Mathematical and numerical analysis for reaction-diffusion systems modeling the spread of early tumors

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Abstract
We prove existence results for a reaction-diffusion system modeling the spread of early tumors. The existence result is proved by the Faedo-Galerkin method, a priori estimates and the compactness method. Moreover, we construct a finite volume scheme to our model, we establish existence of discrete solutions to this scheme, and show that it converges to a weak solution. Finally, some numerical simulations are reported.

Keywords : reaction-diffusion system, weak solution, existence, finite volume scheme.

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1 Introduction
During the past several years mathematical models have been applied to various aspects of cancer dynamics, in particular avascular and vascular tumor growth, invasion, angiogenesis and metastasis. A better understanding of the dynamics of tumor formation can be expected from a mathematical modelling. Mathematical models could improve the clinical outcome by predicting the results of specific forms of treatment administered at specific time points. During the last decade theoreticians have developed a great variety of tumour models covering various morphological and functional aspects of tumour growth. These advances have been recently reviewed \cite{23, 25, 26} with a focus on the classification of mathematical tools and computational algorithms.

At the early growth stage the tumour is relatively harmless and is still avascular, it lacks its own network of blood vessels for supplying nutrients, including oxygen, and for removing wastes. The critical event that converts a self-contained pocket of aberrant cells into a rapidly growing malignancy comes when the tumour becomes vascularized. That means that it has its own blood supply and microcirculation \cite{18, 19}.

The literature contains several approaches to mathematical description of the growth and spread of cancer. One of them is the tumors cords \cite{6, 7, 8, 30} which are similar to the spheroids, although these latter have spherical symmetry \cite{24}. Another approach \cite{31} assumes that tumor invasion occurs inside a tubular structure.

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To state our model, we consider a thin sheet of cells in a physical domain $\Omega \subset \mathbb{R}^3$ over a time span $(0, T)$, $T > 0$. Let $u = u(x, t)$ and $v = v(x, t)$ represent the densities of cells and growth factor molecules respectively, at time $t \in (0, T)$ and location $x \in \Omega$. Besides, it is assumed that cell proliferation is influenced by the growth factor, which is produced by the medium and bound by cells. A prototype of a nonlinear system that governs the spreading of early tumors in a spatial domain is the following reaction-diffusion system:

$$
\begin{align*}
\frac{\partial u}{\partial t} - d_u \Delta u &= F(u, v), \quad \text{in } Q_T, \\
\frac{\partial v}{\partial t} - d_v \Delta v &= G(u, v), \quad \text{in } Q_T, \\
\nabla u \cdot \vec{n} &= 0, \quad \nabla v \cdot \vec{n} = 0, \quad \text{on } \partial \Omega, \\
(\cdot, 0) &= u_0(\cdot) \quad \text{and} \quad v(\cdot, 0) = v_0(\cdot), \quad \text{in } \Omega.
\end{align*}
$$

(1.1)

where $Q_T := \Omega \times (0, T)$, $T > 0$ is a fixed time, and $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial \Omega$ and outer unit normal $\vec{n}$. Herein, $d_u > 0$ and $d_v > 0$ are diffusion constants given. The nonlinearities $F$ and $G$ take the form:

$$
F(u, v) = \left( \frac{a_1(uv)^b}{1 + (uv)^b} - \gamma \right) u,
$$

$$
G(u, v) = h_0 \beta_1 \frac{u^e}{1 + r_1 u^s} - v \left( \alpha + h_0 \alpha \frac{u^e}{1 + r_1 u^s} + \frac{a_1(uv)^b}{1 + (uv)^b} - \gamma \right),
$$

(1.2)

In the model above (as the principal reference we use [22]), the cell cycle is treated as a well-mixed tank, from which cells may either enter division at rate $a(x, t)$ or die at rate $\gamma$. The flux to mitosis is equal to $a(x, t)u(x, t)$, while the flux to death is equal to $\gamma u(x, t)$. Cells divide with efficiency $p \in [0, 1]$, the flux of just divided cells from mitosis back to the cell cycle is equal to $2pa(x, t)u(x, t)$. As a result, the rate of change of $u$ is equal to:

$$
\frac{\partial u}{\partial t} = 2pa - au - \gamma u.
$$

The rate $a$ of cells entering division is a increasing function and take the following form,

$$
a = \frac{a_0 k(uv)^b}{1 + k(uv)^b}.
$$

Combining the last two equations we obtain the kinetics term

$$
F(u, v) = \left[ (2p - 1) \frac{a_0 k(uv)^b}{1 + k(uv)^b} - \gamma \right] u.
$$

(1.3)

Now, if we do not take into account the diffusion of bound growth factor particles over the surface of the sheet of cells, then the kinetics of growth factor is implied by the following assumptions: free growth factor is supplied by a hormonal mechanism extraneous with respect to the sheet of tumor cells. The molecules of the growth factor bind to receptors on the cell’s membrane. Total amount of bound and unbound growth factor is conserved, i.e. loss is balanced by supply. The binding process is derived from a probabilistic model. Let $H(x, t)$ and $L(x, t)$ a number of free growth factor particles and a number of bound
growth factor particles, respectively.
We have the following
\[
\frac{\partial H}{\partial t} = \beta - \delta H - \nu CH,
\]
\[
\frac{\partial L}{\partial t} = \nu CF - \alpha L.
\]
where \(\nu C\) binding of free growth factor particles, \(\beta\) constant influx of free growth factor particles, \(\delta\) loss of free growth factor particles and \(\alpha\) loss of bound growth factor particles.
We assume that the total number of growth factor particles is conserved and so we obtain,
\[
\frac{\partial H}{\partial t} + \frac{\partial L}{\partial t} = \beta - \delta H - \alpha L = 0.
\]
Since we are interested in the number of particles of bound growth factor per cell, we define \(v(x,t) = L(x,t)/u(x,t)\) and obtain:
\[
\frac{\partial v}{\partial t} = \frac{\nu \beta C(u)}{\delta u} - v(x,t) \left[ \alpha + \frac{\nu \alpha}{\delta} C(u) + \frac{\partial u}{\partial t} \frac{1}{u} \right].
\]
We assume also that
\[
C(u) = \frac{ue}{1+ru^s}.
\]
Using the rate of change of \(u\) obtained before, we can rewrite the last equation and get the kinetics model for the growth factor,
\[
G(u,v) = \frac{\nu \beta}{\delta} \frac{ue^{e-1}}{1+ru^s} - v \left[ \alpha + \frac{\nu \alpha}{\delta} \frac{ue}{1+ru^s} + \frac{(2p-1)\nu k(ue)^b}{1+k(ue)^b} - \gamma \right]. \quad (1.4)
\]
To reduce the number of parameters in (1.3) and (1.4) we perform the following rescaling:
\[
a_1 = (2p-1)a_0 \quad u = uk^{1/b} \quad r_1 = rk^{-s/b} \quad h_0 = \frac{\nu}{\delta} k^{-e/b} \quad \beta_1 = \beta k^{1/b},
\]
thus we obtain (1.2).
In this work we assume that the coefficients \(a_1, h_0, \beta_1, \alpha, \gamma, s\) and \(e\) satisfy
\[
a_1, b, \gamma, h_0, \beta_1, r_1, \alpha > 0 \quad \text{and} \quad 1 \leq e \leq s. \quad (1.5)
\]
This assumption will be used to prove the existence of weak solutions.
In [22], the authors, Marciniak-Czochra and Kimmel, said that the model (the system (1.1) with \(d_u = 0\)) can be thought of as representing an early stage of tumor evolution and destabilization of the equilibrium in such system represents an initial invasion of cancer. The authors were looking for a transition from a slightly perturbed equilibrium state to uncontrolled and irregular growth. The idea of their paper was to understand how cellular dynamics of tumor cells can generate pattern formation which may be phenomenologically observed at the macroscopic scale. More sophisticated models of cellular dynamics were developed in [3, 5] to obtain the model they applied the so-called generalized kinetic theory.
which provides a statistical description of large cell populations governed by kinetic type interactions. In [4], the authors presented the survey of different models and methods dealing with multiscale modeling of tumor evolution. We mention also that in [21], the authors have shown the derivation of the macroscopic reaction-diffusion models describing the interplay between the cellular dynamics and the signaling molecules diffusing in the intercellular space using homogenization methods of functional analysis.

Before stating our main results, we give the definition of a weak solution.

