In many real-world systems, the internal states cannot be directly measured and only their outputs are available for control purpose. The observer-based control design which consists in reconstructing the system states and realizing the required feedback is a popular approach to deal with dynamic systems with unavailable states for measurement. The problem of a non-fragile $H_\infty$ observer based control for continuous-time linear systems, with respect to additive norm-bounded controller and observer gain variations, was investigated in [26]. A design of non-fragile
H∞ observer-based control for continuous time delay systems was provided in [27]. In [28], the robust and non-fragile H∞ control problem for (T–S) fuzzy systems with linear fractional parametric uncertainties was proposed. Non-fragile dissipative fuzzy control for nonlinear discrete-time systems via (T–S) model has been considered in [29]. In [30], based on a basis-dependent Lyapunov function a new approach regarding H∞ non-fragile stability analysis and observer-based controller synthesis problem for a class of discrete-time fuzzy systems has been investigated.

To complement these efforts, the current paper tackles the problem of non-fragile observer based H∞ fuzzy control design for a class of continuous-time nonlinear systems. We use an appropriate fuzzy Lyapunov functional to characterize the conditions under which the closed-loop fuzzy system is asymptotically stable with an H∞ gain smaller than a prescribed constant level. With the introduction of some matrix variables, we show that the solution of the observer and the controller design problem can be obtained by solving a set of linear matrix inequalities (LMIs) and can therefore be easily checked by using an available numerical software.

The main result concerns the exploitation of the relaxed approach proposed by Liu and Zhang [31] and the fuzzy Lyapunov function approach [13,12] to obtain the less conservative results in non-fragile control design context.

This paper is organized as follows. System description and preliminaries are presented in Section 2. New H∞ performance conditions for a class of (T–S) fuzzy systems are derived in Section 3. The design of non-fragile observer based H∞ fuzzy controllers for such system is addressed in Sections 4 and 5. In Section 6, three numerical examples are given to illustrate the correctness and effectiveness of the theoretical results. Finally, Section 7 offers some conclusions and remarks.

Notation: For a symmetric matrix \( X, X > 0 \) means that it is positive definite. Symbol \( (\ast) \) within a matrix represents the symmetric entries. sym(X) stands for \( X + X^T \). To avoid clutter, in what follows, \( h_i(t) \) denotes \( h_i(\mu(t)) \).

2. System descriptions and preliminaries

Once it was proved that the (T–S) fuzzy model is a universal approximator for a class of nonlinear systems, this representation has been widely used to represent complex nonlinear systems. A continuous fuzzy model can be described by a set of fuzzy IF–THEN rules as

\[
R_i: \text{if } \mu_i(t) = M_i^1 \text{ and if } \mu_i(t) = M_i^2 \ldots \text{if } \mu_i(t) = M_i^n, \text{ then } \]

\[
X(t) = (A_i + \Delta A_i)x(t) + (B_{2i} + \Delta B_{2i})u(t) + (B_{1i} + \Delta B_{1i})w(t)
\]

\[
z(t) = C_{1i}x(t) + D_{1i}u(t) + D_{1i}w(t)
\]

\[
y(t) = C_{2i}x(t) + D_{2i}w(t), \quad i \in \mathbb{S} = \{1, 2, \ldots, r\}
\]

where \( x(t) \in \mathbb{R}^n \) is the state, \( u(t) \in \mathbb{R}^m \) is the control input, \( w(t) \in \mathbb{R}^l \) is the external disturbance input, \( z(t) \in \mathbb{R}^p \) is the controlled output, \( y(t) \in \mathbb{R}^q \) is the measured output, \( M_i^j \) (\( j = 1, \ldots, v \)) are fuzzy sets and \( \mu(t) = (\mu_1(t), \ldots, \mu_r(t)) \) is the premise variable vector. It is assumed that the premise variables do not depend on input variables \( u(t) \), which are needed to avoid a complicated defuzzification process of fuzzy controllers. System disturbance, \( w(t) \), is assumed to belong to \( L_2(0, \infty) \), that is, \( \int_0^\infty w^T(t)w(t) \, dt < \infty \). This implies that the disturbance has a finite energy. \( A_i, B_{1i}, B_{2i}, C_{1i}, D_{1i}, D_{2i}, C_{2i} \) and \( D_{2i} \) are finite constant matrices that describe the nominal system. Moreover, \( \Delta A_i, \Delta B_{1i} \) and \( \Delta B_{2i} \) are bounded matrices and defined as follows:

\[
[\Delta A_i, \Delta B_{1i}, \Delta B_{2i}] = M_i [N_i, N_{1i}, N_{2i}]
\]

where

\[
\Delta = F(1 - JF)^{-1}, \quad 0 < I - J^TF
\]

\( M_i, N_i, N_{1i} \) and \( N_{2i} \) (\( i = 1, 2 \)) are real matrices of appropriate dimensions.

Given pair \( (x(t), u(t)) \), the overall fuzzy system is inferred as follows:

\[
\begin{align*}
X(t) &= \sum_{i=1}^{r} h_i(t)((A_i + \Delta A_i)x(t) + (B_{2i} + \Delta B_{2i})u(t) + (B_{1i} + \Delta B_{1i})w(t)) \\
z(t) &= \sum_{i=1}^{r} h_i(t)(C_{1i}x(t) + D_{1i}u(t) + D_{1i}w(t)) \\
y(t) &= \sum_{i=1}^{r} h_i(t)(C_{2i}x(t) + D_{2i}w(t))
\end{align*}
\]

where \( h_i(t) \) is the averaging weight for each rule, representing the normalized grade of membership and satisfies

\[
\sum_{i=1}^{r} h_i(t) = 1, \quad \sum_{i=1}^{r} h_i(t) = 0, \quad i \in \mathbb{S}
\]

For the purpose of control design, we consider different feedback schemes including fuzzy state feedback and fuzzy observer-based state feedback. Closed-loop fuzzy system (5) with fuzzy controllers is given by

\[
\begin{align*}
X(t) &= \sum_{i=1}^{r} h_i(t)(A_i + \Delta A_i)x(t) + (B_{2i} + \Delta B_{2i})u(t) + (B_{1i} + \Delta B_{1i})w(t) \\
z(t) &= \sum_{i=1}^{r} h_i(t)(C_{1i}x(t) + D_{1i}u(t) + D_{1i}w(t)) \\
y(t) &= \sum_{i=1}^{r} h_i(t)(C_{2i}x(t) + D_{2i}w(t))
\end{align*}
\]

Our main objective in this work is twofold: the first one is to derive new conditions under which closed-loop fuzzy system (7) is asymptotically stable with \( w(t) = 0 \), and also under zero initial conditions the following \( H_\infty \) performance is satisfied for all non-zero \( w(t) \in L_2([0, \infty)) \) and \( \gamma > 0 \).

\[
\|z(t)\|_2 < \gamma \|w(t)\|_2
\]

The second objective is to provide an LMI-based method for the non-fragile \( H_\infty \) performance synthesis by developing control schemes based on state or output feedbacks for all admissible observer and controller gain variations. We end this section by recalling the following lemma which will be used in the proof of our main results.

Lemma 2.1 (Xie [32]). Given matrices \( Q = Q^T \), \( H \) and \( E \) with appropriate dimensions, if there exists a scalar \( \epsilon > 0 \) such that

\[
\begin{bmatrix}
Q & * & * \\
\epsilon H^T & -\epsilon I & * \\
E & \epsilon J & -\epsilon I
\end{bmatrix} < 0
\]

then \( Q + \text{sym}(H \Delta E) < 0 \) for any \( \Delta \) satisfying

\[
\Delta = F(1 - JF)^{-1}, \quad 0 < I - J^TF
\]

where \( J \) is a known real constant matrix and \( F \) is an uncertain matrix satisfying \( F^TF \leq 1 \).

3. A new \( H_\infty \) performance condition

This section gives a new characterization involving a fuzzy Lyapunov functional to analyze the stability as well as the \( H_\infty \) disturbance attenuation performance for closed-loop fuzzy system (7).

Setting uncertainties in (7) to zero, the following result is addressed for the nominal closed-loop fuzzy system.

