Generalized observers for a class of nonlinear systems

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Abstract—A general class of MIMO nonlinear systems with unknown inputs is considered with a view to observer synthesis. The particularity of this class of systems lies in the fact that the expression of the outputs depends on the unknown inputs. Different situations are considered and in each case a nonlinear observer, synthesized under appropriate hypotheses, is proposed to jointly estimate all state variables together with the unknown inputs. Simulation results are given in order to highlight the performances of the proposed observers.

I. INTRODUCTION

The design of observers for systems with unknown inputs still be the subject of intensive researches. Many results are available for linear systems and in most of these works, the objective is to estimate the non measured state independently of the unknown inputs [15], [14], [17], [11], [8], [3]. In relatively recent results, nonlinear systems with unknown inputs have been considered and many approaches have been proposed for the on-line estimation of the non measured states simultaneously with the unknown inputs [2], [16], [5]. However, in all these works, the expression of the considered outputs is independent of the unknown inputs. In [1] and [9], the authors considered classes of nonlinear systems where the dynamics of the state is composed of a linear part and a nonlinearity satisfying some assumptions which make possible the design of an observer to simultaneously estimate the non measured states and the unknown inputs. In these works, the expression of the considered output does depend on the unknown inputs. In this paper, one considers a general class of nonlinear systems where the output directly depends on the unknown input. Different situations are considered and in each case a nonlinear observer, synthesized under appropriate hypotheses, is proposed to jointly estimate all state variables together with the unknown inputs.

This paper is organized as follows. In the next section, the class of nonlinear systems which is the basis of the observer design is introduced. Section 3 is devoted to the observer synthesis: different situations are analyzed and discussed and in each case an observer synthesized under appropriate assumptions is proposed. In section 4, simulation results are given in order to illustrate the performances of the proposed observers.

II. PROBLEM FORMULATION

Consider MIMO systems of the form:

\[
\begin{aligned}
\dot{x} &= f(u, s, x, v) \\
y &= h(u, x) + W(u, s)v
\end{aligned}
\]

(1)

where the state \(x = \left(\begin{array}{c}x^1 \\X\end{array}\right) \in \mathbb{R}^n\), \(x^1 \in \mathbb{R}^p\), \(X \in \mathbb{R}^{n-p}\), the known input \(u(t) \in U\) the set of bounded absolutely continuous functions with bounded derivatives from \(\mathbb{R}^+\) into \(U\) a compact subset of \(\mathbb{R}^m\); \(v \in \mathbb{R}^m\) with \(m \leq p\) is the unknown input; \(f(u, s, x, v) = \left(\begin{array}{c}f^1(u, s, x, v) \\f_X(u, s, x, v)\end{array}\right) \in \mathbb{R}^n\), \(f^1(u, s, x, v) \in \mathbb{R}^p\), \(f_X(u, s, x, v) \in \mathbb{R}^{n-p}\); the output \(y \in \mathbb{R}^p\), the function \(h(u, x)\) is such that \(\frac{\partial h}{\partial x^1}(u, x)\) is of full rank for all \(x \in \mathbb{R}^n\) and \(u \in U\) (i.e. \(\text{Rank} \left(\frac{\partial h}{\partial x^1}(u, x)\right) = p\) for all \(x \in \mathbb{R}^n\) and \(u \in U\)); \(s(t)\) is a known signal with a bounded time derivative and \(W(u, s)\) is a \(p \times m\) known matrix which assumes the following structure:

\[
W(u, s) = \left(\begin{array}{c}W_1(u, s) \\W_2(u, s)\end{array}\right)
\]

(2)

where \(W_1(u, s)\) is a \(m_0 \times \mu_1\) full rank column matrix and \(W_2(u, s)\) is the remaining \((p - m_0) \times \mu_1\) sub-matrix with \(0 < \mu_1 \leq m_0 \leq m\). This means that:

\[
\forall u \in U; \forall t \geq 0 : \text{rank}(W_1(u, s)) = \mu_1
\]

(3)

Let \(m_1\) be the smallest possible value of \(m_0\). Notice that in the case where the matrix \(W_1\) is constant, one always has \(m_1 = m\). This is generally no longer true if \(W_1(u, s)\) is time varying as it is shown by the following example:

\[
W_1(t) = \left(\begin{array}{c}\sin(t) \\\cos(t)\end{array}\right)
\]

(4)

One has \(m = 1\) and \(m_1 = 2\).