**Definition 1.1** A weak solution of (1.1) is a pair \((u,v)\) of nonnegative functions satisfying the following conditions,

\[
\begin{align*}
 u, v &\in L^2(0,T; H^1(\Omega)) \cap C([0,T]; L^2(\Omega)), \\
 \partial_t u, \partial_t v &\in L^2(0,T,(H^1(\Omega))'), \\
 u(0) &= u_0, \quad v(0) = v_0 \text{ a.e. in } \Omega, \\
 F(u,v), G(u,v) &\in L^2(Q_T), \\
 \end{align*}
\]

and

\[
\begin{align*}
 \int_0^T \langle \partial_t u, \varphi_1 \rangle \, dt + d_u \int_0^T \nabla u \cdot \nabla \varphi_1 \, dx \, dt = \int_0^T \int_{Q_T} F(u,v) \varphi_1 \, dx \, dt, \\
 \int_0^T \langle \partial_t v, \varphi_2 \rangle \, dt + d_v \int_0^T \nabla v \cdot \nabla \varphi_2 \, dx \, dt = \int_0^T \int_{Q_T} G(u,v) \varphi_2 \, dx \, dt,
\end{align*}
\]

for all \(\varphi_1, \varphi_2 \in L^2(0,T; H^1(\Omega))\). Here, \(\langle \cdot, \cdot \rangle\) denotes the duality pairing between \(H^1(\Omega)\) and \((H^1(\Omega))'\).

Our first main result is the following existence theorem for weak solutions.

**Theorem 1.1** Assume conditions (1.5) holds. If \(u_0,v_0 \in L^2(\Omega)\), then the system (1.1) possesses at least one weak solution.

Let us define,

\[
f(u) = \frac{u^{e-1}}{1 + r_1 u^s} \quad \text{and} \quad g(u) = \frac{u^{e}}{1 + r_1 u^s}. \tag{1.8}
\]

For technical reasons, we need also to extend the functions \(F\) and \(G\) so that they become defined for all \((u,v) \in \mathbb{R}^2\). We do this by setting:

\[
F(u,v) = \begin{cases} 
0, & \text{if } u < 0, \, v \geq 0, \\
-\gamma u, & \text{if } u \geq 0, \, v < 0, \\
0, & \text{if } u < 0, \, v < 0.
\end{cases}
\]

\[
G(u,v) = \begin{cases} 
-v(\alpha - \gamma), & \text{if } u < 0, \, v \geq 0, \\
h_0 \beta_1 f(u), & \text{if } u \geq 0, \, v < 0, \\
0, & \text{if } u < 0, \, v < 0.
\end{cases} \tag{1.9}
\]

Observe that for all \(u,v \in \mathbb{R}\)

\[
\begin{align*}
 |F(u,v)| &\leq (a_1 + \gamma) |u|, \\
 |G(u,v)| &\leq K_1 + K_2 |v|,
\end{align*}
\]

where \(K_1\) and \(K_2\) are constants depend on \(e\) and \(s\) (recall that \(1 \leq e \leq s\)).

We prove existence of solutions to the system (1.1) by applying the Faedo-Galerkin method, deriving apriori estimates, and then passing to the limit in the approximate
solutions using compactness arguments.

Now we discretize our problem (1.1). This description follows the framework of [16]. We let $\Omega$ be an open bounded polygonal connected subset of $\mathbb{R}^3$ with boundary $\partial \Omega$. Let $T$ be an admissible mesh of the domain $\Omega$ consisting of open and convex polygons called control volumes with maximum size (diameter) $h$. For all $K \in T$, let us denote by $x_K$ the center of $K$, $N(K)$ the set of the neighbors of $K$ i.e. the set of cells of $T$ which have a common interface with $K$, by $N_{\text{int}}(K)$ the set of the neighbors of $K$ located in the interior of $T$, by $N_{\text{ext}}(K)$ the set of edges of $K$ on the boundary $\partial \Omega$. Furthermore, for all $L \in N(K)$ denote by $d(K,L)$ the distance between $x_K$ and $x_L$, by $\sigma_{K,L}$ the interface between $K$ and $L$, by $\eta_{K,L}$ the unit normal vector to $\sigma_{K,L}$ outward to $K$. For all $K \in T$, we denote by $m(K)$ the measure of $K$. We also need some regularity on the mesh:

$$\min_{K \in T, L \in N(K)} \frac{d(K,L)}{\text{diam}(K)} \geq \alpha$$

for some $\alpha \in (0, \infty)$.

The admissibility of $T$ implies that $\overline{\Omega} = \cup_{K \in T} \overline{K}$, $K \cap L = \emptyset$ if $K, L \in T$ and $K \neq L$, and there exist a finite sequence of points $(x_K)_{K \in T}$ and the straight line $\overline{x_K x_L}$ is orthogonal to the edge $\sigma_{K,L}$.

Now, we let $K \in T$ and $L \in N(K)$ with common vertexes $(a_{\ell,K,L})_{1 \leq \ell \leq I}$ with $I \in \mathbb{N}^*$. Next, let $T_{K,L}$ (respectively $T_{K,\sigma}$ for $\sigma \in N_{\text{ext}}(K)$) be the open and convex polygon with vertexes $(x_K, x_L)$ ($x_K$ respectively) and $(a_{\ell,K,L})_{1 \leq \ell \leq I}$. Observe that $\Omega = \cup_{K \in T} \left( \left( \cup_{L \in N(K)} T_{K,L} \right) \cup \left( \cup_{\sigma \in N_{\text{ext}}(K)} T_{K,\sigma}^{\text{ext}} \right) \right)$.

The approximation $\nabla_h u_h$ of $\nabla u$ is defined by

$$\nabla_h u_h(x) = \begin{cases} 
\frac{m(\sigma_{K,L})}{d(K,L)}(u_L - u_K)\eta_{K,L} & \text{if } x \in T_{K,L}, \\
0 & \text{if } x \in T_{K,\sigma}^{\text{ext}},
\end{cases}$$

for all $K \in T$.

We denote by $\mathcal{D}$ an admissible discretization of $Q_T$, which consists of an admissible mesh of $\Omega$, a time step $\Delta t > 0$, and a positive number $N$ chosen as the smallest integer such that $N \Delta t \geq T$. We set $t^n = n \Delta t$ for $n \in \{0, \ldots, N\}$.

With the notation introduced above, finite volume discretization of our model takes the following form: Find vectors $(u^K_n)_{K \in T}$ and $(v^K_n)_{K \in T}$ for $n \in \{0, \ldots, N\}$, such that for all $K \in T$ and $n \in \{0, \ldots, N - 1\}$

$$u^K_0 = \frac{1}{m(K)} \int_K u_0(x) \, dx, \quad v^K_0 = \frac{1}{m(K)} \int_K v_0(x) \, dx, \quad (1.11)$$

$$m(K) \frac{u^K_{n+1} - u^K_n}{\Delta t} - d_u \sum_{L \in N(K)} \frac{m(\sigma_{K,L})}{d(K,L)} (u^L_{n+1} - u^K_n) - m(K)F(u^K_{n+1}, v^K_{n+1}) = 0, \quad (1.12)$$

$$m(K) \frac{v^K_{n+1} - v^K_n}{\Delta t} - d_v \sum_{L \in N(K)} \frac{m(\sigma_{K,L})}{d(K,L)} (v^L_{n+1} - v^K_n) - m(K)G(u^K_{n+1}, v^K_{n+1}) = 0. \quad (1.13)$$
To simplify the notation we write “$u_h$” and “$v_h$” instead of “$u_{h,t}$” and “$v_{h,t}$” respectively, “$h \to 0$” instead of “$h, \Delta t \to 0$”, and so forth.

Now we introduce the “piecewise constant” functions

$$u_h(t,x) = u_{n+1}^K, \quad v_h(t,x) = v_{n+1}^K,$$

for all $(t,x) \in (n\Delta t, (n+1)\Delta t) \times K$, with $K \in T$ and $n \in \{0, \ldots, N-1\}$. To simplify the notation, let us write $u_h$ for the vector $(u_h, v_h)$. In this paper we assume that the following mild time step condition is satisfied (recall that $K_1$ is defined in (1.10)):

$$\Delta t < \min \left( \frac{(\theta - 1)}{\theta a_1}, \frac{(\theta - 1)}{\theta K_1} \right) \quad \text{for some } \theta > 1.$$  

(1.15)

Our second main result is the following theorem.