Lemma 3.1. Given a scalar \( \gamma > 0 \), fuzzy system (7) is asymptotically stable with \( H_\infty \) norm bound \( \gamma \), if there exist \( n \times n \) matrices \( P_i > 0, G_i \).
$G_1, \theta_i = \theta^0_i$ and $X_{ji}$, $i, j \in \mathbb{S}$, such that the following inequalities hold:

\begin{equation}
P_s + \theta_i > 0, \quad s \in \mathbb{S},
\end{equation}

\begin{equation}
Y_{ji} + X_{ji} < 0,
\end{equation}

\begin{equation}
Y_{ji} + Y_{ji} + \text{sym}(X_{ji}) < 0, \quad j > i,
\end{equation}

\begin{equation}
\begin{bmatrix}
X_{11} & \ldots & X_{1r} \\
0 & \ddots & 0 \\
X_{r1} & \ldots & X_{rr}
\end{bmatrix}
\end{equation}

\begin{equation}
> 0
\end{equation}

where

\begin{equation}
Y_{ji} = \begin{bmatrix}
\sum_{s=1}^{s} \phi_s(P_s + \theta_i) + \text{sym}(G_1 A_{ji}) & * & * \\
G_1 A_{ji} - G_1 + P_i & -\text{sym}(G_2) & * & *
\end{bmatrix}
\begin{bmatrix}
X_{ij} \\
B_1^T G_1
\end{bmatrix}

\begin{equation}
X_{ji} = \begin{bmatrix}
X_{ji}^{11} & X_{ji}^{12} & X_{ji}^{13} & 0 \\
X_{ji}^{21} & X_{ji}^{22} & X_{ji}^{23} & 0 \\
X_{ji}^{31} & X_{ji}^{32} & X_{ji}^{33} & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\end{equation}

\begin{equation}
n, i \neq j.
\end{equation}

\textbf{Proof.} Under the conditions of the theorem, we first establish the stability of the system in (7). To this end, we consider (7) with $w(t) = 0$; that is

\begin{equation}
x(t) = \sum_{i=1}^{i} \sum_{j=1}^{j} h_i(t) h_j(t) A_{ji} x(t)
\end{equation}

In the sequel, we choose a fuzzy based-dependent Lyapunov function candidate for system (15) as follows:

\begin{equation}
V(x(t)) = x^T(t) P(t) x(t)
\end{equation}

where $P(t)$ is a fuzzy weighting-dependent Lyapunov function

\begin{equation}
P(t) = \sum_{i=1}^{i} h_i(t) P_i
\end{equation}

The derivative of (17) provides information about time derivative of the membership functions. Then, we assume that

\begin{equation}|h_i(t)| \leq \phi_s
\end{equation}

where $\phi_s \geq 0, \ s \in \mathbb{S}$.

Then, the derivative of Lyapunov function (16) gives

\begin{equation}
V(x(t)) = 2x^T(t) P(t) x(t) + x^T(t) \dot{P}(t) x(t)
\end{equation}

Considering the properties in (6), we have

\begin{equation}
\dot{P}(t) = \sum_{i=1}^{i} h_i(t) \dot{P}_i = \left( \sum_{i=1}^{i} h_i(t) \right) \sum_{i=1}^{i} h_i(t) \dot{P}_i = \sum_{i=1}^{i} \sum_{j=1}^{j} h_i(t) \dot{h}_j(t) P_s
\end{equation}

Once again considering (6), the following equation holds:

\begin{equation}
\sum_{i=1}^{i} \sum_{j=1}^{j} h_i(t) \dot{h}_j(t) \theta_i = 0
\end{equation}

where $\theta_i, \ i \in \mathbb{S}$, are slack matrices with appropriate dimensions.

Assuming that assumption (18) and (11) hold, it is easy to have

\begin{equation}
\dot{P}(t) \leq \sum_{i=1}^{i} \sum_{j=1}^{j} h_i(t) \phi_j (P_s + \theta_i)
\end{equation}

Considering (15), the following equality holds for matrices $G_1 \in \mathbb{R}^{r \times n}$ and $G_2 \in \mathbb{R}^{r \times n}$:

\begin{equation}
2(x^T(t) G_1^T + x^T(t) G_2^T) \left( -x(t) + \sum_{i=1}^{i} \sum_{j=1}^{j} h_i(t) h_j(t) A_{ji} x(t) \right) = 0
\end{equation}

Then, adding null terms (21) and (23) to (19) and replacing $\dot{P}(t)$ by the upper bound in (22), it follows that

\begin{equation}
V(x(t)) \leq \sum_{i=1}^{i} \sum_{j=1}^{j} h_i(t) h_j(t) \Xi_{ij} \zeta(t)
\end{equation}

\begin{equation}
= \sum_{i=1}^{i} \sum_{j=1}^{j} h_i(t) h_j(t) (\Xi_{ij} + \Xi_{ji}) \zeta(t)
\end{equation}

where

\begin{equation}
\zeta(t) = \begin{cases}
x(t) \quad \Xi_{ij} \\zeta(t) = \begin{bmatrix}
\sum_{i=1}^{i} \phi_j(P_s + \theta_i) + \text{sym}(G_1 A_{ji}) & * \\
G_1 A_{ji} - G_1 + P_i & -\text{sym}(G_2)
\end{bmatrix}
\end{cases}
\end{equation}

Let

\begin{equation}
X_{ii} = \begin{bmatrix}
X_{ii}^{11} & * & * \\
X_{ii}^{21} & X_{ii}^{22} & * \\
X_{ii}^{31} & X_{ii}^{32} & X_{ii}^{33}
\end{bmatrix}, \quad X_{jj} = \begin{bmatrix}
X_{jj}^{11} & * \\
X_{jj}^{21} & X_{jj}^{22}
\end{bmatrix}
\end{equation}

From (12) and (13), it is easy to obtain that

\begin{equation}
\Xi_{ij} + X_{ii} < 0, \quad i \in \mathbb{S}
\end{equation}

\begin{equation}
\Xi_{ii} + X_{ij} + \text{sym}(X_{ji}) < 0, \quad j > i
\end{equation}

Thus, from (24) and (27) we have

\begin{equation}
V(x(t)) \leq \zeta(t) \left( -\sum_{i=1}^{i} h_i^2(t) X_{ii} - \sum_{i=1}^{i} \sum_{j=1}^{j} h_i(t) h_j(t) (X_{ji} + X_{ij}) \right) \zeta(t)
\end{equation}

\begin{equation}
= -\left( h_i(t) \zeta(t) \right)^T \begin{bmatrix}
X_{ii}^{11} & \ldots & X_{ii}^{1r} \\
X_{ii}^{21} & \ldots & X_{ii}^{2r} \\
X_{ii}^{31} & \ldots & X_{ii}^{3r}
\end{bmatrix} h_i(t) \zeta(t)
\end{equation}

\text{It is obvious from (14) that for all } \zeta(t) \neq 0, \ V(x(t)) < 0. \ Hence, closed-loop system (7) is asymptotically stable when there is no disturbance.

Next, we show that for any nonzero $w(t) \in I_2[0, \infty)$ system (7) satisfies (8) under the zero initial condition. To this end, we introduce the following index:

\begin{equation}
J = \int_0^{\infty} (x^T(t) z(t) - x^T(t) w(t)) dt
\end{equation}

For any nonzero $w(t) \in I_2[0, \infty)$ and zero-initial conditions, we can get

\begin{equation}
J \leq \int_0^{\infty} (x^T(t) z(t) - x^T(t) w(t)) dt + V(\infty)
\end{equation}

\begin{equation}
= \int_0^{\infty} (x^T(t) z(t) - x^T(t) w(t)) dt + V(\infty)
\end{equation}

where $V(x(t))$ is given in (16).

