Journal of Interdisciplinary Mathematics

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/tjim20

An analysis of hereditary and pure hereditary ring using a triangular matrix ring

Mosst. Asma Akter

To cite this article: Mosst. Asma Akter (2022) An analysis of hereditary and pure hereditary ring using a triangular matrix ring, Journal of Interdisciplinary Mathematics, 25:7, 2053-2061, DOI: 10.1080/09720502.2022.2133232

To link to this article: https://doi.org/10.1080/09720502.2022.2133232

Published online: 08 Dec 2022.

Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
Vol. 25 (2022), No. 7, pp. 2053–2061
DOI : 10.1080/09720502.2022.2133232

An analysis of hereditary and pure hereditary ring using a triangular matrix ring

Mosst. Asma Akter
Department of Quantitative Science
International University of Business Agriculture and Technology (IUBAT)
Dhaka
Bangladesh

Abstract

In this paper the hereditary ring is considered, defined, classified and studied which is associative with unit where an ideal of R is T. Considering the ring $B = B'/J$ is regular ring (where let B' as a regular ring that is commutative but not semi-simple and J be the ideal of B'). First of all, R has shown as a left Hereditary and Noetherian Hereditary ring. A triangular matrix ring $\begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$ has been taken which is shown as left hereditary forming with two left Hereditary Ring R and S. Finally we have declared that although the ring R is not a right semi-Hereditary but must be left hereditary **[2]**. Accordingly, we can say that R is not a right pure-Hereditary ring.

Subject Classification: *1.60E+61.*

Keywords: *Hereditary ring, Pure hereditary ring, Semi-simple ring.*

1. Introduction

We will start this section by recalling some well-known publications on hereditary rings. Like [11] has considered a ring which is right Hereditary as well as PI but can't be left Hereditary, where the considering ring is associative with unit, has left global dimension three and the ring contains some more axioms of not becoming left hereditary. [3] Described some of the group rings which are not hereditary using the idea of group theory. The concept of right hereditary, completely reducible, Von Neumann regular ring, right Noetherian ring has been used. After all to describe the

E-mail: **asma.akter@iubat.edu**

group rings a graph of connected groups which is Fundamental also has been used. In the paper [6], the writer considered a semiprime ring which is right Hereditary PI ring has shown as a generated finitely through the center of Ring must be a left Hereditary ring. As a consequence **[7]** the authors considered a affine PI rings which is right Hereditary that rings are not necessary to be a Noetherian. Claus Michael Ringel has collected several conditions for a ring being hereditary.

In this paper, first of all, we have taken some definitions which are related to the Hereditary ring.

Semi-simple ring, hereditary ring, pure hereditary ring and related example have taken. Then we have taken a proposition to show a condition for a ring *P* being left hereditary and left global dimension of the ring at most one. The ring is associative with unit, triangular matrix ring and the same way Noetherian hereditary ring also. We have included a lemma related to left hereditary ring.

This paper aims to demonstrate that a ring is left. Hereditary as well as left pure hereditary but not a right semi-hereditary ring . Accordingly the ring does not satisfy the property of being Right pure hereditary. For this purpose we have taken a triangular matrix ring which can be left hereditary or right hereditary but not Pure Hereditary.

2. Hereditary ring

Consider a ring *R* which is called as a right Hereditary when each sub-module of projective module being called projective (as right *R*-module) and any module which is factor module of an injective module considered as injective. If ring *R* is not only right hereditary but also left hereditary then we may consider the ring is a Hereditary ring. For example:

i) Triangular matrix ring $\begin{bmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{bmatrix}$ is right Hereditary and semi-hereditary (left) but not hereditary (Left) because radical of this triangular matrix ring is (flat), but not projective. On the other hand the matrix ring $\begin{bmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Z} \end{bmatrix}$ is left Hereditary.

ii) The triangular matrix ring $\begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$ is also left hereditary.

iii) Every semi-simple ring is Hereditary (**Semi-simple ring**) The ring *R* is said as a semi-simple (left) if ring is semi-simple as a module (left) through itself A semi-simple (left) ring appears to be right as well semi-simple and proportionally. Hence, one often drops the left/ right quantifier altogether and simply speaks of semi-simple rings. Considering *R* as a base ring which is semi-simple, then the group of R-modules would be automatically semi-simple. Furthermore, every R-module which is Simple must be isomorphic mapping into a left ideal and minimal of *R* [14].

iv) The property is already established that if and only if , the ring is right Hereditary is right and left Noetherian ring is left hereditary.