In the sequel, \(h(u, x) = \left(\begin{array}{c}h_1(u, x) \\h_2(u, x)\end{array}\right)\) shall denote the partition of \(h(u, x)\) induced by that of \(W(u, s)\) under the form (2).

Our objective consists in synthesizing an observer to simultaneously estimate the state \(x\) and the unknown input \(v\).
III. OBSERVER DESIGN

As stated above, two situations will be considered depending on whether \( \mu_1 = m \) or \( \mu_1 < m \). In each case, appropriate additional hypothesis shall be assumed and an observer shall be designed for the simultaneous estimation of the state and the unknown inputs.

A. Observer synthesis with \( \text{Rank}(W(u, s)) = \mu_1 = m \)

Consider the following change of coordinates:

\[
T : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^{n+m}
\]

\[
\xi = \begin{pmatrix} x \\ v \end{pmatrix} \quad \rightarrow \quad T(\xi) = \begin{pmatrix} \bar{x} = x \\ \hat{v} = h_1(u, x) + W_1(u, s)v \end{pmatrix}
\]

Simple calculations show that the map \( T \) puts system (1) under the following form:

\[
\begin{cases}
\dot{x} = f(u, s, \bar{x}, W_1^+(y_1 - h_1(u, \bar{x}))) \\
\dot{\hat{v}} = \hat{y}_1 \\
y_2 = h_2(u, \bar{x}) - W_2(u, s)W_1^+(u, s)h_1(u, \bar{x}) + W_2(u, s)W_1^+(u, s)y_1
\end{cases}
\]

(6)

It is clear that the second equation in (6) is not useful and is to drop. The remaining system becomes:

\[
\begin{cases}
\dot{x} = f(u, s, \bar{x}, W_1^+(y_1 - h_1(u, \bar{x}))) \\
y_2 = h_2(u, \bar{x}) - W_2(u, s)W_1^+(u, s)h_1(u, \bar{x}) + W_2(u, s)W_1^+(u, s)y_1
\end{cases}
\]

(7)

Again, one shall consider two situations depending on whether \( m_1 < p \) or \( m_1 = p \) where \( m_1 \) is the smallest possible value of \( m_0 \) as stated previously in section 2.

A.1. Observer synthesis with \( \text{Rank}(W_1(u, s)) = m \) and \( m_1 < p \)

The initial observation problem becomes standard since the objective is now to design a conventional observer for the following class of nonlinear systems:

\[
\begin{cases}
\dot{x} = \tilde{f}(u, s, y_1, \bar{x}) \\
y = f^0(y_1, \bar{x})
\end{cases}
\]

where \( \bar{x} \) is the state, \( (s^T, u^T, y_1^T)^T \) is the input, \( y \) is the output with \( \tilde{f}(u, s, y_1, \bar{x}) \triangleq f(u, s, \bar{x}, W_1^+(y_1 - h_1(u, \bar{x}))) \) and \( f^0(y_1, \bar{x}) = h_2(u, \bar{x}) - W_2(u, s)W_1^+(u, s)h_1(u, \bar{x}) + W_2(u, s)W_1^+(u, s)y_1 \).

It is well known that such a problem still be open in spite of the big amount of available results (see e.g. [12], [13], [7], [10]). One can refer for example to these works to achieve the estimation of \( \bar{x} \). Notice that, as soon as an observer is designed for the estimation of \( \bar{x} = x \), an estimation \( \hat{v} \) of the unknown input \( v \) can be recovered by coming back the original coordinates. Indeed, one has:

\[
\hat{v}(t) = W_1^+(u, s)(y_1 - h_1(u, \bar{x}))
\]

(9)

where \( \hat{x} \) is the estimate of \( x \).