**Theorem 1.2**  Assume that (1.15) holds and $u_0, v_0 \in L^2(\Omega)$. Then the finite volume solution $u_h$, generated by (1.11), (1.12) and (1.13), converges along a subsequence to $u = (u,v)$ as $h \to 0$, where $u$ is a weak solution of (1.1) in the following sense:

$$u_h \to u \text{ and } v_h \to v \text{ strongly in } L^2(Q_T) \text{ and a.e. in } Q_T,$$

$$\nabla_h u_h \to \nabla u \text{ and } \nabla_h v_h \to \nabla v \text{ weakly in } (L^2(Q_T))^3,$$

where $(u,v)$ satisfies

$$- \int_{\Omega} u_0(x) \varphi(0,x) \, dx - \int \int_{Q_T} u \partial_t \varphi \, dx \, dt + d_u \int \int_{Q_T} \nabla u \cdot \nabla \varphi \, dx \, dt = \int \int_{Q_T} F(u,v) \varphi \, dx \, dt,$$

(1.16)

$$- \int_{\Omega} v_0(x) \psi(0,x) \, dx - \int \int_{Q_T} v \partial_t \psi \, dx \, dt + d_v \int \int_{Q_T} \nabla v \cdot \nabla \psi \, dx \, dt = \int \int_{Q_T} G(u,v) \psi \, dx \, dt,$$

(1.17)

for all $\varphi, \psi \in D([0,T] \times \overline{\Omega})$.

The plan of this paper is as follows: In section 2 we prove existence of solutions for the system (1.1). In Section 3 we prove that discrete solutions converge, as the discretization parameter tends to zero, to weak solutions. In section 4 we give some numerical examples.

## 2 Existence of weak solutions

This section is devoted to prove existence of solution to the problem (1.1). For the proof we use Faedo-Galerkin method, a priori estimates, and the compactness method.

Consider the following spectral problem: Find $w \in H^1(\Omega)$ and a number $\lambda$ such that

$$\begin{cases}
(\nabla w, \nabla \varphi)_{L^2(\Omega)} = \lambda (w, \varphi)_{L^2(\Omega)}, & \forall \varphi \in H^1(\Omega), \\
\nabla w \cdot \eta = 0, & \text{on } \partial \Omega,
\end{cases}$$

(2.18)
where \((\cdot, \cdot)_{L^2(\Omega)}\) denotes the inner product of \(L^2(\Omega)\). The problem (2.18) possesses a sequence of eigenvalues \(\{\lambda_i\}_{i=1}^\infty\) and the corresponding eigenfunctions form a sequence \(\{e_i\}_{i=1}^\infty\) that is orthogonal in \(H^1(\Omega)\) and orthonormal in \(L^2(\Omega)\). Furthermore, we assume without loss of generality that \(\lambda_1 = 0\).

We look for finite dimensional approximate solution to the problem (1.1) as sequences \((u_n)_{n>1}, (v_n)_{n>1}\) defined for \(t \geq 0\) and \(x \in \Omega\) by

\[
\begin{align*}
  u_n(t, x) &= \sum_{l=1}^n b_{n,l}(t)e_l(x), \\
  v_n(t, x) &= \sum_{l=1}^n c_{n,l}(t)e_l(x).
\end{align*}
\]

(2.19)

The next step is to determine the coefficients \((b_{n,l}(t))_{l=1}^n, (c_{n,l}(t))_{l=1}^n\) such that for \(k = 1, \ldots, n\)

\[
\begin{align*}
  (\partial_t u_n, e_k)_{L^2(\Omega)} + d_u \int_\Omega \nabla u_n \cdot \nabla e_k \, dx \, dt &= \int_\Omega F(u_n, v_n)e_k \, dx, \\
  (\partial_t v_n, e_k)_{L^2(\Omega)} + d_v \int_\Omega \nabla v_n \cdot \nabla e_k \, dx \, dt &= \int_\Omega G(u_n, v_n)e_k \, dx,
\end{align*}
\]

(2.20)

and, with reference to the initial conditions,

\[
\begin{align*}
  u_n(0, x) &= u_{0,n}(x) := \sum_{l=1}^n b_{n,l}(0)e_l(x), \\
  v_n(0, x) &= v_{0,n}(x) := \sum_{l=1}^n c_{n,l}(0)e_l(x).
\end{align*}
\]

(2.21)

By our choice of basis, \(u_n\) and \(v_n\) satisfy the boundary condition in (1.1). Observe that since \(u_0, v_0 \in L^2(\Omega)\), it is clear that, as \(n \to \infty\), \(u_{0,n} \to u_0\) and \(v_{0,n} \to v_0\) in \(L^2(\Omega)\), respectively. Using the normality of the basis, we can write (2.20) as a system of ordinary differential equations:

\[
\begin{align*}
  b_{n,k}'(t) + d_u \int_\Omega \nabla u_n \cdot \nabla e_k \, dx &= \int_\Omega F(u_n, v_n)e_k \, dx, \\
  c_{n,k}'(t) + d_v \int_\Omega \nabla v_n \cdot \nabla e_k \, dx &= \int_\Omega G(u_n, v_n)e_k \, dx.
\end{align*}
\]

(2.22)

Let \(F\) and \(G\) be functions defined as follow:

\[
\begin{align*}
  F(t, (b_{n,l}(t))_{l=1}^n, (c_{n,l}(t))_{l=1}^n) &= \int_\Omega F(u_n, v_n)e_k \, dx - d_u \int_\Omega \nabla u_n \cdot \nabla e_k \, dx, \\
  G(t, (b_{n,l}(t))_{l=1}^n, (c_{n,l}(t))_{l=1}^n) &= \int_\Omega G(u_n, v_n)e_k \, dx - d_v \int_\Omega \nabla v_n \cdot \nabla e_k \, dx.
\end{align*}
\]

(2.23)

Proceeding exactly as in [1], we prove that \(F\) and \(G\) are Caratheodory functions and the existence interval \([0, t')\) for the Faedo-Galerkin solutions \(u_n\) and \(v_n\) defined by (2.19).

To prove global existence of the solutions we derive \(n\)-independent a priori estimates bounding \(u_n, v_n\) in various Banach spaces. Given some continuous coefficients \(d_{1,n,l}(t)\) and \(d_{2,n,l}(t)\), we form the functions \(\varphi_{1,n}(t, x) := \sum_{l=1}^n d_{1,n,l}(t)e_l(x)\) and \(\varphi_{2,n}(t, x) := \sum_{l=1}^n d_{2,n,l}(t)e_l(x)\). Now our Faedo-Galerkin solutions satisfy the following weak formulations:

\[
\begin{align*}
  \int_\Omega \partial_s u_n \varphi_{1,n} \, dx + d_u \int_\Omega \nabla u_n \cdot \nabla \varphi_{1,n} \, dx &= \int_\Omega F(u_n, v_n)\varphi_{1,n} \, dx, \\
  \int_\Omega \partial_s v_n \varphi_{2,n} \, dx + d_v \int_\Omega \nabla v_n \cdot \nabla \varphi_{2,n} \, dx &= \int_\Omega G(u_n, v_n)\varphi_{2,n} \, dx.
\end{align*}
\]

(2.24)

(2.25)
Lemma 2.1 There exist constants $c_1, c_2 > 0$ not depending on $n$ such that for $t \in [0, t')$
\[
\|u_n\|_{L^2(0,t; H^1(\Omega))} + \|v_n\|_{L^2(0,t; H^1(\Omega))} \leq c_1, \tag{2.26}
\]
\[
\|\partial_s u_n\|_{L^2(0,t; (H^1(\Omega))')} + \|\partial_s v_n\|_{L^2(0,t; (H^1(\Omega))')} \leq c_2. \tag{2.27}
\]

Proof. Substituting $\varphi_{1,n} = u_n$ in (2.24), we get from (1.10)
\[
\frac{1}{2} \frac{d}{ds} \int_\Omega |u_n|^2 \, dx + d_u \int_\Omega |\nabla u_n|^2 \, dx \leq \left( a_1 + \gamma \right) \int_\Omega |u_n|^2 \, dx. \tag{2.28}
\]
Using the nonnegativity of the second term of the left-hand side (2.28) and the Gronwall’s inequality, we obtain
\[
\int_\Omega |u_n(x,s)|^2 \, dx \leq c_3 \text{ for all } s \in (0, t], \tag{2.29}
\]
for some constant $c_3 > 0$. Integrating (2.28) over $(0, t)$ and using (2.29), we obtain:
\[
\int_\Omega |u_n(x,t)|^2 \, dx + d_u \int_0^t \int_\Omega |\nabla u_n|^2 \, dx \leq c_4, \tag{2.30}
\]
for some constant $c_4 > 0$, which proves
\[
\|u_n\|_{L^2(0,t; H^1(\Omega))} \leq c_5,
\]
where $c_5 > 0$ is a constant independent of $n$.