3. Pure hereditary ring

The ring R is called a pure hereditary (left) ring if each R module (left) is pure injective as like as right pure hereditary can be defined also. A pure hereditary (left) ring is must a hereditary (left) too.

Example: i) $R = \mathbb{Z},$ where the ring $\mathbb{Z}$ is all integers. Although this is pure hereditary not pure semi-simple. ii) Let $R = \mathbb{Z}/n\mathbb{Z},$ where n is free of Square. Then this R is semi-simple Artinian ring and so, it must be pure hereditary.

Proposition 1: For the ring R, if each ideal of ring R which is left ideal is projective then the ring R is hereditary (Left) and the global dimension (left) of ring R is at most one.

Proof: Consider the mapping $\varphi : I \to M,$ where I considered as an injective module of R also φ is surjective. We have to establish that for each. $R^L \leq R^R$ left ideal and every map $\theta : L \to M$ the extension of mapping will be $\omega : R \to M.$ Consider the following diagram

Since L is projective thus $\overline{f} : L \to I$ exists, and I is injective that's why $\tilde{f}$ also possible. Now $\omega = \varphi\overline{f} : R \to M$ is an extension of *f*. According to the injective test lemma, we may say *M* is injective, which proves that R is a left Hereditary. Now if a ring R is left Hereditary then surely the left global dimension of *R* is at most 2 because each R module (left) has an injective sensitivity to length at most 1. Thus the $Ext_R^2(-, N) = 0$ for every $N \in R$ -mod. Thus $l\,gl\dim R = 1.$

Corollary 2: *The submodule of projective R modules must be projective in order for the ring R to be right hereditary [13].*

Proposition 3 : A triangular Matrix ring R is a Noetherian Hereditary ring.

Proof: Let $\begin{pmatrix} r & m \\ 0 & s \end{pmatrix}$ and $\begin{pmatrix} r' & m' \\ 0 & s' \end{pmatrix}$ be two triangular matrix rings where the addition and multiplication are routine computation -wise.

Consider the ring R is defined by $e'Re = 0,\ e' = 1 - e$ and e is an idempotent element of R and $f(a) = \begin{pmatrix} ere & ere' \\ 0 & e're' \end{pmatrix}$ is isomorphic. According to the definition of the Noetherian Hereditary ring if N(R) is collection of ideal which is maximal nilpotent of Noetherian hereditary ring then $eN(R)e' = 0$ [13]. Hence, ring R is Noetherian Hereditary ring also.

Lemma 4: *The ring* $\begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$ *is left hereditary (left) if*

(a) *Rings R and S considered as left hereditary,*

(b) *Module M_s must flat,*

(c) *For each left ideal I of the ring S, the R-module M | M1 must be R-projective. And vice versa [6]*

Proof: Let $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $f = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ for every $r \in R, fr = frf$.

This implies $a \Rightarrow b$ Since R is semi hereditary on the left it is the same for $ere \simeq R$ and $frf = S$. Hence rings R and S must be Hereditary (left). Let T be the ideal defined by $\begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix}$; because of being the Ring R is semi–Hereditary left. The module T_R is flat. We can easily deduce that Tf is a flat straight fRf-module. This is equivalent to saying M_s is flat.

As R is semi-hereditary on the left, erf is a semi-hereditary left eRe and consequently R^M is semi-hereditary. Let $m_i; i$ belongs to $I, s_i; i \in I$ be considered as the families of components of M and S respectively. We presume for each $i \in I : Y_i = \begin{pmatrix} 0 & m_i \\ 0 & s_i \end{pmatrix};\ \ Z_i = \begin{pmatrix} 0 & 0 \\ 0 & s_i \end{pmatrix} = fY_i$ and $K = \sum_{i \in I} RY_i;$ $L = \sum_{i \in I} RZ_i$, after all if R^K is projective, R^L is a factor of R^K (direct).

Consider $(\varnothing_i)_{i \in I}$ as a collection of linear forms on R^K such a way for each and every $Y \in K$, $[\varnothing_i(Y)]_{i \in I}$ be finite support and $Y = \sum_{i \in I} \varnothing_i(Y) Y_i;$

We define homomorphism $\varnothing$ of R^K in R^L by assuming: $\varnothing(Y) = \sum_{i \in I} \varnothing_i(Y) Z$

On the other hand, we assume for every *j* contains in *I*:

$$Z_j = \sum_i \varnothing_i(Z) Y_i = \sum_i \varnothing_i(fZ_j) Y_i = \sum_{i \in I} f \varnothing_i(Z_j) fY_i = \sum_{i \in I} \varnothing_i(Z_j) Z_i = \varnothing(Z_j)$$

Thus $\varnothing$ induces the identity on R^L and R^L is indeed a direct factor of R^K.