Setting $\zeta(t) = [x^T(t) x^T(t) w^T(t)]$ and $\Xi_{ij} = [C_{ij} 0 D_{ij}]$, we have

\begin{equation}
J \leq \int_0^{\infty} \left( \sum_{i=1}^{i} \sum_{j=1}^{j} h_i(t) h_j(t) \Xi_{ij} \zeta(t) + 2x^T(t) P(t) x(t)
\end{equation}

\begin{equation}
+ x^T(t) P(t) x(t) - x^T(t) w(t) \right) dt
\end{equation}

Noting $\Xi = [G_1 \ G_2 0]$. The following equation holds:

\begin{equation}
2 \zeta(t) \Xi^T(t) \zeta(t) = \left( -x(t) + \sum_{i=1}^{i} \sum_{j=1}^{j} h_i(t) h_j(t) A_{ji} x(t) + B_j w(t) \right) = 0
\end{equation}
It can be established from (22) and (33) that
\[ J \leq \int_{0}^{\infty} \left\{ \xi(t) \left( \sum_{i=1}^{r} \sum_{j=1}^{r} h_{ij}(t)\phi_{ii} + \sum_{i=1}^{r} \sum_{j=1}^{r} h_{ij}(t)\Phi_{ij} + -\text{sym}(G_{2}) \right) \xi(t) \right\} \, dt \] (35)
where
\[ \Phi_{ij} = \begin{bmatrix} \sum_{i=1}^{r} \phi_{ij}(P_{i} + \theta_{i}) + \text{sym}(G_{1})A_{ij} & 0 & 0 \\ -\text{sym}(G_{2}) & 0 & 0 \\ B_{i}^{T}G_{1} & -\sum_{i=1}^{r} \sum_{j=1}^{r} h_{ij}(t)h_{ij}(t) & y^{2}l \end{bmatrix} \]

If (12) and (13) hold, we have
\[ \sum_{i=1}^{r} \sum_{j=1}^{r} h_{ij}(t)h_{ij}(t)Y_{ii} = \sum_{i=1}^{r} h_{ii}^{2}(t)Y_{ii} + \sum_{i=1}^{r} \sum_{j=1}^{r} h_{ij}(t)h_{ij}(t)(Y_{ii} + Y_{jj}) \] (36)
\[ \sum_{i=1}^{r} \sum_{j=1}^{r} h_{ij}(t)h_{ij}(t)Y_{ij} = -\sum_{i=1}^{r} h_{ii}^{2}(t)X_{ii} - \sum_{i=1}^{r} \sum_{j=1}^{r} h_{ij}(t)h_{ij}(t)(X_{ij} + X_{ji}) \] (37)
\[ \sum_{i=1}^{r} \sum_{j=1}^{r} h_{ij}(t)h_{ij}(t)Y_{jj} = \begin{bmatrix} h_{11}(t) & \cdots & h_{1r}(t) \\ \vdots & \ddots & \vdots \\ h_{r1}(t) & \cdots & h_{rr}(t) \end{bmatrix} \begin{bmatrix} X_{11} & \cdots & X_{1r} \\ \vdots & \ddots & \vdots \\ X_{r1} & \cdots & X_{rr} \end{bmatrix} \begin{bmatrix} h_{11}(t) \\ \vdots \\ h_{r1}(t) \end{bmatrix} < 0 \] (38)

Applying the Schur complement, it is easy to get
\[ Y_{ij} = \Phi_{ij} + C_{ij}^{T}C_{ij} < 0 \] (39)

This inequality together with (35) shows that for all \( \xi(t) \neq 0, J < 0 \), which implies \( \|x\| < \gamma \|w\| \) for any nonzero \( w_{k} \in I_{2}(0,\infty) \). This completes the proof. \( \Box \)

**Remark 3.1.** The main feature of Lemma 3.1 is that its derivation has combined the relaxation approach proposed by Liu and Zhang [31] with the fuzzy Lyapunov approach [14,12], which can further reduce the conservativeness of the conditions.

In the sequel, we proceed to derive sufficient conditions for the robust \( H_{\infty} \) stabilization problem of underlying system (7) for all admissible uncertainties.

**Lemma 3.2.** Given a scalar \( \gamma > 0 \), fuzzy system (7) is asymptotically stable with \( H_{\infty} \) norm bound \( \gamma \), if there exist \( n \times n \) matrices \( P_{i} > 0, G_{1}, G_{2}, \theta_{i} = \theta_{i}^{T}, X_{ij} \), and positive scalars \( \epsilon_{ij} \), \( i, j \in \mathbb{S} \), such that the following inequalities hold:
\[ P_{i} + \theta_{i} > 0, \quad s \in \mathbb{S}, \]
\[ \begin{bmatrix} Y_{ii} & X_{ij} & 0 & 0 \\ \epsilon_{ij}h_{ij} & -\epsilon_{ij}I & 0 & 0 \\ \epsilon_{ij} & -\epsilon_{ij}I & -\epsilon_{ij}I & 0 \\ 0 & 0 & 0 & -\epsilon_{ij}I \end{bmatrix} < 0, \quad (40) \]
\[ \begin{bmatrix} Y_{ij} & Y_{ji} + \text{sym}(X_{ij}) & 0 & 0 \\ \epsilon_{ij} & -\epsilon_{ij}I & 0 & 0 \\ \epsilon_{ij} & -\epsilon_{ij}I & -\epsilon_{ij}I & 0 \\ 0 & 0 & 0 & -\epsilon_{ij}I \end{bmatrix} < 0, \quad j > i, \quad (41) \]
\[ \begin{bmatrix} X_{11} & \cdots & X_{1r} \\ \vdots & \ddots & \vdots \\ X_{r1} & \cdots & X_{rr} \end{bmatrix} > 0 \quad (42) \]

where
\[ M_{ii}^{T} = \begin{bmatrix} M_{ii}^{T}G_{1} & M_{ii}^{T}G_{2} & 0 & 0 \\ M_{ii}^{T}G_{1} & M_{ii}^{T}G_{2} & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} N_{ij} & 0 & 0 & 0 \\ 0 & 0 & N_{ij} & 0 \end{bmatrix}, \quad I = \text{diag}(I, I), \quad J = \text{diag}(J, J) \]

**Proof.** Considering (3), the uncertainty part in (7) can be written as
\[ \Delta A_{ij} = M_{ii}\Delta A_{ij}, \quad \Delta B_{ij} = M_{ii}\Delta B_{ij} \]
(44)

According to Lemma 2.1, we get (42)
\[ Y_{ij} + Y_{ji} + \text{sym}(X_{ij}) + \text{sym}(M_{ii}\Delta A_{ij}) + \text{sym}(M_{ii}\Delta B_{ij}) < 0, \quad j > i, \quad (45) \]

which can be written as
\[ (Y_{ii} + \Delta Y_{ii}) + (Y_{jj} + \Delta Y_{jj}) + \text{sym}(X_{ij}) < 0 \]
(46)

likewise we get
\[ (Y_{ii} + \Delta Y_{ii}) + X_{ii} < 0 \]
(47)

where
\[ \Delta Y_{ii} = \begin{bmatrix} \text{sym}(G_{1}^{T}\Delta A_{ij}) & \Delta A_{ij}^{T}G_{2} & G_{1}^{T}\Delta B_{ij} & 0 \\ \Delta A_{ij}^{T}G_{2} & 0 & G_{1}^{T}\Delta B_{ij} & 0 \\ \Delta B_{ij}^{T}G_{1} & \Delta B_{ij}^{T}G_{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{diag}(\Delta, \Delta) = \left( \begin{array}{ccc} F & 0 \\ 0 & F \end{array} \right) \left( \begin{array}{ccc} I & 0 \\ 0 & I \end{array} \right) \left( \begin{array}{ccc} I & 0 \\ 0 & I \end{array} \right) \left( \begin{array}{ccc} I & 0 \\ 0 & F \end{array} \right)^{-1} \]

In the light of Lemma 3.1, fuzzy system (7) is robustly stable with \( H_{\infty} \) disturbance attenuation level \( \gamma \). \( \Box \)

4. **Non-fragile \( H_{\infty} \) fuzzy controller design**

For practical implementation, the following non-fragile state feedback fuzzy controller is considered to stabilize and achieve the disturbance attenuation for system (5).
\[ u(t) = \sum_{i=1}^{r} h_{ii}(t)(K_{i} + \Delta K_{i})w(t) \]
(48)

with \( K_{i} \in \mathbb{R}^{n \times m}, i \in \mathbb{S} \), being the constant controller gains to be determined, \( \Delta K_{i} \) are perturbed matrices. Two perturbation types are considered in this section.