A.2. Observer synthesis with \( \text{Rank}(W_1(u, s)) = m \) and \( m_1 = p \)

\( y_2 \) is to drop from system (7) since \( W_1(u, s) \) and \( y_1 \) respectively coincide with \( y \) and \( W(u, s) \). System (7) becomes:

\[
\dot{\bar{x}} = \tilde{f}(u, s, \bar{x}, \bar{y})
\]

(10)

where \( \tilde{f}(u, s, \bar{x}, \bar{y}) \triangleq f(u, s, \bar{x}, W^+(u, s)(y - h(u, \bar{x}))) \). To design an observer for system (10), one assumes the following detectability condition:

(C) There exists a time varying matrix \( P(t) \) such that:

(i) \( \exists \alpha_1, \alpha_2 > 0 ; \forall t \geq 0 : \alpha_1 I_n \leq P(t) \leq \alpha_2 I_n \)

(11)

(ii) \( \exists \alpha_3 > 0 ; \forall x \in \mathbb{R}^n ; \forall u \in U ; \forall t \geq 0 : \]

\[
P(t) + P(t) \frac{\partial \tilde{f}}{\partial u}(u, s, y_1, \bar{x}) + \left( \frac{\partial \tilde{f}}{\partial y_1}(u, s, y_1, \bar{x}) \right)^T P(t) \leq -\alpha_3 I_n
\]

(12)

Under condition (C), one shows that the following dynamical system is an exponential observer for system (10):

\[
\dot{\bar{x}} = \tilde{f}(u, s, \bar{x}, \bar{y})
\]

(13)

where \( \bar{x} \in \mathbb{R}^n \).

It is easy to see that observer (13) can be written in the original coordinates as follows:

\[
\dot{x} = f(u, s, \bar{x}, W^+(y - h(u, \bar{x})))
\]

where \( \bar{x} \in \mathbb{R}^n \). As previously, the unknown input can be recovered from equation (9).

Remark 3.1: It is easy to show that Part (ii) of Condition (C) can be formulated using the original coordinates as follows:

\[
P(t) + P(t) \frac{\partial f}{\partial x}(u, s, x, W^+(u, s)(y - h(u, x))) - \frac{\partial f}{\partial v}(u, s, x, W^+(u, s)(y - h(u, x)))W^+(u, s) \frac{\partial h}{\partial x}(u, x) + \left( \frac{\partial f}{\partial v}(u, s, x, W^+(u, s)(y - h(u, x))) \right)^T P(t) \leq -\alpha_3 I_n
\]

(14)

B. Observer synthesis with \( \text{Rank}(W(u, s)) = \mu_1 < m \)

To simplify, one shall assume that \( W(u, s) \) is under the following form:

\[
W(u, s) = \begin{pmatrix} W_1(u, s) & 0 \\ W_2(u, s) & 0 \end{pmatrix}
\]

(15)

where \( W_1(u, s) \) is a \( m_1 \times m_1 \) full rank column matrix with \( 0 < \mu_1 \leq m_1 < m \) and \( W_2(u, s) \) is the remaining \( (p - m_1) \times \mu_1 \) sub-matrix.
Set \( v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \) with \( v_1 \in \mathbb{R}^{n_1}, v_2 \in \mathbb{R}^{n-m_1} \) and \( G(u, s) = [G_1(u, s) \quad G_2(u, s)] \) where \( G_1(u, s) \) and \( G_2(u, s) \) are respectively \( n \times m_1 \) and \( n \times (m-m_1) \) matrices and respectively denote the partitions of \( v \) and \( G(u, s) \) issued from that of \( W \) under form (15). Now, consider the following change of coordinates:

\[
T : \mathbb{R}^{n+m_1} \longrightarrow \mathbb{R}^{n+m_1} \\
\xi = \begin{pmatrix} x \\ v_1 \end{pmatrix} \Rightarrow T(\xi) = \begin{pmatrix} \hat{x} = x \\ \hat{v}_1 = h_1(u, x) + W_1(u, v_1) \end{pmatrix}
\]