Otherways $K = \begin{pmatrix} 0 & \Sigma Rm_i + \Sigma Ms_i \\ 0 & \Sigma Ss_i \end{pmatrix}$; $L = \begin{pmatrix} 0 & \Sigma Ms_i \\ 0 & \Sigma Ss_i \end{pmatrix}$, We deduce that if $R^K = R^L \oplus L'$. L' is the form $\begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix}$ where N is the submodule of R^M such that $\sum_i Rm_i + \sum_i Ms_i = N \oplus (\sum_i Ms_i)$. Therefore if R^K is projective then the submodule $\frac{\Sigma Rm_i + \Sigma Ms_i}{\Sigma Ms_i}$ of $R^{\left(\frac{M}{\Sigma Ms_i}\right)}$ is isomorphic of N. Then immediately we conclude c).For each ideal (left) of S is J, the R-module $M|MI$ is R-Projective.Conversely ii) $\Rightarrow$ i) As R^R is semi hereditary the ideal $Re = \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix}$ is semi- hereditary;

Let us show that it is the same for *Rf*. Let $\left[Y_i = \begin{pmatrix} 0 & m_i \\ 0 & s_i \end{pmatrix}, i \in I \right]$ be a finite family element of *Rf*. According to c) we can find elements m_i, where i is in I of M such that $\Sigma_i Rm_i + (\Sigma_i Ms_i) = \Sigma_i Rm'_i \oplus (\Sigma_i Ms_i)$. Then we have $\Sigma_{i \in I} RY_i = \Sigma_i R \begin{pmatrix} 0 & m'_i \\ 0 & 0 \end{pmatrix} \oplus \left[S_i R \begin{pmatrix} 0 & 0 \\ 0 & s_i \end{pmatrix} \right]$; We will consider for every i, $Z_i = \begin{pmatrix} 0 & 0 \\ 0 & s_i \end{pmatrix}$

The submodule $\Sigma_i Rm'_i$ of R^M being projective, the same is true for the left ideal of $R : \Sigma_i R \begin{pmatrix} 0 & m'_i \\ 0 & 0 \end{pmatrix}$. Else the hypothesis made on S leads to the existence of a family of linear shapes $(f_i)_{i \in I}$ on the left ideal $= \Sigma_{i \in I} Ss_i$, such as for every $s \in T, (f_i(s))_{i \in I}$ at finite $s = \Sigma_i f_i(s) s_i$ either f is a linear form on S^T; As M_S is a flat relation of the $\Sigma_i \mu_i s_i = 0$ or $(\mu_i) \in M^{(I)}$; Continuing $\Sigma_i \mu_i f(s_i) = 0$. Therefore, for every $i \in I$, Form a linear form F_i. Surely ΣRZ_j by asking $F_i(Z_j) = \begin{pmatrix} 0 & 0 \\ 0 & f_i(s_j) \end{pmatrix}$; For every $Z \in \Sigma RZ_j$, $(F_i(Z))_{i \in I}$ is finite support and $Z = \Sigma_i F_i(Z) Z_i$. The ideal on the left $\Sigma_i RZ_i$ and therefore the ideal has left ΣRY_i are therefore Projective. Consider S as a finite ideal which is left of R. Considering the sequence which is exact $0 \to S \cap Rf \to S \to Se \to 0$. And the fact is that *Re* and *Rf* can be semi hereditary left ideal show that R^S must be projective.

Proposition 5: For any associative ring R with the unit is left semi Hereditary but not semi-Hereditary (right) although its will be right Hereditary. [2]

Proof: Let R is the ring of all two-by-two matrices $R = \begin{pmatrix} r & m \\ 0 & s \end{pmatrix}$ where $r \in R$, $m \in M$ and $s \in S$; Consider T is a collection of components of $R$of the structure $T = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}$. A mapping $\alpha : R' \to T$ defined by $\alpha(m) = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}$ must be an isomorphism.

Set $B = B' / J$ is a regular ring, where B' is a ring which is commutative and regular although not a semi-simple ring [2]. And B' has an ideal namely J. Here the right module of B' is B. Because of B being finitely generated and cyclic over B' which gives by the definition of mapping R' and $R'(B)$ is finitely generated. Thus T is generated finitely and a right ideal of ring R. we may consider $hd_R T = hd_R T' = hd_{M'} M > 0.$

This says that the ring R can't be semi-hereditary (right).