Next, substituting $\varphi_{2,n} = v_n$ in (2.25), we get from (1.10)
\[
\frac{1}{2} \frac{d}{ds} \int_\Omega |v_n|^2 \, dx + d_v \int_\Omega |\nabla v_n|^2 \, dx \leq K_1 \int_\Omega |v_n| \, dx + K_2 \int_\Omega |v_n|^2 \, dx \leq \frac{K_1}{2} |\Omega| + \left( \frac{K_1}{2} + K_2 \right) \int_\Omega |v_n|^2 \, dx, \tag{2.31}
\]
where we have used Young’s inequality. This implies
\[
\frac{d}{ds} \int_\Omega |v_n|^2 \, dx + d_v \int_\Omega |\nabla v_n|^2 \, dx \leq c_6 \int_\Omega |v_n|^2 \, dx + c_7. \tag{2.32}
\]
By the Gronwall’s inequality,
\[
\int_\Omega |v_n(x,s)|^2 \, dx \leq c_8 \quad \forall s \in (0, t], \tag{2.33}
\]
where $c_8 > 0$ is a constant not depending on $n$. Integrating (2.32) over $(0, t)$ and taking into account the previous inequality, we obtain:
\[
\int_\Omega |v_n(x,t)|^2 \, dx + d_v \int_0^t \int_\Omega |\nabla v_n|^2 \, dx \leq c_9, \tag{2.34}
\]
for some constant $c_9 > 0$ (not depending on $n$), which proves what we wanted
\[
\|v_n\|_{L^2(0,t; H^1(\Omega))} \leq c_{10}.
\]
for some constant $c_{10} > 0$ independent of $n$.

Finally, we let $\varphi_1 \in L^2(0, t; H^1(\Omega))$. Using the weak formulation (2.24), we obtain

\[
\left| \int_0^t (\partial_t u_n, \varphi_1) \, ds \right| \\
\leq \left| \int_0^t \int_\Omega F(u_n, v_n) \varphi_1 \, dx \, ds \right| + |d_u| \int_0^t \int_\Omega \nabla u_n \cdot \nabla \varphi_1 \, dx \, ds \\
\leq (a + \gamma) \left( \int_0^t \int_\Omega |u_n|^2 \, dx \, ds \right)^{1/2} \left( \int_0^t \int_\Omega |\varphi_1|^2 \, dx \, ds \right)^{1/2} \\
+ d_u \left( \int_0^t \int_\Omega |\nabla u_n|^2 \, dx \, ds \right)^{1/2} \left( \int_0^t \int_\Omega |\nabla \varphi_1|^2 \, dx \, ds \right)^{1/2} \\
\leq c_{11} \|\varphi_1\|_{L^2(0,T;H^1(\Omega))},
\]

where we have used Hölder’s inequality. This implies

\[
\|\partial_t u_n\|_{L^2(0,t;H^1(\Omega)^')} \leq c_{12}.
\tag{2.35}
\]

Reasoning along the same lines for $v_n$, yields (2.35) for $v_n$. \hfill \blacksquare

The next is to show that the local solution constructed above can be extended to the whole time interval $[0,T)$ (independent of $n$) but this can be done as in [1], so we omit the details.

We have the following result:

**Lemma 2.2** The solution $(u_n,v_n)$ of the system (1.1) is nonnegative.

**proof.** The proof of Lemma 2.2 is based on the choice of test functions $\varphi_{1,n} = -u_n^-$, $\varphi_{2,n} = -v_n^-$ in (2.24) and (2.25), respectively, where $u_n^- = \max(0,-u_n)$ and $v_n^- = \max(0,-v_n)$. Then integrating over $(0,t)$ with $0 < t \leq T$, we obtain

\[
\frac{1}{2} \int_\Omega |u_n^-(x,t)|^2 \, dx + d_u \int_0^t \int_\Omega |\nabla u_n^-|^2 \, dx \, dt = \int_\Omega |u_n^-(x,0)|^2 \, dx - \int_0^t \int_\Omega F(u_n,v_n)u_n^- \, dx \, dt = 0,
\]

and

\[
\frac{1}{2} \int_\Omega |v_n^-(x,t)|^2 \, dx + d_v \int_0^t \int_\Omega |\nabla v_n^-|^2 \, dx \, dt = \int_\Omega |v_n^-(x,0)|^2 \, dx - \int_0^t \int_\Omega G(u_n,v_n)v_n^- \, dx \, dt \leq 0,
\]

where we have used the nonnegativity of the initial condition $(u_0,v_0)$ and (1.9). This implies that $u_n^- = 0$ and $v_n^- = 0$ a.e. in $\Omega$. \hfill \blacksquare

From Lemma 2.1 we have that $(u_n)_{n>1}$, $(v_n)_{n>1}$, are bounded in $L^2(0,T;H^1(\Omega))$ and $(\partial_t u_n)_{n>1}$, $(\partial_t v_n)_{n>1}$, are bounded in $L^2(0,T;(H^1(\Omega))^{'})$. Therefore, by standard compactness results [28] we can extract subsequences, which we do not relabel and we can assume that there exist limit functions $u,v$ such that as $n \to \infty$

\[
\begin{align*}
    u_n & \to u, v_n \to v \text{ strongly in } L^2(Q_T) \text{ and a.e. in } Q_T, \\
    u_n & \to u, v_n \to v \text{ weakly in } L^2(0,T;H^1(\Omega)), \\
    \partial_t u_n & \to \partial_t u, \partial_t v_n \to \partial_t v \text{ weakly in } L^2(0,T;(H^1(\Omega))^{'}) \\
    F(u_n,v_n) & \to F(u,v), G(u_n,v_n) \to G(u,v) \text{ a.e. in } Q_T.
\end{align*}
\tag{2.36}
\]

The following lemma is a consequence of (1.10), (2.36) and Vitali’s theorem.
Lemma 2.3 As \( n \to \infty \), \( F(u_n, v_n) \) and \( G(u_n, v_n) \) converge strongly to \( F(u, v) \) and \( G(u, v) \), respectively, in \( L^q(Q_T) \) for all \( 1 \leq q \leq 2 \).

Finally, we prove that the limits \( u \) and \( v \) in (2.36) obey the initial data in (1.1) but this can be done as Lemma 5.3 in [1], so we omit the details.

Keeping in mind (2.36) and Lemma 2.3, and using the following weak formulation:

\[
\int_0^T \langle \partial_t u_n, \varphi_1 \rangle \ dt + d_u \int_Q \nabla u_n \cdot \nabla \varphi_1 \ dx \ dt = \int_Q F(u_n, v_n) \varphi_1 \ dx \ dt, \tag{2.37}
\]

\[
\int_0^T \langle \partial_t v_n, \varphi_2 \rangle \ dt + d_v \int_Q \nabla v_n \cdot \nabla \varphi_2 \ dx \ dt = \int_Q G(u_n, v_n) \varphi_2 \ dx \ dt, \tag{2.38}
\]

for all \( \varphi_1, \varphi_2 \in L^2(0, T; H^1(\Omega)) \), we can let \( n \to \infty \) and obtain a weak solution.

### 3 Finite volume scheme

In this section, we denote by \( H_h(\Omega) \subset L^2(\Omega) \) the space of functions which are piecewise constant on each control volume \( K \in T \). For all \( u_h \in H_h(\Omega) \) and for all \( K \in T \), we denote by \( u_K \) the constant value of \( u_h \) in \( K \). For \( (u_h, v_h) \in (H_h(\Omega))^2 \), we define the following inner product:

\[
\langle u_h, v_h \rangle_{H_h} = \frac{1}{2} \sum_{K \in T} \sum_{L \in N(K)} \frac{m(L)}{d(K, L)} (u_L - u_K)(v_L - v_K),
\]

with a norm in \( H_h(\Omega) \)

\[
\| u_h \|_{H_h(\Omega)} = (\langle u_h, u_h \rangle_{H_h})^{1/2}.
\]

We also define \( L_h(\Omega) \subset L^2(\Omega) \) the space of functions which are piecewise constant on each control volume \( K \in T \) with the associated norm

\[
\langle u_h, v_h \rangle_{L_h(\Omega)} = \sum_{K \in T} m(K) u_K v_K, \quad \| u_h \|^2_{L_h(\Omega)} = \sum_{K \in T} m(K) |u_K|^2,
\]

for \( (u_h, v_h) \in (L_h(\Omega))^2 \). Next, we introduce the Hilbert spaces

\[
E_h := (H_h(\Omega) \cap L_h(\Omega))^2,
\]

under the norm

\[
\| u_h \|^2_{E_h} := \| u_h \|^2_{H_h(\Omega)} + \| v_h \|^2_{H_h(\Omega)} + \sum_{K \in T} m(K) |u_K|^2 + \sum_{K \in T} m(K) |v_K|^2,
\]

where \( u_h = (u_h, v_h) \).
3.1 Existence of solutions to the finite volume scheme

The existence for the finite volume scheme is given in the following proposition.

**Proposition 3.1** Let \( D \) be an admissible discretization of \( Q_T \). Then the problem (1.11)-(1.13) admits at least one solution \((u^n_K, v^n_K)_{(K,n)\in\mathcal{T}\times\{0,...,N\}}\).