Type 1: Perturbations depending on matrix \( K_{i} \), where \( \Delta K_{i} \) is with the norm-bounded multiplicative form
\[ \Delta K_{i} = H_{ci}\Delta E_{ci}K_{i} \]
(49)

Type 2: Perturbations depending on matrix \( K_{i} \), where \( \Delta K_{i} \) is with the norm-bounded additive form
\[ \Delta K_{i} = H_{ci}\Delta E_{ci} \]
(50)

Two types of perturbations are considered in the following:
\[ \Delta c = F_{c}(I - J_{c}F_{c})^{-1}, \quad 0 < I - J_{c}^{T}F_{c} \]
(51)

where \( H_{ci}, E_{ci} \) and \( J_{c} \) are known real constant matrices of appropriate dimensions and \( F_{c} \) is an unknown time-varying matrix function satisfying \( F_{c}^{T}F_{c} \leq I \).
The application of control law (48) to system (5) yields the following perturbed closed-loop system:

\[
\begin{align*}
\dot{X}(t) &= \sum_{i=1}^{2} \sum_{j=1}^{2} h_i(t) h_j(t) (A_{ij} X(t) + B_{1ij} w(t)) \\
2(t) &= \sum_{i=1}^{2} \sum_{j=1}^{2} h_i(t) h_j(t) (C_{ij} x(t) + D_{11i} w(t)),
\end{align*}
\]

where

\[A_{ij} = A_i + B_2 K_j + M_i \Delta (N_1 + N_2 K_j), \quad B_{1ij} = B_{1i} + M_j \Delta N_{1i}, \quad C_{ij} = C_i + D_{12j} K_j, \quad K_j = K_j + \Delta K_j.
\]

(53)

Now, for both cases of controller gain uncertainties, we provide our results for the \(H_{\infty}\) non-fragile state feedback control problem. For the first case, the following theorem shows how fuzzy controller (48) with uncertainty (49) can be designed.

**Theorem 4.1.** Consider system (5) with the non-fragile state feedback fuzzy control (48) with multiplicative form (49). Given a scalar \(\gamma > 0\), if \(n \times n\) matrices \(P_i > 0, U, G, \Theta_i = \Theta_i^T, X_i\), and some positive scalars \(\varepsilon_{ij}, \varepsilon_{ij}^2\) exist for a value of design parameter \(\lambda\) such that the following LMIs hold for \(i, j \in \mathbb{S}\):

\[
\begin{align*}
\mathbf{P}_i + \Theta_i &> 0, \quad s \in \mathbb{S}, \\
\begin{bmatrix}
\gamma_i^2 \mathbf{T}_{ii} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} &< 0, \quad i \in \mathbb{S}, \\
\begin{bmatrix}
\gamma_{ij}^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} &< 0, \quad j > i,
\end{align*}
\]

(56)

\[
\begin{align*}
\mathbf{X}_{11} &< 0, \quad \cdots \quad \mathbf{X}_{1r} \\
\vdots & \quad \vdots \\
\mathbf{X}_{r1} &< 0, \quad \cdots \quad \mathbf{X}_{rr}
\end{align*}
\]

(58)

where

\[
\begin{align*}
\gamma_i^2 &> 0, \quad \gamma_{ij}^2 > 0, \quad i, j \in \mathbb{S}, \\
\begin{bmatrix}
\mathbf{P}_{ii} + \mathbf{X}_{ii} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} &< 0, \quad \gamma_{ij}^2 > 0, \quad i, j \in \mathbb{S}, \\
\begin{bmatrix}
\mathbf{P}_{ij} + \mathbf{X}_{ij} & \mathbf{P}_{ji} & \mathbf{X}_{ij} & \mathbf{X}_{ji} \\
\mathbf{P}_{ji}^T & \mathbf{P}_{ii} & \mathbf{X}_{ji} & \mathbf{X}_{ji} \\
\mathbf{X}_{ij} & \mathbf{X}_{ji} & \mathbf{P}_{ii} & \mathbf{X}_{ii} \\
\mathbf{X}_{ji} & \mathbf{X}_{ji} & \mathbf{X}_{ii} & \mathbf{P}_{ii}
\end{bmatrix} &< 0, \quad i, j \in \mathbb{S}
\end{align*}
\]

(57)

Then closed-loop fuzzy system (52) is asymptotically stable with \(H_{\infty}\) norm bound \(\gamma\), and the fuzzy local feedback gains can be taken as \(K_i = Y_i U^{-1}, \quad i \in \mathbb{S}\)

(59)

**Proof.** Under the conditions of Theorem 4.1, a feasible solution satisfies condition sym(U) > 0, which implies that \(U\) is nonsingular.

Using Lemma 2.1, we can easily formulate the following inequalities from (56) and (57), respectively:

\[
\begin{align*}
\sum_{s=1}^{n} \phi_s (P_{i} + \Theta_i) + sym(\gamma_{1i} \mathbf{A}_i) &< 0, \\
\sum_{s=1}^{n} \phi_s (P_{ij} + \Theta_{ij}) + sym(\gamma_{1j} \mathbf{X}_{ij}) &< 0
\end{align*}
\]

(60)

\[
\begin{align*}
\sum_{s=1}^{n} \phi_s (P_{ij} + \Theta_{ij}) + sym(\gamma_{2i} \mathbf{A}_i) &< 0, \\
\sum_{s=1}^{n} \phi_s (P_{ij} + \Theta_{ij}) + sym(\gamma_{2j} \mathbf{X}_{ij}) &< 0
\end{align*}
\]

(61)

with

\[
\begin{align*}
\mathbf{U} = \left( \begin{array}{cccc}
F_c & 0 & 0 & 0 \\
0 & F_c & 0 & 0 \\
0 & 0 & F_c & 0 \\
0 & 0 & 0 & F_c
\end{array} \right) > 0.
\end{align*}
\]

(62)

Since \(U\) is nonsingular, we let \(U = G^{-1}, Y_i = K_i G^{-1}, P_i = G^{-1} P_i G^{-1}, \Theta_i = G^{-1} \Theta_i G^{-1}, X_i = P_i \hat{X}_i, \Pi_i = \Pi_i G, \Pi_c = \Pi_c G^{-1}, G^{-1} I_1, I_2, \ldots, I_n\).

\[
\mathbf{U} = \left( \begin{array}{cccc}
F_c & 0 & 0 & 0 \\
0 & F_c & 0 & 0 \\
0 & 0 & F_c & 0 \\
0 & 0 & 0 & F_c
\end{array} \right)^{-1} > 0.
\]

(63)

Checking a congruence transformation to the previous inequalities by diag(G, G, I_2, I_3, G, G, I_3, I_4) and diag(G, G, I_2, G, I_3, G, G, I_3, I_4, I_5) hold with

\[
\begin{align*}
A_j = A_i + B_2 K_j, \\
B_j = B_{1i}, \\
C_j = C_i + D_{12j} K_j, \\
D_j = D_{11i}, \\
M_A = M_i, \\
N_j = N_i + N_2 K_j, \\
G_1 = G, \\
G_2 = G, \\
B_j = B_{1i}
\end{align*}
\]

(64)

Multiplying (58) from the left and right, respectively, by diag(Pi, Pi, Pi, Pi, Pi) and its transpose, condition (43) holds. According to Lemma 3.2, non-fragile state feedback fuzzy controller (48) is the \(H_{\infty}\) fuzzy control of system (5) with disturbance attenuation level \(\gamma\).

Now, when the perturbation in the form (50) is considered, we obtain the following results.

**Theorem 4.2.** Consider system (5) with non-fragile state feedback fuzzy control (48) and (50). For a given scalar \(\gamma > 0\), if there exist matrices \(P_i > 0, U, G, \Theta_i = \Theta_i^T, X_i\) and \(\varepsilon_{ij}, \varepsilon_{ij}^2\) of appropriate dimensions, and some positive scalars \(\varepsilon_{ij} > 0, \varepsilon_{ij}^2 > 0\) such that (55)–(58) hold with \(\varepsilon_{ij} = [E_{ij} U 0 0 0]\). Then closed-loop fuzzy system (52) is asymptotically stable with \(H_{\infty}\) norm bound \(\gamma\), and the fuzzy local feedback gains are given by (59).