Now we want to establish that R is right hereditary for this purpose consider a right ideal of R [2] is

$\begin{pmatrix} r & a \\ 0 & b \end{pmatrix}$ Where $r \in K$, $(a, b) \in Q$ where an ideal (right) of R is K and the subspace of $F \oplus B$ is Q, in this way $KM \subset Q$. Considering an ideal (right) I is in this, implies are no elements $i = \begin{pmatrix} r & a \\ 0 & b \end{pmatrix}$ with $r \neq 0$, then I is projective by Chase. Now let a right ideal $N = \begin{pmatrix} r & c \\ 0 & 0 \end{pmatrix}$ where $r \in K$, $c \in C$.

Here B has a subspace called C. By Kaplansky theorem there exist an idempotents seequence $e_1, e_2, \ldots e_n \ldots$ in K which shows that $e_1 R \subset e_2 R \subset \ldots \subset e_n R \subset \ldots$

And $K = \cup_{n=1} e_n R$ set $L_n = \begin{pmatrix} e_n & 0 \\ 0 & 0 \end{pmatrix} N$ now it can be easily seen that $L_1 \subset L_2 \subset \ldots \subset L_n \subset \ldots$ Here each L_n is a direct summand of L_{n+1}.

Again, by Kaplansky's results $N_1 = \cup_{n=1}^{\infty} L_n$ is projective and $N_2 = \begin{pmatrix} 0 & \overline{c} \\ 0 & 0 \end{pmatrix}$ where $\overline{c}$ is containing an orthogonal complement of IC in C.

In the same way N_2 is projective. That means $N = N_1 + N_2$ and N is projective. Thus according to the definition R is a ring which is right Hereditary.

Last part of this proposition is R is left semi-Hereditary. For this purpose, consider $P = (\beta_1, \beta_2, \ldots \beta_n)$ as a left ideal which is generated finitely in R, where $\beta_i = \begin{pmatrix} a_i & m_i \\ 0 & bi \end{pmatrix} a_i$, $m_i \in B$ & $b_i \in B'$

Since B' is regular we may write $B'b_1 + B'b_2 + \ldots + B'b_n = B'e$ for some idempotent e in B'. Now $\mu_1 b_1 + \mu_2 b_2 + \ldots + \mu_n b_n = e$ whereas $\mu_1 \in B'$;

Then $\begin{pmatrix} 0 & 0 \\ 0 & \mu_i b_i \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \mu_i \end{pmatrix}\begin{pmatrix} a_i & m_i \\ 0 & b_i \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \mu_i \end{pmatrix}\beta_i \in P$. Considering $\varepsilon = \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix}$; We have ε is in P and so $R\varepsilon \subseteq P$; where $R\varepsilon$ contains all elements of R of the form $\begin{pmatrix} 0 & me \\ 0 & be \end{pmatrix}$; $m \in B,\ b \in B'$ …(i). Since $b_i = b_i e$ for all $i = 1, 2, \ldots n$. Thus $\begin{pmatrix} a_i & m_i(1-e) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a_i & m_i \\ 0 & b_i \end{pmatrix} - \begin{pmatrix} a_i & m_i e \\ 0 & b_i e \end{pmatrix} \in P$ ….(ii). Let B be the left sub-module of $B \oplus B$ generated with its components $(a_i, m_i(1-e))$; $i = 1, 2, \ldots n$. Consider the mapping $f : U(L) \oplus R\varepsilon \to R$. Thus $\{(m, n) + \beta\varepsilon\} = \left\{\begin{pmatrix} m & n \\ 0 & 0 \end{pmatrix} + \beta\varepsilon\right\} = \begin{pmatrix} m & n+xe \\ 0 & be \end{pmatrix}$; Where $\beta = \begin{pmatrix} a & x \\ 0 & b \end{pmatrix}$; $a, x \in B,\ b \in B'$. We may verify this easily that *f* is a left R-module homomorphism and Image $(f) \subseteq P$ and $f(a_i, m_i(1-e)) = \begin{pmatrix} a_i & m_i(1-e) \\ 0 & 0 \end{pmatrix}$. That means $R\varepsilon \subseteq \text{Image}(f)$ Now from (i) and (ii) $\text{Image}(f) = P$. Consider $f\{(m, n) + \beta\varepsilon\} = 0$ where $(m, n) \in L$. By the definition of $Lv = v(1-e)$ and we may write $f\{(m, n) + \beta\varepsilon\} = \begin{pmatrix} m & n \\ 0 & 0 \end{pmatrix} + \beta\varepsilon = \begin{pmatrix} m & n(1-e)+xe \\ 0 & be \end{pmatrix} = 0$. Then $m = be = n(1-e) + xe = 0$. Which gives $n(1-e) = 0$ & $xe = 0$. Because e is an idempotents elements of B'. Thus $(m, n) = 0$ and $\beta\varepsilon = \begin{pmatrix} a & x \\ 0 & b \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix} = \begin{pmatrix} 0 & xe \\ 0 & be \end{pmatrix} = 0$. Accordingly $(m, n) + \beta\varepsilon = 0$. Which implies *f* is an isomorphism; That is $P \approx U(L) \oplus R\varepsilon$ as a left R –module. Here L is a sub-module which is generated finitely of left module B and free $B \oplus B$ [8] .Because B is a regular ring also semi-hereditary (left) [2]. Since L is B-projective that means $R\varepsilon$ is a direct summand of $= R(\varepsilon$ is an idempotent) By Chase $hd_R P = hd_R U(L) = hd_B L = 0$. Thus every finely generated left ideal of R projective and therefore R is left semi hereditary [9]