**proof.** Let \( \Phi_h = (\varphi_h, \psi_h) \in E_h \) and define the discrete bilinear forms (recall that \( u^n_h = (u_h, v_h) \))

\[
P_h(u^n_h, \Phi_h) = \left( u^n_h, \varphi_h \right) + \left( v^n_h, \psi_h \right),
\]

\[
B_h(u^{n+1}_h, \Phi_h) = \sum_{K \in \mathcal{T}} m(K) F(u^{n+1}_K, \varphi_K) + \sum_{K \in \mathcal{T}} m(K) G(u^{n+1}_K, v^{n+1}_K) \psi_K,
\]

and

\[
A_h(u^{n+1}_h, \Phi_h) = \frac{d_u}{2} \sum_{K,L \in \mathcal{N}(K)} \frac{m(\sigma_{K,L})}{d(K,L)} (u^{n+1}_L - u^{n+1}_K)(\varphi^{n+1}_L - \varphi^{n+1}_K)
\]

\[
+ \frac{d_v}{2} \sum_{K,L \in \mathcal{N}(K)} \frac{m(\sigma_{K,L})}{d(K,L)} (v^{n+1}_L - v^{n+1}_K)(\psi^{n+1}_L - \psi^{n+1}_K).
\]

Multiplying (1.12) and (1.13) by \( \varphi_K \) and \( \psi_K \) respectively, we get the equation

\[
\frac{1}{\Delta t} \left( P_h(u^{n+1}_h, \Phi_h) - P_h(u^n_h, \Phi_h) \right) + A_h(u^{n+1}_h, \Phi_h) - B_h(u^{n+1}_h, \Phi_h) = 0.
\]

We define the mapping \( \mathcal{M} \) from \( E_h \) into itself

\[
[\mathcal{M}(u^{n+1}_h), \Phi_h] = \frac{1}{\Delta t} (P_h(u^{n+1}_h, \Phi_h) - P_h(u^n_h, \Phi_h)) + A_h(u^{n+1}_h, \Phi_h) - B_h(u^{n+1}_h, \Phi_h),
\]

for all \( \Phi_h \in E_h \). It is easy to check that the mapping \( \mathcal{M} \) is continuous.

Now we show that for a sufficiently large \( \rho \)

\[
[\mathcal{M}(u^{n+1}_h), u^{n+1}_h] > 0 \quad \text{for} \quad \|u^{n+1}_h\|_{E_h} = \rho > 0.
\]  

(3.39)

Which implies (see for e.g. [20] and [29]): there exists \( u^{n+1}_h \in E_h \) with \( \|u^{n+1}_h\| \leq \rho \) such that

\[
\mathcal{M}(u^{n+1}_h) = 0.
\]

We observe that

\[
[\mathcal{M}(u^{n+1}_h), u^{n+1}_h] = \frac{1}{\Delta t} \sum_{K \in \mathcal{T}} m(K) \left| u^{n+1}_K \right|^2 + \frac{1}{\Delta t} \sum_{K \in \mathcal{T}} m(K) \left| v^{n+1}_K \right|^2
\]

\[
+ A_h(u^{n+1}_h, u^{n+1}_h) - B_h(u^{n+1}_h, u^{n+1}_h)
\]

\[
- \frac{1}{\Delta t} \sum_{K \in \mathcal{T}} m(K) u^{n+1}_K u^{n+1}_K - \frac{1}{\Delta t} \sum_{K \in \mathcal{T}} m(K) v^{n+1}_K v^{n+1}_K.
\]  

(3.40)
It follows that from (3.40) and Young’s inequality that
\[
[M(u_{h}^{n+1}, u_{h}^{n+1}) - a_1 \sum_{K \in T} m(K) |u_{K}^{n+1}|^2 + \frac{1}{\Delta t} \sum_{K \in T} m(K) |u_{K}^{n+1}|^2 + d_u \left\| u_{h}^{n+1} \right\|_{H_h(\Omega)}^2 + d_v \left\| v_{h}^{n+1} \right\|_{H_h(\Omega)}^2
\]
\[
- a_1 \sum_{K \in T} m(K) |u_{K}^{n+1}|^2 - C_1(K_1, \Delta t, \theta) - \frac{1}{2\theta \Delta t} \sum_{K \in T} m(K) |v_{K}^{n+1}|^2
\]
\[
- K_1 \sum_{K \in T} m(K) |v_{K}^{n+1}|^2 - \frac{1}{\theta \Delta t} \sum_{K \in T} m(K) |u_{K}^{n+1}|^2 - C_2(\theta, \Delta t) \sum_{K \in T} m(K) |v_{K}^{n+1}|^2
\]
\[
- \frac{1}{2\theta \Delta t} \sum_{K \in T} m(K) |v_{K}^{n+1}|^2 - C_3(\theta, \Delta t) \sum_{K \in T} m(K) |v_{K}^{n+1}|^2,
\]
where we have used the definitions of \( F \) and \( G \). This implies that
\[
[M(u_{h}^{n+1}, u_{h}^{n+1}) - a_1 \sum_{K \in T} m(K) |u_{K}^{n+1}|^2 + \frac{1}{\Delta t} \sum_{K \in T} m(K) |u_{K}^{n+1}|^2 + d_u \left\| u_{h}^{n+1} \right\|_{H_h(\Omega)}^2 + d_v \left\| v_{h}^{n+1} \right\|_{H_h(\Omega)}^2
\]
\[
- a_1 \sum_{K \in T} m(K) |u_{K}^{n+1}|^2 - C_1(K_1, \theta, \Delta t)
\]
\[
- C_2(\theta, \Delta t) \sum_{K \in T} m(K) |u_{K}^{n+1}|^2 - C_3(\theta, \Delta t) \sum_{K \in T} m(K) |v_{K}^{n+1}|^2
\]
\[
\geq \min \left\{ \frac{(\theta - 1)}{\theta \Delta t} - a_1, \frac{(\theta - 1)}{\theta \Delta t} - K_1, d_u, d_v \right\} \left\| u_{h}^{n+1} \right\|_{E_h}^2 + C_1(K_1, \theta, \Delta t)
\]
\[
- C_2(\theta, \Delta t) \sum_{K \in T} m(K) |u_{K}^{n+1}|^2 - C_3(\theta, \Delta t) \sum_{K \in T} m(K) |v_{K}^{n+1}|^2.
\]

Finally, for a given \( u_{h}^{n} \) and \( v_{h}^{n} \) we deduce from (3.41) and (1.15) that (3.39) holds for \( \rho \) large enough (recall that \( \left\| u_{h}^{n+1} \right\|_{E_h} = \rho \)). Then, we obtain the existence of at least one solution to the scheme (1.11)-(1.13).

\[\blacksquare\]

### 3.2 Nonnegativity

We have the following lemma.

**Lemma 3.1** Let \((u_{K}^{n}, v_{K}^{n})_{K \in T, n \in \{0, \ldots, N\}}\) be a solution of the finite volume scheme (1.11), (1.12) and (1.13). Then, \((u_{K}^{n}, v_{K}^{n})_{K \in T, n \in \{0, \ldots, N\}}\) is nonnegative.

**proof.** Multiplying (1.12) by \(-\Delta t u_{1,K}^{n+1}\), we find that
\[
- m(K) u_{K}^{n+1} (u_{K}^{n+1} - u_{K}^{n}) + d_u \sum_{L \in N(K)} \frac{m(s_{K,L})}{d(K,L)} (u_{L}^{n+1} - u_{K}^{n+1}) u_{K}^{n+1}
\]
\[
+ m(K) \Delta t F (u_{K}^{n+1}, v_{K}^{n+1}) u_{K}^{n+1} = 0.
\]
We know that $u_{K}^{n+1} = u_{K}^{n+1^+} - u_{K}^{n+1^-}$ and $(a^+ - b^+)(a^- - b^-) \leq 0$ for all $a, b \in \mathbb{R}$. Thus we deduce
\[ d_u \sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}} \sum_{L \in N(K)} m(\sigma_{KL}) \frac{u_{L}^{n+1} - u_{K}^{n+1}}{d(K, L)} (u_{L}^{n+1} - u_{K}^{n+1})u_{K}^{n+1^-} \]
\[ = \frac{d_u}{2} \sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}} \sum_{L \in N(K)} m(\sigma_{KL}) \frac{u_{L}^{n+1} - u_{K}^{n+1}}{d(K, L)} (u_{L}^{n+1} - u_{K}^{n+1})(u_{L}^{n+1} - u_{K}^{n+1}) \quad (3.43) \]
\[ \geq \frac{d_u}{2} \sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}} \sum_{L \in N(K)} m(\sigma_{KL}) \frac{u_{L}^{n+1} - u_{K}^{n+1}}{d(K, L)} |u_{L}^{n+1} - u_{K}^{n+1}|^2 \geq 0, \]
and
\[ \sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}} m(K) F(u_{K}^{n+1})u_{K}^{n+1^-} = 0. \quad (3.44) \]