**Remark 4.1.**

1. Obviously, the off-diagonal matrix blocks in (58) allow us to be non-symmetric, then there are \(n^2 \left( n_i = 2m \right) \) elements in each matrix. However, when they are symmetric, each matrix has \(\frac{1}{2} (n_i + 1) n_i \) variables. Therefore, the LMI conditions in Theorems 4.1 and 4.2 introduce rather excessive free weighting matrices which can reduce the conservatism (see Example 1).

2. Note that LMIs (56) and (57) are linear on scalar \(\gamma^2\). This indicates that it can be included as an optimization variable,
which can be exploited to reduce the attenuation level bound. Then, the minimum (in terms of the feasibility of LMIs) attenuation level of \( H_\infty \) performance can be obtained by the MATLAB script \( \text{mincx} \).

For additive perturbations in the controller coefficients, the problem of \( H_\infty \) non-fragile state feedback fuzzy control design is also treated in [28]. In order to compare with our approach, we recall the following lemma, which summarizes this result.

**Lemma 4.1** (Zhang et al. [28]). For a given scalar \( \gamma > 0 \), there exists a state feedback fuzzy controller in the form of (48) such that the resulting closed-loop system is robustly stable with an \( H_\infty \) performance \( \gamma \), if there exist a scalar \( \varepsilon > 0 \), matrices \( X > 0 \), \( Y, Z \), and \( Q_i \) such that the following LMIs hold for \( i, j \in \mathcal{S} \),

\[
\begin{bmatrix}
\text{sym}(AX + B_2Y) & * & * & * & * & * & * & * & * & * \\
B_i^T \gamma - \gamma^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_iX + D_{ij} \gamma^T & D_{ij} \gamma^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
e\mu_j^T & e\mu_j^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
eX + E_{2j} \gamma^T & E_{2j} \gamma^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
N_iX & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} < 0
\]

(62)

In this case, the controller gains are given by

\[ K_i = Q_iX^{-1}, \quad i \in \mathcal{S} \]  

(63)

**5. Non-fragile \( H_\infty \) observer-based fuzzy control**

In the sequel, we focus our attention on the fuzzy observer-based non-fragile \( H_\infty \) control problem for (T-S) continuous fuzzy model (5) and we wish to develop a single-step LMI method to solve this problem. Based on **Lemma 3.2**, the corresponding result is immediate.

To construct the state observer for system (5), we assume that all \( \mu_i(t), i = 1, 2, \ldots, v \), are measurable. Through the parallel distributed compensation, we suggest the following overall observer-based control law:

\[
\begin{aligned}
\dot{x}(t) &= \sum_{i=1}^{v} h_i(\mu_i(t))[A_i\dot{x}(t) + B_2u(t) + (L_i + \Delta L_i)y(t)] - \hat{y}(t)), \\
\dot{y}(t) &= \sum_{i=1}^{v} h_i(\mu_i(t))C_i\dot{x}(t), \\
u(t) &= \sum_{i=1}^{v} h_i(\mu_i(t))(K_i + \Delta K_i)\dot{x}(t), \quad \hat{x}(0) = 0
\end{aligned}
\]

(64)

where \( \dot{x}(t) \) is the state estimation of \( x(t) \), \( \hat{y}(t) \) is the observer output, \( L_i \in \mathbb{R}^{n \times q} \) and \( K_i \in \mathbb{R}^{n \times m}, i \in \mathcal{S} \), are, respectively, the observer and the controller constant gain matrices to be determined, \( \Delta K_i \) and \( \Delta L_i \) are multiplicative gain perturbation matrices represented by

\[
\Delta K_i = H_{0i} \Delta E_i K_i, \quad \Delta L_i = H_{0i} \Delta E_i L_i,
\]

(65)

where \( \Delta \) is defined as in (51). \( H_{ci}, H_{0i}, E_{ci} \) and \( E_{0i} \) are known real constant matrices of appropriate dimensions.

Let us denote the estimation error as \( e(t) = x(t) - \hat{x}(t) \). Combining (5) with (64), the augmented closed-loop fuzzy system is written as

\[
\begin{aligned}
\dot{x}(t) &= \sum_{i=1}^{v} h_i(\mu_i(t))[\bar{A}_{ii}\dot{x}(t) + \bar{B}_{ii}u(t)] + \bar{D}_{ii}e(t), \\
\dot{z}(t) &= \sum_{i=1}^{v} h_i(\mu_i(t))[\bar{C}_i\hat{y}(t) + \bar{D}_{1i}e(t)] + \bar{D}_{2i}e(t)
\end{aligned}
\]

(66)

with

\[
\begin{aligned}
\hat{y}(t) &= [x^T(t) \ y^T(t)], \quad \bar{A}_{ii} = A_{ii} - \bar{A}_{ii}, \\
\bar{B}_{ii} &= \bar{B}_{ii}, \\
\bar{C}_i &= [C_{ii} + D_{ij} K_i - D_{ij} K_i], \\
\bar{D}_{ii} &= \bar{D}_{ii}, \\
\bar{D}_{1i} &= D_{1i}, \\
\bar{D}_{2i} &= D_{2i}
\end{aligned}
\]

Considering (65), the previous matrices can be expressed as

\[
\bar{A}_{ii} = A_{ii} + \hat{A}_{ii} diag(\Delta_c, \Delta_c), \quad \bar{B}_{ii} = B_{ii} + \hat{B}_{ii} diag(\Delta_c, \Delta_c), \\
\bar{C}_i = \tilde{C}_i + \hat{C}_i diag(\Delta_c, \Delta_c), \quad \bar{D}_{ii} = \bar{D}_{ii}
\]

(67)

where

\[
\tilde{C}_i = C_{ii} + D_{ij} K_i - D_{ij} K_i,
\]

(68)

Based on **Lemma 3.1**, we have the following result for non-fragile observer-based \( H_\infty \) fuzzy control design problem.

**Theorem 5.1.** Consider fuzzy system (5) with non-fragile observer-based control law (64) with multiplicative form (65). Given a positive scalar \( \gamma > 0 \), if matrices \( \hat{P}_i = \hat{P}_i, \hat{Q}_i = \hat{Q}_i, \hat{D}_i = \hat{D}_i \), \( \hat{Y}_i \) of appropriate dimensions, and some positive scalars \( \varepsilon_i, \varepsilon_i \) exist for a value of design parameter \( \lambda \) such that the following LMIs hold:

\[
\begin{bmatrix}
\tilde{\Omega}_{ii}^{11} & * & * & * \\
\tilde{\Omega}_{ii}^{21} & -\varepsilon_i \tilde{\Omega}_{ii} & * & * \\
\tilde{\Omega}_{ii}^{21} & -\varepsilon_i \tilde{\Omega}_{ii} & * & * \\
\tilde{\Omega}_{ii}^{21} & -\varepsilon_i \tilde{\Omega}_{ii} & * & *
\end{bmatrix} < 0, \quad i \in \mathcal{S}
\]

(70)

\[
\begin{bmatrix}
\tilde{\Omega}_{ij}^{11} & * & * & * \\
\tilde{\Omega}_{ij}^{21} & -\varepsilon_j \tilde{\Omega}_{ij} & * & * \\
\tilde{\Omega}_{ij}^{21} & -\varepsilon_j \tilde{\Omega}_{ij} & * & * \\
\tilde{\Omega}_{ij}^{21} & -\varepsilon_j \tilde{\Omega}_{ij} & * & *
\end{bmatrix} < 0, \quad j > i
\]

(71)
where
\[
\begin{align*}
\Omega_{n}^{1} &= \begin{bmatrix}
\Psi_{ii} + \frac{\varepsilon_{ii}}{c_{ii}} \ast & 
\end{bmatrix} , \\
\Omega_{m}^{12} &= \begin{bmatrix}
\frac{\varepsilon_{ii} M_{i}}{N_{ij}} \\
\end{bmatrix} ,
\end{align*}
\]

\[
\Omega_{q}^{22} = \begin{bmatrix}
diag(l, l, I) \ast \\
-diag(j, j, I) \ast \\
diag(l, l, I) \\
\end{bmatrix}
\]

Therefore, \( \bar{U}_{1} \) and \( \bar{U}_{2} \) are non-singular matrices.