Simple calculations show that the map \( T \) puts system (1) under the following form:

\[
\begin{align*}
\dot{\hat{x}} &= f(u, s, \hat{x}, s, W_1^+(y_1-h_1(u, \hat{x})), v_2) \\
\dot{\hat{v}}_1 &= y_1 \\
y_2 &= h_2(u, \hat{x}) - W_2(u, s)W_1^+(u, s)h_1(u, \hat{x}) \\
&+ W_2^{1}(u, s)W_1^{r}(u, s)y_1
\end{align*}
\]

Again, the second equation of system (17) is not useful for the observation problem and is dropped. The remaining subsystem is:

\[
\begin{align*}
\dot{\tilde{x}} &= f(u, s, \tilde{x}, s, W_1^+(y_1-h_1(u, \tilde{x})), v_2) \\
y_2 &= h_2(u, \tilde{x}) - W_2(u, s)W_1^+(u, s)h_1(u, \tilde{x}) \\
&+ W_2(u, s)W_1^+(u, s)y_1
\end{align*}
\]

Notice that, the unknown input \( v_2 \) intervenes in a nonlinear manner in (18) and the problem of designing an observer for the simultaneous estimation of the state \( x \) and the unknown input \( v_2 \) still be open. We shall propose a solution for this problem in the case where the dependence between \( f \) on \( v_2 \) is affine. Indeed, one assumes that:

(A) The function \( f(u, s, \tilde{x}, W_1^+(y_1-h_1(u, \tilde{x})), v_2) \) is such that:

\[
\frac{\partial f}{\partial v_2}(u, s, \tilde{x}, W_1^+(y_1-h_1(u, \tilde{x})), v_2) = G_2(u, s)
\]

Assumption (A) means that the function \( f(u, s, \tilde{x}, W_1^+(y_1-h_1(u, \tilde{x})), v_2) \) can be written under the following form:

\[
f(u, s, \tilde{x}, W_1^+(y_1-h_1(u, \tilde{x})), v_2) \Delta f(u, s, y_1, \tilde{x}) + G_2(u, s)v_2
\]

Under Assumption (A), system (18) becomes:

\[
\begin{align*}
\dot{\tilde{x}} &= f(u, s, y_1, \tilde{x}) + G_2(u, s)v_2 \\
y_1 &= \tilde{y} = f_0(y_1, \tilde{x})
\end{align*}
\]

where \( \tilde{x} \) is the state, \( (s^T, u^T, y_1^T)^T \) is the known input, \( \tilde{y} \) is the output, \( v_2 \) is the unknown input with:

\[
f_0(y_1, x) \Delta h_2(u, \tilde{x}) - W_2(u, s)W_1^+(u, s)h_1(u, \tilde{x}) \\
+ W_2(u, s)W_1^+(u, s)y_1
\]

Now, system (21) belongs to the class of systems considered in [4] for which the authors have proposed an observer to simultaneously estimate the state \( \tilde{x} \) and the unknown input \( v_2 \). Of course, an estimate \( \hat{v}_1 \) of the the unknown input \( v_1 \) can be then recovered by inverting (17) i.e.

\[
\hat{v}_1(t) = W_1^+(u, s)(y_1 - h_1(u, \tilde{x}))
\]

where \( \hat{x} \) is the estimate of \( x = x \).

### IV. Examples

Two examples are given in this section to illustrate the performance of the proposed observers.

#### A. Example 1

Consider the well-known Lorenz chaotic system:

\[
\begin{align*}
\dot{x} &= Ax + Bu + g(x, v) \\
y &= x_1 + v
\end{align*}
\]

where the state \( x = (x_1, x_2, x_3)^T \in \mathbb{R}^3 \), the unknown input \( v \in \mathbb{R}, A = \begin{pmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -8 \end{pmatrix}, B = \begin{pmatrix} 30 \\ 28 \end{pmatrix} \)

\[
g(x, v) = \begin{pmatrix} -(x_1 + v)x_3 \\ (x_1 + v)x_2 \end{pmatrix}
\]