Let $f \in C^2$ function. By using a Taylor expansion we find
\[ f(b) = f(a) + f'(a)(b-a) + \frac{1}{2} f''(\xi)(b-a)^2, \quad (3.45) \]
for some $\xi$ between $a$ and $b$. Using the Taylor expansion (3.45) on the sequence $f(u_{K}^{n+1})$ with $f(\rho) = \int_{0}^{\rho} s \, ds, a = u_{K}^{n+1}$ and $b = u_{K}^{n}$. We find
\[ u_{K}^{n+1^-}(u_{K}^{n+1} - u_{K}^{n}) = \frac{|u_{K}^{n+1^-}|^2}{2} - \frac{|u_{K}^{n+1}|^2}{2} - \frac{1}{2} f''(\xi) (u_{K}^{n+1} - u_{K}^{n})^2. \quad (3.46) \]

We observe from the definition of $f$ that $f''(\rho) \geq 0$, which implies
\[ u_{K}^{n+1^-}(u_{K}^{n+1} - u_{K}^{n}) \leq \frac{|u_{K}^{n+1^-}|^2}{2} - \frac{|u_{K}^{n+1}|^2}{2}. \quad (3.47) \]

Now, using (3.43)-(3.47) to deduce from (3.42)
\[ \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} m(K) \left( \frac{|u_{K}^{n+1^-}|^2}{2} - \frac{|u_{K}^{n}|^2}{2} \right) + d_u \sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}} \sum_{L \in N(K)} m(\sigma_{KL}) \frac{u_{L}^{n+1} - u_{K}^{n+1}}{d(K, L)} |u_{L}^{n+1} - u_{K}^{n+1}|^2 \leq 0. \quad (3.48) \]
This implies that
\[ \frac{1}{2} (|u_{K}^{N+1^-}|^2 - |u_{K}^{0^-}|^2) \leq 0. \quad (3.49) \]
Note that (3.49) is also true if we replace $N$ by $n_0 \in \{1, \ldots, N\}$, so we have established
\[ |u_{K}^{n_0^-}|^2 \leq |u_{K}^{0^-}|^2. \quad (3.50) \]

Since $u_{K}^{0}$ is nonnegative, the result is $u_{K}^{n+1^-} = 0$ for all $0 \leq n \leq N - 1$ and all $K \in \mathcal{T}$. Along the same lines as $u_{K}^{n+1}$, we obtain the nonnegativity of the discrete solution $v_{K}^{n+1}$ for all $0 \leq n \leq N - 1$ and all $K \in \mathcal{T}$. \[ \blacksquare \]
3.3 A priori estimates

Now we establish several a priori estimates for the finite volume scheme.

Proposition 3.2 Let \((u^n_K, v^n_K)_{K \in T, n \in \{0, \ldots, N\}}\) be a solution of the finite volume scheme (1.11), (1.12) and (1.13). Then there exist constants \(C_1, C_2, C_3 > 0\), depending on \(\Omega, T, u_0, v_0\) and \(\alpha\) such that

\[
\max_{n \in \{0, \ldots, N\}} \sum_{K \in \Omega_R} m(K)(|u^n_K|^2 + |v^n_K|^2) \leq C_1, \tag{3.51}
\]

\[
\frac{1}{2} \sum_{n=0}^{N-1} \Delta t \sum_{K \in T} \sum_{L \in N(K)} \frac{m(\sigma_{K,L})}{d(K,L)} \left( |u^{n+1}_K - u^{n+1}_L|^2 + |v^{n+1}_K - v^{n+1}_L|^2 \right) \leq C_2, \tag{3.52}
\]

and

\[
\sum_{n=0}^{N-1} \Delta t \sum_{K \in \Omega_R} m(K) \left( |F(u^{n+1}_K, v^{n+1}_K)| + |G(u^{n+1}_K, v^{n+1}_K)| \right) \leq C_3. \tag{3.53}
\]

proof. We multiply (1.12) and (1.13) by \(\Delta t u^{n+1}_K\) and \(\Delta t v^{n+1}_K\), respectively, and add together the outcomes. Summing the resulting equation over \(K\) and \(n\) yields

\[
S_1 + S_2 = S_3,
\]

where

\[
S_1 = \sum_{n=0}^{N-1} \sum_{K \in T} m(K)(u^{n+1}_K - u^n_K)u^{n+1}_K + \sum_{n=0}^{N-1} \sum_{K \in T} m(K)(v^{n+1}_K - v^n_K)v^{n+1}_K,
\]

\[
S_2 = -\sum_{n=0}^{N-1} \Delta t \sum_{K \in T} \sum_{L \in N(K)} \left( d_L m(\sigma_{K,L}) (u^{n+1}_K - u^n_K)u^{n+1}_K + d_L m(\sigma_{K,L}) (v^{n+1}_K - v^n_K)v^{n+1}_K \right),
\]

\[
S_3 = \sum_{n=0}^{N-1} \Delta t \sum_{K \in \Omega_R} m(K) \left( F(u^{n+1}_K, v^{n+1}_K)u^n_K + G(u^{n+1}_K, v^{n+1}_K)v^n_K \right).
\]

Using the inequality \(a(a - b) \geq \frac{1}{4}(a^2 - b^2)\), we obtain

\[
S_1 = \sum_{n=0}^{N-1} \sum_{K \in T} m(K)(u^{n+1}_K - u^n_K)u^{n+1}_K + \sum_{K \in T} m(K)(v^{n+1}_K - v^n_K)v^{n+1}_K
\]

\[
\geq \frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in T} m(K) \left( |u^{n+1}_K|^2 - |u^n_K|^2 \right) + \frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in T} m(K) \left( |v^{n+1}_K|^2 - |v^n_K|^2 \right)
\]

\[
= \frac{1}{2} \sum_{K \in T} m(K) \left( |u^N_K|^2 - |u^0_K|^2 \right) + \frac{1}{2} \sum_{K \in T} m(K) \left( |v^N_K|^2 - |v^0_K|^2 \right).
\]

Gathering by edges, we obtain

\[
S_2 = \frac{d_L}{2} \sum_{n=0}^{N-1} \Delta t \sum_{K \in T} \sum_{L \in N(K)} \frac{m(\sigma_{K,L})}{d(K,L)} |u^{n+1}_K - u^{n+1}_L|^2
\]

\[
+ \frac{d_L}{2} \sum_{n=0}^{N-1} \Delta t \sum_{K \in T} \sum_{L \in N(K)} \frac{m(\sigma_{K,L})}{d(K,L)} |v^{n+1}_K - v^{n+1}_L|^2.
\]
An application of Young inequality, we get
\[
S_3 \leq a_1 \sum_{n=0}^{N-1} \Delta t \sum_{K \in T} m(K) |u_K^{n+1}|^2 + C(K_1) + C' \sum_{n=0}^{N-1} \Delta t \sum_{K \in T} m(K) |v_K^{n+1}|^2
\]

\[
+ K_2 \sum_{n=0}^{N-1} \Delta t \sum_{K \in T} m(K) |v_K^{n+1}|^2
\]

\[
\leq a_1 \sum_{n=0}^{N-1} \Delta t \sum_{K \in T} m(K) |u_K^{n+1}|^2 + (C' + K_2) \sum_{n=0}^{N-1} \Delta t \sum_{K \in T} m(K) |v_K^{n+1}|^2 + C(K_1).
\]

Collecting the previous inequalities we obtain
\[
\frac{1}{2} \sum_{K \in T} m(K) \left( |u_K^N|^2 - |u_K^0|^2 \right) + \frac{1}{2} \sum_{K \in T} m(K) \left( |v_K^N|^2 - |v_K^0|^2 \right)
\]

\[
+ \frac{d_u}{2} \sum_{n=0}^{N-1} \Delta t \sum_{K \in T} \sum_{L \in N(K)} \frac{m(\sigma_{K,L})}{d(K,L)} |u_K^{n+1} - u_L^{n+1}|^2
\]

\[
+ \frac{d_u}{2} \sum_{n=0}^{N-1} \Delta t \sum_{K \in T} \sum_{L \in N(K)} \frac{m(\sigma_{K,L})}{d(K,L)} |v_K^{n+1} - v_L^{n+1}|^2
\]

\[
\leq a_1 \sum_{n=0}^{N-1} \Delta t \sum_{K \in T} m(K) |u_K^{n+1}|^2 + (C' + K_2) \sum_{n=0}^{N-1} \Delta t \sum_{K \in T} m(K) |v_K^{n+1}|^2 + C(K_1).
\]