Defining
\[
\begin{align*}
\bar{U}_{1} &= \begin{bmatrix} U_{1} & 0 \\
0 & I \\
\end{bmatrix} , \\
\bar{U}_{2} &= \begin{bmatrix} 0 & I \\
U_{2}^{-1} & 0 \\
\end{bmatrix}.
\end{align*}
\]

From Lemma 2.1, we have
\[
\begin{align*}
\frac{\Omega_{m}^{11}}{\Omega_{m}^{11} - \varepsilon_{m1}^{2}} &+ \text{sym}(\hat{H}_{1i} \hat{A}_{1} \hat{E}_{1i}) < 0 \\
\frac{\Omega_{m}^{11}}{\Omega_{m}^{11} - \varepsilon_{m1}^{2}} &+ \text{sym}(\hat{H}_{2j} \hat{A}_{22} \hat{E}_{2j}) < 0
\end{align*}
\]
with
\[
\begin{align*}
\hat{A}_{1} &= \text{diag}(\Delta_{1}, \Delta_{2}, \Delta_{3}, \Delta_{4}, \Delta_{5}, \Delta_{6}) , \\
\hat{A}_{22} &= \text{diag}(\Delta_{1}, \Delta_{2}, \Delta_{3}, \Delta_{4}, \Delta_{5}, \Delta_{6})
\end{align*}
\]

Without loss of generality, we set
\[
\begin{align*}
\hat{\xi}_{1} &= \hat{H}_{1}^{T} \hat{F}_{0}, \\
\hat{\xi}_{2} &= \hat{H}_{2}^{T} \hat{F}_{0}, \\
\hat{\xi}_{y} &= \hat{H}_{y}^{T} \hat{F}_{y}
\end{align*}
\]

First, performing a congruence transformation to (75) and (76) by \( \text{diag}(\bar{U}_{1}, \bar{U}_{2}, \bar{U}_{1}, \bar{U}_{2}, \bar{U}_{1}, \bar{U}_{2}) \) and \( \text{diag}(\bar{U}_{1}, \bar{U}_{2}, \bar{U}_{1}, \bar{U}_{2}, \bar{U}_{1}, \bar{U}_{2}) \), respectively, and then by \( \text{diag}(\bar{U}_{1}, \bar{U}_{2}, \bar{U}_{1}, \bar{U}_{2}, \bar{U}_{1}, \bar{U}_{2}) \) and \( \text{diag}(\bar{U}_{1}, \bar{U}_{2}, \bar{U}_{1}, \bar{U}_{2}, \bar{U}_{1}, \bar{U}_{2}) \), respectively. Inequalities (41) and (42) hold with

\[
\begin{align*}
A_{y} &= \bar{A}_{y}, \\
B_{y} &= \bar{B}_{y}, \\
C_{y} &= \bar{C}_{y}, \\
D_{y} &= \bar{D}_{y}, \\
M_{y} &= \bar{M}_{y}, \\
N_{y} &= \bar{N}_{y}, \\
G_{1} &= \bar{G}_{1}, \\
G_{2} &= \bar{G}_{2}, \\
B_{y} &= \bar{B}_{y}
\end{align*}
\]

Furthermore, defining \( \bar{H}_{G} = \text{diag}(\bar{H}_{G}, \bar{H}_{G}, \bar{H}_{G}) \) and checking a congruence transformation by \( \text{diag}(\bar{H}_{G}, \ldots, \bar{H}_{G}) \) to (72) show that
\[
\begin{align*}
\bar{X}_{11} &> 0 \\
\vdots &\vdots \\
\bar{X}_{1r} &> 0
\end{align*}
\]

Then, from Lemma 3.2, closed-loop system (66) is asymptotically stable with attenuation level \( \gamma \).

Then closed-loop fuzzy system (66) is asymptotically stable with attenuation level \( \gamma \).
6. Numerical examples

In order to indicate the efficiency of the proposed approaches, three examples are presented in this section.

Example 1. Consider the (T–S) fuzzy system defined by two rules [13]:

\( R_1 \): If \( x_1 \) is \( F_1 \) then
\[
\begin{align*}
    & x(t) = A_1 x(t) + B_1 u(t) + B_{11} w(t) \\
    & z(t) = C_1 x(t) + D_{12} u(t) + D_{11} w(t), \\
\end{align*}
\]
where
\[
A_1 = \begin{bmatrix} 3.6 & -1.6 \\
                    6.2 & -4.3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -a & -1.6 \\
                    6.2 & -4.3 \end{bmatrix}, \quad B_{21} = \begin{bmatrix} -0.45 \\
                     -3 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} -b \\
                     -3 \end{bmatrix}
\]
\[
B_{11} = \begin{bmatrix} 0 \\
1 \end{bmatrix}, \quad C_{11} = [1 \ 1], \quad D_{12} = 0.5, \quad D_{11} = 0, \ i = 1, 2.
\]

This example, which is considered as a benchmark for the (T–S) fuzzy system control, has been dealt with by many different methods. Assume that controller gains have type 2 additive perturbations

\( (50) \) with

\( H_{c1} = 0.7, \quad H_{c2} = -0.7, \quad E_{c1} = [-0.5 \ -0.5], \quad E_{c2} = [-0.5 \ 0.5]. \)

Under the assumption of the derivatives of membership functions with \( \phi_{12} = 1, \) stabilization conditions with \( H_{c} \) performance \( \gamma = 10 \) via several methods are compared with others by calculating the feasible values of \( b \) with respect to different values of \( a. \)

The area filled with circles “○” corresponds to feasible solutions with Lemma 4.1. However, symbols “•” and “+” indicate, respectively, the regions of feasibility obtained by Theorem 4.2 for a first case where the off-diagonal matrix blocks in (58) are considered to be non-symmetric and for a second case where they are allowed to be symmetric. Fig. 1 shows that with the parameters in these intervals, Theorem 4.2 is able to stabilize the system with a larger region than the one given by Lemma 4.1. Thus, it can be concluded that the stabilization conditions in Theorem 4.2 lead to less conservative results than Reference [28].

Example 2. We consider the following problem of balancing an inverted pendulum on a cart. The dynamic equations of motion of the pendulum are given as follows [31,33]:

\[
\begin{align*}
    & \dot{x}_1 = x_2 \\
    & \dot{x}_2 = \frac{g}{4l/3 - a ml^2} \sin(2x_1) - a \cos(x_1) u + \frac{80}{4l/3 - a ml^2} w \\
    & z(t) = x_1 + x_2 + 0.001 u(t) \\
    & y(t) = x_1 + 0.01 w(t)
\end{align*}
\]

where \( x_1 \) denotes the angle of the pendulum from the vertical axis, and \( x_2 \) is the angular velocity, \( g = 9.8 \text{ m/s}^2 \) is the gravity constant, \( m \) is the mass of the pendulum, \( 2l \) is the length of the pendulum, \( a = 1/(m+M) \), \( M \) is the mass of the cart, and \( u \) is the force applied to the cart. In this simulation, the pendulum parameters are chosen as \( m = 2 \text{ kg}, M = 8 \text{ kg}, \) and \( 2l = 1 \text{ m}. \) Let us consider the following fuzzy model to design a non-fragile observer-based fuzzy controller that achieves \( H_{c} \) performance

\[
\begin{align*}
    & \dot{x}(t) = \sum_{i=1}^{2} h_i(x_1)((A_i + \Delta A_i)x(t) + (B_{2i} + \Delta B_{2i})u(t) + (B_{1i} + \Delta B_{1i})w(t)) \\
    & z(t) = \sum_{i=1}^{2} h_i(x_1)(C_{1i}x(t) + D_{12i}u(t)) \\
    & y(t) = \sum_{i=1}^{2} h_i(x_1)(C_{2i}x(t) + D_{21i}w(t))
\end{align*}
\]

where

\[
A_1 = \begin{bmatrix} 17.2941 & 0 \\
                    0 & 12.6305 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\
                    12.6305 & 0 \end{bmatrix},
\]
\[
B_{21} = \begin{bmatrix} 0 \\
-0.1765 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} 0 \\
-0.0779 \end{bmatrix},
\]
\[
B_{11} = \begin{bmatrix} 0 \\
1 \end{bmatrix}, \quad C_{11} = [1 \ 1], \quad D_{12i} = 0.001, \quad C_{2i} = [1 \ 0],
\]
\[
D_{21i} = 0.01 \quad i = 1, 2.
\]

Fig. 1. Stabilization region based on Lemma 4.1(○) and Theorem 4.2 (• for the first case) and (+ for the second case).