Proposition 6: A ring R is hereditary (left) iff R is left pure hereditary and left semi-hereditary [4]

Proposition 7: Ring R can not be a right Pure Hereditary Ring.[1]

4. Conclusion

The triangular matrix ring [*P*, *I*] which is associative with units not a Pure (right) Hereditary Ring although this can be a left pure Hereditary. Despite the fact that this is my first paper ever on Hereditary ring but I wish to work with different examples and application of hereditary ring in real life as well as medical science for instance in cancer treatment.

5. Acknowledgment

This research is funded by Miyan Research Institute (MRI), International University of Business Agriculture and Technology (IUBAT), Bangladesh.

References

[1] Armendariz, Efraim P., and C. R. Hajarnavis. "On Prime Ideals in Hereditary PI-Rings." *Journal of Algebra*, vol. 116, no. 2, (1988), pp. 502–05.

[2] Chase, Stephen U. "A Generalization of the Ring of Triangular Matrices." *Nagoya Mathematical Journal*, vol. 18, (1961), pp. 13–25.

[3] Dicks, Warren. "Hereditary Group Rings." *Journal of the London Mathematical Society*, vol. s2-20, no. 1, (1979), pp. 27–38.

[4] Geng, Yuxian, and Nanqing Ding. "Pure Hereditary Rings." *Communications in Algebra*, vol. 37, no. 6, (2009), pp. 2127–41.

[5] Guil Asensio, Pedro A., and Dilek Pusat. "Hereditary Rings with Countably Generated Cotorsion Envelope." *Journal of Algebra*, vol. 403,(2014), pp. 19–28.

[6] Kirkman, Ellen, and James Kuzmanovich. "Right Hereditary Affine PI Rings Are Left Hereditary." *Glasgow Mathematical Journal*, vol. 30, no. 1, (2009), pp. 115–20.

[7] Kirkman, Ellen E., and James Kuzmanovich. "Hereditary Finitely Generated Algebras Satisfying a Polynomial Identity." *Proceedings of the American Mathematical Society*, vol. 83, no. 3, (1981), p. 461.

[8] Krylov, P. A., and A. A. Tuganbaev. "Modules over Discrete Valuation Domains. II." *Journal of Mathematical Sciences*, vol. 151, no. 5, (2008), pp. 3255–371.

[9] Moradzadeh-Dehkordi, A., and S. H. Shojaee. "Rings in Which Every Ideal Is Pure Projective or FP-Projective." *Journal of Algebra*, vol. 478, (2017), pp. 419–36.

[10] Page, Annie. "Sur Les Anneaux Hereditaires Ou Semi-Hereditaires." *Communications in Algebra*, vol. 6, no. 11, (1978), pp. 1169–86.

[11] Small, Lance W. "AN EXAMPLE IN NOETHERIAN RINGS." *Proceedings of the National Academy of Sciences*, vol. 54, no. 4,(1965), pp. 1035–36.

[12] Small, Lance W. "HEREDITARY RINGS." *Proceedings of the National Academy of Sciences*, vol. 55, no. 1, (1966), pp. 25–27.

[13] T.Y.Lam. "Lecture on modules and rings." Springer Science and business media LLC, (1999)

[14] http://en.wikipedia.org/wiki/Semisimple_module,(2010).

[15] Mohammad Saleh, and Hasa. Yousef , "The Number of Ring Homomorphisms From $Z_{m1} \times \ldots \times Z_{mr}$ into $Z_{k1} \times \ldots \times Z_{ks}$." The American Mathematical Monthly, 105:3, (1998),pp.259-60.