Using this to deduce
\[
\sum_{K \in T} m(K) |u_K^N|^2 + \sum_{K \in T} m(K) |v_K^N|^2
\]

\[
\leq C_4 + C_5 \sum_{n=0}^{N-1} \Delta t \sum_{K \in T} m(K) |u_K^{n+1}|^2 + C_6 \sum_{n=0}^{N-1} \Delta t \sum_{K \in T} m(K) |v_K^{n+1}|^2,
\]

for some constants $C_4, C_5, C_6 > 0$. Herein we have used
\[
\sum_{K \in T} m(K) |u_K^0|^2 + \sum_{K \in T} m(K) |v_K^0|^2 \leq \|u_0\|_{L^2(\Omega)}^2 + \|v_0\|_{L^2(\Omega)}^2.
\]

Thus by the discrete Gronwall inequality (see e.g. [17]), we obtain from (3.55)
\[
\sum_{K \in T} m(K) |u_K^n|^2 + \sum_{K \in T} m(K) |v_K^n|^2 \leq C_7,
\]

for any $n_0 \in \{1, \ldots, N\}$ and some constant $C_7 > 0$. Then
\[
\max_{n \in \{0, \ldots, N\}} \sum_{K \in T} m(K) |u_K^n|^2 + \max_{n \in \{0, \ldots, N\}} \sum_{K \in T} m(K) |v_K^n|^2 \leq C_7.
\]

Moreover, we obtain from (3.54) and (3.56) the existence of a constant $C_8 > 0$ such that
\[
\sum_{n=0}^{N-1} \Delta t \sum_{K \in T} \sum_{L \in N(K)} \frac{m(\sigma_{K,L})}{d(K,L)} |u_K^{n+1} - u_L^{n+1}|^2
\]

\[
+ \sum_{n=0}^{N-1} \Delta t \sum_{K \in T} \sum_{L \in N(K)} \frac{m(\sigma_{K,L})}{d(K,L)} |v_K^{n+1} - v_L^{n+1}|^2 \leq C_8.
\]
Finally a consequence of (3.51) is that
\[ \|F(u_h, v_h)\|_{L^1(Q_T)} + \|G(u_h, v_h)\|_{L^1(Q_T)} \leq C_9, \]
for some constant \( C_9 > 0 \).

### 3.4 Convergence of the finite volume scheme

In this subsection we derive estimates on differences of space and time translates of the function \( v_h \) which imply that the sequence \( v_h \) is relatively compact in \( L^2(Q_T) \).

**Lemma 3.2** There exists a positive constant \( C > 0 \) depending on \( \Omega, T, u_0, v_0, d_u \) and \( d_v \) such that
\[ \int_{\Omega' \times (0, T)} |w_h(t, x + y) - w_h(t, x)|^2 \, dx \, dt \leq C |y| (|y| + 2h), \quad w_h = u_h, v_h, \quad (3.57) \]
for all \( y \in \mathbb{R}^3 \) with \( \Omega' = \{ x \in \Omega, [x, x + y] \subset \Omega \} \), and
\[ \int_{\Omega' \times (0, T - \tau)} |w_h(t + \tau, x) - w_h(t, x)|^2 \, dx \, dt \leq C(\tau + \Delta t), \quad w_h = u_h, v_h, \quad (3.58) \]
for all \( \tau \in (0, T) \).

The proof of Lemma 3.2 will be omitted since it is similar to that of Lemma 4.3 and Lemma 4.6 in [16] (see also Lemma 6.1 in [2]).

The consequence of Lemma 3.2 and Lemma 3.2 is the following lemma.

**Lemma 3.3** There exists a subsequence of \( u_h = (u_h, v_h) \), not relabeled, such that, as \( h \to 0 \),
\[ (i) \ u_h \to u \text{ and } v_h \to v \text{ strongly in } L^2(Q_T) \text{ and a.e. in } Q_T, \]
\[ (ii) \ \nabla_h u_h \to \nabla u \text{ and } \nabla_h v_h \to \nabla v \text{ weakly in } (L^2(Q_T))^3, \]
\[ (iii) \ F(u_h, v_h) \to F(u, v) \text{ and } G(u_h, v_h) \to G(u, v) \text{ strongly in } L^1(Q_T). \]

**proof.** The claim (i) in (3.59) follows from Lemma 3.2 and Kolmogorov’s compactness criterion (see, e.g., [9], Theorem IV.25). The proof of the claim (ii) will be omitted since it is similar to that of Lemma 4.4 in [10], we refer to the proof of this lemma for more details. The claim (iii) follows from Vitali’s theorem.

Our final goal is to prove that the limit functions \( u, v \) constructed in Lemma 3.3 constitute a weak solution of the system (1.1).

Let \( T \) be a fixed positive constant and \( \varphi \in D([0, T] \times \Omega) \). We multiply the discrete equations (1.12) and (1.13) by \( \Delta t \varphi_1(t^n, x_K) \) and \( \Delta t \varphi_2(t^n, x_K) \), respectively, for all \( K \in T \) and \( n \in \{0, \ldots, N\} \). Summing the result over \( K \) and \( n \) yields
\[ T_1 + T_2 = T_3 \text{ and } T_1' + T_2' = T_3', \quad (3.60) \]
where
\[ T_1 = \sum_{K \in T} m(K)(u_{K}^{n+1} - u_K^n)\varphi_1(t^n, x_K), \]
\[ T_2 = -d_u \Delta t \sum_{K \in T} \sum_{L \in N(K)} \frac{m(\sigma_{K,L})}{d(K, L)} (u_{L}^{n+1} - u_L^n)\varphi_1(t^n, x_K), \]
\[ T_3 = \Delta t \sum_{K \in T} m(K)F(u_{K}^{n+1}, v_K^{n+1})\varphi_1(t^n, x_K), \]
and
\[ T'_1 = \sum_{K \in T} m(K)(v_{K}^{n+1} - v_K^n)\varphi_2(t^n, x_K), \]
\[ T'_2 = -d_v \Delta t \sum_{K \in T} \sum_{L \in N(K)} \frac{m(\sigma_{K,L})}{d(K, L)} (v_{L}^{n+1} - v_L^n)\varphi_2(t^n, x_K), \]
\[ T'_3 = \Delta t \sum_{K \in T} m(K)G(u_{K}^{n+1}, v_K^{n+1})\varphi_2(t^n, x_K), \]
Knowing in mind (3.59) and using the formulations (3.60), we can let \( h \to 0 \) and obtain (1.16)-(1.17).

4 Numerical Results

4.1 Pattern Formation

The purpose of this section is to study the Turing Instability to stationary spatial patterns in reaction-diffusion system (1.1). The mathematical Turing formulation is introduced, the Turing instability is discussed using mathematical terms and analyzed by employing linear stability analysis.

First, using the length scale \( L \) an time scale \( \frac{d}{d_v} \), the governing system (1.1) in the dimensionless form become
\[
\begin{align*}
\partial_t u - d \Delta u &= \mu F(u, v) \quad \text{in} \quad Q_T \\
\partial_t v - \Delta v &= \mu G(u, v) \quad \text{in} \quad Q_T
\end{align*}
\]
where \( \mu = \frac{L^2}{d_v} \) and \( d = \frac{d}{d_v} \).

\[
\begin{align*}
F(u, v) &= \left( \frac{a_1(uv)^b}{1 + (uv)^b} - \gamma \right) u, \\
G(u, v) &= h_0\beta_1 \frac{ue^{-1}}{1 + r_1u^e} - v \left( \alpha + h_0\alpha \frac{ue}{1 + r_1u^e} + \frac{a_1(uv)^b}{1 + (uv)^b} - \gamma \right).
\end{align*}
\]

Let us consider \( U^* = (u^*, v^*) \) to be a positive steady state for the system without the diffusion terms. That is \( F(U^*) = F(u^*, v^*), G(u^*, v^*) = (0, 0) \).