Fig. 2. Closed-loop responses of pendulum system. (a) Response and estimate of the state \( x_1(t) \) and (b) response and estimate of the state \( x_2(t). \)
Moreover, matrices $M_i, N_i, N_{11}$ and $N_{21}, i=1,2,$ are defined as

$$M_1 = M_2 = \begin{bmatrix} 0 & 0 \\ 0.1 & 1 \end{bmatrix}, \quad N_1 = N_2 = [3.5 \ 0], \quad N_{21} = N_{22} = 0.1,$$

$N_{11} = N_{12} = 0, \quad J = 0.1.$

We use the following membership functions:

$$h_1(x_1) = 1 - \frac{1}{1 + \exp(-7(x_1 - \frac{x}{4}))}, \quad h_2(x_1) = 1 - h_1(x_1).$$

The operating domain of the nonlinear plant are assumed to be $x_1(t) \in [-1/2, 1/2]$ and $x(t) = x_2(t) \in [-2, 2]$. Thus, based on the derivative of the membership function, we choose $\phi_{1,2} = 10.$

Assume that there are additive perturbations of type 1 in the controller and observer coefficients. We give the known parameters in (65) as:

$$H_{11} = 2, \quad H_{22} = 2, \quad E_{oi} = 2, \quad E_{o} = 0.015, \quad I_e = 0.5,$$

$$H_{o1} = \begin{bmatrix} -0.15 & 0 \\ 0 & 0.15 \end{bmatrix}, \quad H_{o2} = \begin{bmatrix} 0.15 & 0 \\ 0 & -0.2 \end{bmatrix},$$

$$E_{o1} = \begin{bmatrix} 0.12 & 0 \\ 0 & -0.12 \end{bmatrix}, \quad E_{o2} = \begin{bmatrix} 0.17 & 0 \\ 0 & 0.15 \end{bmatrix}.$$

Then, setting $c_{11} = 25, \ c_{21} = 15, \ c_{22} = 30.5, \ c_{e1} = 5, \ c_{e2} = 3.5$ and $v_{c2} = 7.25$. With $\lambda = 0.1$, Theorem 5.1 produces a set of feasible solutions to corresponding LMI’s with a lower bound of $H_\infty$ performance level $\gamma = 0.6231$ and

$$\hat{P}_1 = \begin{bmatrix} 0.5129 & -1.7691 & 0.6838 & -0.0870 \\ -1.7691 & 7.5104 & -0.0870 & 0.6796 \\ 0.6838 & -0.0870 & 28.1762 & 3.8316 \\ -0.0870 & 0.6796 & 3.8316 & 3.1168 \end{bmatrix},$$

$$\hat{P}_2 = \begin{bmatrix} 0.5046 & -1.7358 & 0.7296 & -0.1338 \\ -1.7358 & 7.3432 & -0.1338 & 0.9578 \\ 0.7296 & -0.1338 & 28.1142 & 3.5178 \\ -0.1338 & 0.9578 & 3.5178 & 2.7350 \end{bmatrix},$$

$$\hat{p}_1 = \begin{bmatrix} -0.5014 & 1.7256 & -0.7202 & 0.1283 \\ 1.7256 & -7.2877 & 0.1283 & -0.9592 \\ -0.7202 & 0.1283 & -27.8886 & -3.7256 \\ 0.1283 & -0.9592 & -3.7256 & -2.5383 \end{bmatrix},$$

$$\hat{p}_2 = \begin{bmatrix} -0.4754 & 1.6374 & -0.7261 & 0.1169 \\ 1.6374 & -7.0111 & 0.1169 & -0.9013 \\ -0.7261 & 0.1169 & -27.7408 & -3.4899 \\ 0.1169 & -0.9013 & -3.4899 & -2.7220 \end{bmatrix},$$

$$\hat{U}_1 = \begin{bmatrix} 0.4858 & -1.2328 \\ -1.6399 & 5.3507 \end{bmatrix}, \quad \hat{U}_2 = \begin{bmatrix} 6.5663 & 1.7029 \\ -1.2318 & 2.1067 \end{bmatrix},$$

$$K_1 = [284.9800 \ 88.4142], \quad K_2 = [488.8091 \ 163.1839], \quad L_1 = \begin{bmatrix} 28.0386 \\ 29.4848 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 24.8235 \\ 28.1001 \end{bmatrix}.$$

For the simulation, we select $F(t) = \sin(6\pi t), \ F_c(t) = 0.8 + 0.2 \sin(10\pi t)$ and the disturbance input as $w(t) = \sin(3t)e^{-0.05t}$.

The simulation results depicted in Figs. 2 and 3 show that:

- For initial condition $x_0 = [-\pi/3, 0]$, Fig. 2(a) and (b) depicts the trajectories of the pendulum system states and their estimation, respectively. It is shown that the estimate of the angle tracks closely its state. However, the estimation of the unmeasurable state (the angular velocity) can track its state after a long time. We also observe that the states exhibit some oscillations, with reduced effects, due to the presence of the input disturbance.
- From Fig. 3(a), the ratio of $\int_0^t z(s)z(s) \ ds / \int_0^t w(s)w(s) \ ds$ is less than 0.3033 under zero-initial condition, which reveals that the $H_\infty$ disturbance attenuation level is less than the required $\gamma = 0.785$. i.e. $\sqrt{0.3033} = 0.5507 < 0.6231$.

The evolution of the membership derivative shown in Fig. 3 (b) allows us to verify that Assumption 18 is verified.

To illustrate the efficiency and application of the proposed procedure, we consider the following two controllers:

- A classical controller designed for an ideal case where non-perturbations that affect the model and the controller. Setting $\lambda = 0.01$ and $\phi_{1,2} = 10$, according to Theorem 5.1 (without uncertainty terms) we can find a feasible solution as:

$$K_1 = [656.44 \ 516.55], \quad K_2 = [1320.3 \ 1067],$$

$$L_1 = \begin{bmatrix} 5.7197 \\ 101.16 \end{bmatrix}, \quad L_2 = \begin{bmatrix} 4.6272 \\ 86.142 \end{bmatrix}.$$

- A non-fragile controller designed for a real case when the uncertainty solicits the controller and observer gains. Choose $H_{11} = 6, \ H_{22} = 6.25, \ E_{o1} = 0.1, \ E_{o2} = 0.085, \ J_e = 0,$

$$H_{o1} = \begin{bmatrix} -0.015 & 0 \\ 0 & 0.015 \end{bmatrix}, \quad H_{o2} = \begin{bmatrix} 0.015 & 0 \\ 0 & -0.02 \end{bmatrix},$$

$$E_{o1} = \begin{bmatrix} 0.012 \\ 0 \end{bmatrix}, \quad E_{o2} = \begin{bmatrix} 0.017 \\ 0 \end{bmatrix}.$$

With $c_{ij} = 15 (i,j = 1, 2)$ and $\lambda = 0.045$ to solve the LMIs in Theorem 5.1, we can obtain the following controller and observer gain matrices:

$$K_1 = [969.37 \ 272.76], \quad K_2 = [685.76 \ 197.41].$$

**Fig. 3.** Ratio and membership derivative evolutions. (a) Satisfaction of $H_\infty$ performance and (b) evolution of $|h_1(t)|$. 
Simulations were carried out for initial conditions $\frac{\pi}{4} = 4.767$ and the uncertainty function $F_c = \alpha e^{-0.5t}$. For $\alpha = 1, 1.4$ and 1.45, Figs. 4 and 5 show the state trajectories of the system.

From the plotted graphs, we record the following observations. First, with non-fragile controller (83) the state trajectories still stable for predefined values of $\alpha$. Second, injecting the fragile controller (82) yields divergent responses.