It is clear that system (23) is under form (1) with \( f(x, v) = Ax + Bu + g(x, v), h(u, x) = Cx \) with \( C = [1 \ 0 \ 0] \) and \( W = 1 \). Since \( \text{Rank}(W) = m = m_1 = p = 1 \), let us check (14). One has:

\[
\begin{pmatrix}
\frac{\partial f}{\partial x}(x, v) - \frac{\partial f}{\partial v}(x, v)W^{-1}C \\
A(y) + A^T(y)
\end{pmatrix} = \begin{pmatrix} -40 & 10 & 0 \\ 0 & -1 & -y \\ 0 & 8 & -3 \end{pmatrix} = \begin{pmatrix} -40 & 10 & 0 \\ 0 & -1 & -y \\ 0 & 8 & -3 \end{pmatrix} \leq -\alpha_3 I_3
\]

where \( \alpha_3 = 41 - \sqrt{1621} > 0 \). Thus, conditions (11) and (14) are satisfied with \( P = I_3 \) and an observer of the form (13) can be used for the estimation of \( x \) and \( v \). The equations of the observer are:

\[
\begin{align*}
\dot{\hat{x}} &= A\hat{x} + B\hat{v} + g(\tilde{x}, y - \hat{x}) \\
\hat{v} &= y - \hat{x}_1
\end{align*}
\]

where \( \hat{x} = [\hat{x}_1 \hat{x}_2 \hat{x}_3]^T \in \mathbb{R}^3 \) and \( \hat{v} \in \mathbb{R} \).

For simulation purposes, the following expression of \( v \) is used: \( v = 0.5 \sin(60\pi t) \). Estimation results provided by this observer are given in figure 1 which clearly shows the good performance of the observers since the estimations errors quickly decay to zero.

#### B. Example 2

Consider the following dynamical system:

\[
\begin{align*}
\dot{x}_1 &= (2 + \sin(x_1))x_2 - 2x_1 + v \\
\dot{x}_2 &= -x_2^3 - v \\
y_1 &= x_1 - v \\
y_2 &= x_2^3 + v
\end{align*}
\]

where the state \( x = (x_1, x_2)^T \in \mathbb{R}^2 \) and the unknown input \( v \in \mathbb{R} \). One has: \( W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} \) with \( W_1 = -1, W_2 = 1 \).
This means that $\text{Rank}(W_1) = m = m_1 = 1 < p = 2$. Simple calculations show that system (25) can be written under form (8) as follows:

$$\begin{align*}
\dot{x}_1 &= (2 + \sin(x_1))x_2 - x_1 - y_1 \\
\dot{x}_2 &= -x_2^3 + y_1 - x_1 \\
y_2 &= x_1^3 + x_1 - y_1
\end{align*}$$

System (26) is under a nonlinear canonical that characterizes uniformly observable systems [6] and the following observer can be used to estimate $x_1$, $x_2$:

$$\begin{align*}
\dot{\hat{x}}_1 &= (2 + \sin(\hat{x}_1))\hat{x}_2 - \hat{x}_1 - y_1 \\
\dot{\hat{x}}_2 &= -\hat{x}_2^3 - \hat{x}_1 + y_1 \\
&\quad - \frac{2\theta}{(1 + 3\hat{x}_1^2)}(\hat{x}_1^3 + \hat{x}_1 - y_1 - y_2)
\end{align*}$$

The unknown input can then be recovered using equation (9) which specializes here as follows:

$$\hat{v} = \hat{x}_1 - y_1$$

For simulation purposes, the following expression of $v$ is used: $v = 5\sin(60\pi t)$. Estimation results given in Fig2 clearly show the good performance of observer II since the estimation errors quickly decay to zero in spite of the relatively large magnitude and frequency of the unknown input signal.

V. CONCLUSION

Nonlinear observers have been proposed for a general class of nonlinear systems where the considered outputs are directly correlated to the unknown inputs. Different situations have been considered and appropriate observers have been proposed to jointly estimate the non measured state and the unknown inputs. The performance of some of the proposed observers are illustrated in simulation through two illustrative examples.

\textbf{REFERENCES}