Let \( J(U^*) \) be Jacobian at the steady state \( U^* \)
\[
J(U^*) = \begin{pmatrix} F_u & F_v \\ G_u & G_v \end{pmatrix}
\]
where

\[ F_u = \frac{a_1 b(u^*v^*)^b}{(1 + (u^*v^*)^b)^2} + \frac{a_1 (u^*v^*)^b}{1 + (u^*v^*)^b} - \gamma \]

\[ F_v = \frac{a_1 b(u^*v^*)^b u^*}{(1 + (u^*v^*)^b)^2 v^*} \]

\[ G_u = h_0 \beta_1 \frac{(e - 1)(u^*)^{c + r_1(e - s - 1)(u^*)^{c+s}}}{(1 + r_1(u^*)^s)^2(u^*)^2} - h_0 \alpha \frac{e(u^*)^c + r_1(e - s)(u^*)^{c+s} v^*}{(1 + r_1(u^*)^s)^2 u^*} \]

\[ G_v = -\alpha - h_0 \alpha \frac{(u^*)^c}{1 + r_1(u^*)^s} - \frac{a_1 b(u^*v^*)^b}{(1 + (u^*v^*)^b)^2} - \frac{a_1 (u^*v^*)^b}{1 + (u^*v^*)^b} + \gamma \]

(4.63)

Linear Stability. Let \( W = U - U^* \) a small perturbation added to \( U^* \). The steady state \( W \) is linearly stable if \( \text{Re} \lambda < 0 \) (\( \lambda \) is an eigenvalue of \( J(U^*) \)) since in this case the perturbation \( W \) goes to zero as \( t \) goes to infinity. A simple calculation, we deduce that linear stability is guaranteed if

\[ F_u + G_v < 0 \]  
(4.64)

\[ F_u G_v - F_v G_u > 0 \]  
(4.65)

Turing Conditions. Let us recall the general condition for diffusion driven instability (for details see [25]). Now consider the full reaction-diffusion system (4.61) and again linearise about the steady state (\( W = 0 \)) to get

\[ \partial_t W = \mu J(U^*) W + \mathcal{D} \Delta W \]

with

\[ \mathcal{D} = \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \]

In the linear analysis the spatial and time variance are taken into account by substituting a solution of the form \( w(x, t) = \sum_k c_k e^{\lambda t} w_k(x) \) into the linearized system in the presence of diffusion. Herein, \( w_k \) are the eigenfunctions of the time independent eigenvalue problem defined by

\[ -\Delta w = k^2 w \quad \text{in} \quad Q_T \quad \text{and} \quad \frac{\partial w}{\partial \eta} = 0 \quad \text{in} \quad Q_T \]  
(4.66)

The eigenvalues \( \lambda(k) \) as functions of the wavenumber \( k \) is

\[ |\lambda \mathcal{I} - \mu J + k^2 \mathcal{D}| = 0, \]  
(4.67)

with

\[ \mathcal{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

The eigenvalues \( \lambda(k) \) as functions of the wavenumber \( k \) as the roots of

\[ \lambda^2 + \lambda(k^2(d + 1) - \mu(F_u + G_v)) + H(k^2) = 0, \]  
(4.68)
where

\[ H(k^2) = dk^4 - \mu(F_u + dG_v)k^2 + \mu^2(G_u F_v - F_u G_v). \]

The only way that the steady state to be unstable to spatial distribution, is one of the roots of (4.68) satisfies \( \text{Re}(\lambda(k)) > 0 \) for some \( k \neq 0 \). This occurs if either the coefficient \( k^2(d + 1) - \mu(F_u + G_v) < 0 \) or if \( H(k^2) < 0 \) for some \( k \neq 0 \). Note that from the stability condition \( F_u + G_v < 0 \), and \( k^2(d + 1) > 0 \) for all \( k \neq 0 \), so \( (k^2(d + 1) - \mu(F_u + G_v)) > 0 \); then the case for which the steady state can be unstable steady state is only if \( H(k^2) < 0 \) for some \( k \neq 0 \). So we must have the first Turing condition

\[ F_u + dG_v > 0 \quad (4.69) \]

This implies, by comparison with (4.64), that \( d \neq 1 \), and \( F_u, G_v \) must have opposite signs. The inequality (4.69) is necessary but not sufficient for \( \text{Re}(\lambda(k)) > 0 \). For \( H(k^2) \) to be negative, the minimum must be negative

\[ H_{\text{min}} = H(k_{\text{min}}^2) = -\frac{\mu^2(F_u + dG_v)^2}{4d} + \mu^2(F_u G_v - F_v G_u) \quad (4.70) \]

with

\[ k_{\text{min}}^2 = \frac{\mu(F_u + dG_v)}{2d} \quad (4.71) \]

Then, we can write the second Turing condition as follows

\[ (F_u + dG_v)^2 - 4d(F_u G_v - F_v G_u) > 0. \quad (4.72) \]

Let us define a critical diffusion ratio \( d_c \) for which \( H_{\text{min}} = 0 \), we require

\[ (F_u + d_c G_v)^2 - 4d_c(F_u G_v - F_v G_u) = 0. \quad (4.73) \]

and the critical wave number \( k_c \) is then given by

\[ k_c^2 = \frac{\mu(F_u + d_c G_v)}{2d_c} = \mu \sqrt{\frac{F_u G_v - F_v G_u}{d_c}} \quad (4.74) \]

The second Turing condition (4.72) is equivalent to

\[ G_v^2 d^2 + 2(2F_v G_u - 4F_u G_v)d + F_u^2 > 0. \quad (4.75) \]

From (4.68), with \( d > d_c \), and

\[ d_c = \frac{-(2F_v G_u - F_v G_v) + \sqrt{(2F_v G_u - F_v G_v)^2 - F_u^2 G_v^2}}{G_v^2} \quad (4.76) \]

the range of unstable wavenumbers

\[ k_1^2 < k^2 < k_2^2 \quad (4.77) \]
is obtained from the zeros $k_1^2$ and $k_2^2$ of $H(k^2) = 0$ as
\[
    k_1^2 = \frac{\mu}{2d} \left( F_u + dG_v - \sqrt{(F_u + dG_v)^2 - 4d(F_uG_v - F_vG_u)} \right),
\]
\[
    k_2^2 = \frac{\mu}{2d} \left( F_u + dG_v + \sqrt{(F_u + dG_v)^2 - 4d(F_uG_v - F_vG_u)} \right). \tag{4.78}
\]

**Steady State.** The steady states are solutions of the system
\[
    F(u^*, v^*) = \left( \frac{a_1(u^*v^*)^b}{1 + (u^*v^*)^c} - \gamma \right) u^* = 0,
\]
\[
    G(u^*, v^*) = \frac{h_0\beta_1}{1 + r_1(u^*)^e} - v^* \left( \alpha + h_0\alpha \frac{(u^*)^c}{1 + r_1(u^*)^e} + \frac{a_1(u^*v^*)^b}{1 + (u^*v^*)^b} - \gamma \right) = 0. \tag{4.79}
\]

**Turing Space.** The conditions on the parameters $\mu, d$, to generate spatial patterns are given by Turing conditions (4.64), (4.65), (4.69) and (4.72).

### 4.2 Numerical Simulations

We now show some numerical experiments with the scheme proposed. The test problem is the system (1.1) in the square domain $\Omega = [0, 1] \times [0, 1]$.

We consider for our problem a uniform mesh given by a Cartesian grid with $N_x \times N_y$ control volumes and choosing $N_x = N_y = 200$ ($\Delta x = \Delta y = 1/200$). Obviously, it is possible to consider also unstructured meshes, but for simplicity we will use uniform mesh. For our simulations, we use a time step $\Delta t = 0.0025$.

Let us first precise the initial conditions
\[
    u(x, y, 0) = u^*(1 + \epsilon_u \cos(n\pi x) \cos(m\pi y)), \quad (x, y) \in \Omega
\]
\[
    v(x, y, 0) = v^*(1 + \epsilon_v \cos(n\pi x) \cos(m\pi y)), \quad (x, y) \in \Omega \tag{4.80}
\]
with $\epsilon_u = 0.1, \epsilon_v = 0.006$.

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### References


Reaction-diffusion systems modelling the spread of early tumors

<table>
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<tr>
<th>Parameters</th>
<th>$a_1 = 0.2$, $b = 2$, $\gamma = 0.09$, $h_0 = 1$, $\beta_1 = 1$, $r_1 = 10/3$</th>
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<td>$s = 2$, $\alpha = 0.1$, $e = 1$, $d_u = 0.0004$, $d_v = 0.0001$</td>
</tr>
</tbody>
</table>

Figure 1: Patterns for cells (left) and growth factor molecules (right), with $u^* = 2.91366$, $v^* = 0.31045$ and $m = n = 2$ in the initial condition. The surface densities of cells ($u$) and growth factor molecules ($v$) vary between $[3.65, 4.45]$ and $[8.93, 9.14]$ respectively.


Figure 2: Patterns for cells (left) and growth factor molecules (right), with $u^* = 2.91366$, $v^* = 0.31045$ and $n = 0, m = 2$ in the initial condition.


Figure 3: Patterns for cells (left) and growth factor molecules (right), with $u^* = 0.3857$, $v^* = 2.3452$ and $m = n = 4$ in the initial condition. The surface densities of cells ($u$) and growth factor molecules ($v$) vary between [0.36, 0.47] and [3.15, 3.47] respectively.


Figure 4: Patterns for cells (left) and growth factor molecules (right), with $u^* = 0.3857$, $v^* = 2.3452$ and $m = n = 2$ in the initial condition.