**Example 3.** To evaluate the effectiveness of the proposed observer-based controller design approach, we present an application to improve the cornering stability of the vehicle lateral dynamics described by the following two-degree-of-freedom bicycle model [34,35]:

$$\begin{align*}
\dot{\beta} &= \frac{2F_f + 2F_r}{mU} - \frac{r}{I_z} \\
\dot{r} &= \frac{2a_1F_f - 2a_2F_r + M_z}{I_z}
\end{align*}$$

where $\beta$ denotes the slide slip angle, $r$ is the yaw velocity, $F_f$ is the nonlinear cornering force of the two front tires, $F_r$ is the nonlinear cornering force of the two rear tires and $M_z$ is yaw moment. $U$ is the vehicle velocity, $I_z$ is the yaw moment of inertia, $m$ is the vehicle mass.

As in [36,37], nonlinear cornering forces $F_f$ and $F_r$ have been approximated by (T–S) fuzzy model as follows:

$$\begin{align*}
F_f &= h_1(\alpha_f)C_1 \alpha_f + h_2(\alpha_f)C_2 \alpha_f \\
F_r &= h_1(\alpha_r)C_1 \alpha_r + h_2(\alpha_r)C_2 \alpha_r
\end{align*}$$

$\dot{x} = \begin{bmatrix} 0 & 1 \\ -27.132 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} w$

$A_1 = \begin{bmatrix} 0 & 1 \\ -27.132 & 0 \end{bmatrix}$

$A_2 = \begin{bmatrix} 0 & 1 \\ -27.132 & 0 \end{bmatrix}$

$L_1 = \begin{bmatrix} 26.767 \\ 33.096 \end{bmatrix}$, $L_2 = \begin{bmatrix} 23.939 \\ 25.108 \end{bmatrix}$. (83)
where \( \alpha_1, \alpha_2 \) are the front and rear tire side slip, respectively, and \( h_i (i=1,2) \) are bell curve membership functions satisfying constraints (86) and have the following expressions: the membership functions are defined as

\[
h_1(\alpha_i) = \frac{\beta_1(\alpha_i)}{\beta_1(\alpha_i)+\beta_2(\alpha_i)}, \quad h_2(\alpha_i) = \frac{\beta_2(\alpha_i)}{\beta_1(\alpha_i)+\beta_2(\alpha_i)}
\]

(86)

\[
\beta_1(\alpha_i) = \frac{1}{1 + \left[\frac{|\alpha_i - C_1}{a_1}\right]^{2n_i}}; \quad \beta_2(\alpha_i) = \frac{1}{1 + \left[\frac{|\alpha_i - C_2}{a_2}\right]^{2n_i}}
\]

(87)

where \( a_i, b_i \) and \( c_i (i=1,2) \) are given parameters.

Using (85) and considering that

\[
\begin{align*}
\delta_r &\equiv -\beta - \frac{\alpha r}{U}\delta_f;
\delta_r &\equiv -\beta - \frac{\alpha r}{U}
\end{align*}
\]

(88)

nonlinear model of vehicle lateral dynamic (80) can be represented by

\[
\begin{align*}
\dot{x} &= \sum_{i=1}^{2} h_i(\alpha_i)(A_1 x + B_2 u + B_3 w) \\
y &= \sum_{i=1}^{2} h_i(\alpha_i)(C_{1x} x + D_{2i} w)
\end{align*}
\]

(89)

where

\[
A_i = \begin{bmatrix}
\frac{-2\phi_{x1}+C_{1x} \mu}{m_U} & -\frac{2\phi_{x1}+C_{1x} \mu}{m_U} -1 \\
\frac{-2\phi_{x1}+C_{1x} \mu}{m_U} & -\frac{2\phi_{x1}+C_{1x} \mu}{m_U} -1
\end{bmatrix}, \quad B_{2i} = \begin{bmatrix}
\frac{C_{1x}}{m_U} \\
\frac{C_{1x}}{m_U}
\end{bmatrix}, \quad i = 1, 2,
\]

\[
C_{2i} = \begin{bmatrix}
-\frac{-C_{1x}+C_{1a}}{m_U} & 0 \\
-\frac{-C_{1x}+C_{1a}}{m_U} & 1
\end{bmatrix}, \quad D_{2i} = \begin{bmatrix}
\frac{C_{2a}}{m_U} \\
0
\end{bmatrix}
\]

(90)

\[\delta_f, \delta_r, \alpha_1 \text{ and } \alpha_2 \text{ are, respectively, the front steer angle, the rear steer angle, the slip angle of the front tires, and the slip angle of the rear tires. The output vector of system } y \text{ consists of measurements of lateral acceleration } \phi_x \text{ and the yaw rate about the center of gravity } r.\]

The control design purpose of this example is to design an observer-based controller in order to improve the vehicle stability and maneuverability when this latter is subject to lane changing maneuver. To validate the effectiveness of the proposed vehicle model described by (84)–(85).

The used parameters for the vehicle model are \( m = 1740 \text{ kg}, \quad l_z = 3000 \text{ kg m}^2, \quad a_1 = 1.3, \quad a_2 = 1.2 \text{ and } U = 20 \text{ m/s}. \) For \( \mu = 0.7 \) the parameters of membership functions are \( a_1 = 0.0908, \quad a_2 = 23.3421, \quad b_1 = 0.7237, \quad b_2 = 204.0533, \quad c_1 = 0.0415 \text{ and } c_2 = 23.4094.\)

Assume that \( C_{11} = 163 \text{ 540 N rad}^{-1}, \quad C_{12} = 3170 \text{ N rad}^{-1}, \quad C_{1r} = 142 \text{ 260 N rad}^{-1} \text{ and } C_{2r} = 1980 \text{ N rad}^{-1}; \]

\[
H_{c1} = 20, \quad H_{c2} = 15, \quad E_{c1} = 0.15, \quad E_{c2} = 0.02, \quad J_c = 0.1.
\]

\[
E_{c1} = \begin{bmatrix}
-0.1 & 0 \\
0 & -0.1
\end{bmatrix}, \quad H_{c2} = \begin{bmatrix}
0.1 & 0 \\
0 & 0.1
\end{bmatrix}, \quad \mathbf{F}_{c1} = \begin{bmatrix}
0.01 & 0 \\
0 & -0.01
\end{bmatrix}, \quad \mathbf{F}_{c2} = \begin{bmatrix}
-0.01 & 0 \\
0 & -0.01
\end{bmatrix}
\]

Then, using the parameters \( \theta_{ij} = 35 (i,j=1,2), \quad \lambda = 0.002 \text{ and } \phi_{1.2} = 5 \) to solve the LMIs in Theorem 5.1, we obtain the following controller and observer gain matrices:

\[
K_1 = \begin{bmatrix}
70.253 & -83.723
\end{bmatrix}, \quad K_2 = \begin{bmatrix}
3694.9 & -4483.9
\end{bmatrix}
\]

The control design purpose of this example is to design an observer-based non-fragile controller in order to improve the vehicle stability and maneuverability when this latter is subject to lane changing maneuver. The control system can tolerate the presence of controller and observer gain variations. To validate the effectiveness of the designed observer-based controller, the simulation has been applied to the nonlinear vehicle model described by (84).

Fig. 6 shows vehicle state variables and their estimated signals. We can see that the vehicle maintains its stability despite the variation of the front steering angle \( \delta_f \). Fig. 7 shows the required yaw moment.

\[
L_1 = \begin{bmatrix}
-8.9639 & 20.667 \\
7.5819 & 715.2
\end{bmatrix}, \quad L_2 = \begin{bmatrix}
-836.23 & -52.025 \\
100.9 & 631.77
\end{bmatrix}
\]

(91)

7. Conclusion

As a study in the fuzzy theoretical analysis, LMI-based design procedures have been developed for state and observer-based \( H_{\infty} \) non-fragile control of continuous-time (T–S) fuzzy system. Based on a fuzzy Lyapunov function approach, new relaxed conditions of non-quadratic stabilizability have been provided and the merits and less conservatism of our method have been illustrated in comparison with existing works. For unmeasurable states, a fuzzy observer is constructed to generate the state estimates and by one-step procedure, a \( H_{\infty} \) non-fragile controller has been designed for a certain (T–S) fuzzy systems. The simulation results have shown that the addressed control schemes yield good system performance while maintaining the closed-loop stability.

It is worth mentioning that, due to the capacity limitation of signal converters, the problem of fragility can be found in control systems via networks. The non-fragile output feedback control scheme addressed in this paper can be extended to the networked control systems. This deserves further study for our future works.

References


