MIMO radar signal design to improve the MIMO ambiguity function via maximizing its peak

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Abstract
One of the important obstacles in MIMO (Multiple Input Multiple Output) radars is the issue of designing proper transmit signals. Indeed, the capability of signal design is a significant advantage in MIMO radars, through which, the system can achieve much better performance. Many different aspects of this performance improvement have been considered yet, and the transmit signals have been designed to attain such goal, e.g., getting higher SNR or better detector’s performance at the receiver. However, an important tool for evaluating the radar’s performance is its ambiguity function. In this paper, we consider the problem of transmit signal design, in order to optimize the ambiguity function of a distributed MIMO radar system via maximizing its peak.

1. Introduction
Recently, there has been significant interest towards MIMO radar systems with multiple transmit and receive antennas, capable of transmitting arbitrary signals. Such capability has been achievable only with the rapidly developing computers and embedded processors.

Generally, MIMO radars can be categorized into two areas: systems based on the use of widely separated antennas [1] and systems that use colocated antennas [2]. In the case of widely separated antennas, multiple transmitters and multiple receivers that are widely separated are used in order to detect the targets. As shown in [1], the concepts of spatial diversity and multiplexing gain emerge in this context in a dual manner to the MIMO communication. Indeed, by looking at a target from different angles, the probability of missed detection decreases, that is known as spatial diversity concept in MIMO communication. In [3,4], it is shown that such diversity gain is available not only in the signal processing part but also in the data processing part. On the other hand, in the case of colocated antennas, such configuration is similar to phased array schemes, whereas, here the signal emitted by each antenna is arbitrary and can be totally uncorrelated with other transmit signals. The studies have shown that we can have enhanced detection and estimation performance [5,6], and higher resolution localization [7] with a widely separated antennas scheme, which is the focus of this paper.

The design of radar waveforms to optimize the radar’s performance is a longstanding research topic in Single Input Single Output (SISO) scenario (see e.g., [8–14]). Subsequently, several papers have been published on the topic of MIMO radar waveform design. In [15–17], the covariance matrix of the transmit waveforms has been designed to achieve a desired transmit beam pattern. The
reference [18] suggests an iterative adaptive approach method to improve the range-Doppler resolution and reduce the interference for radar imaging purposes in a colocated MIMO radar. Also, the MIMO radar ambiguity function is studied in [19], introducing some bounds on its volume and height distributions. There, it is shown that in a MIMO radar, the clear area is increased as compared with that of the SISO radar by exploiting spatially diverse waveforms.

In [20–23], mutual information (MI) between the target impulse response and the reflected radar signal has been used as a criterion to design radar waveforms of a colocated MIMO configuration. In these works, some prior knowledge of the impulse response of the target is assumed.

Furthermore, in [24,25], the problem of robust waveform design for MIMO radars when the target scattering covariance matrix is unknown is addressed. There, assuming an extended target and following a min–max approach, the transmit code matrix is designed to minimize the worst-case cost under all possible target covariance matrices.

In [26], an algorithm for designing the orthogonal frequency-hopping waveforms is proposed, in order to reduce the sidelobes in the corresponding MIMO radar ambiguity function and make the energy of the ambiguity function spread evenly in the range and angular dimensions. However, the MIMO configuration assumed in [26] is colocated such that the radar cross sections observed by the transmitting paths are identical. In addition, this configuration results in that all waveforms experience the same delay from the transmit antennas to the target–to–the receive antennas, an assumption which is not true in the widely separated configuration. In another recent work [27], power allocation is done independently of the waveform design, in order to improve the Neyman–Pearson detector’s performance of a MIMO radar. Moreover, maximizing the SINR\(^1\) at the receiver has been another goal for the transmit signal optimization (see e.g. [28,29]).

In this paper, our goal is to improve the ambiguity function (AF) of the MIMO radar with widely separated antennas by designing and choosing proper transmit signals. In more details, we seek the conditions that should be satisfied by the signals, so that the peak of the MIMO ambiguity function is maximized. It is known that ambiguity or energy (volume under the surface) can be moved around in the delay-Doppler plane but not be removed [30,31]. Therefore, by such maximization, the total level of other unwanted ambiguities will decrease too.

The remainder of this paper is organized as follows. In Section 2, the system model is explained. Section 3 includes optimization of the MIMO AF’s peak for three cases: (1) MIMO AF defined for a single target, (2) MIMO AF defined for multiple targets, and (3) multiple prioritized single target MIMO AFs. In Section 4, the MIMO gain obtained by signal design is discussed. Section 5 is dedicated to the simulations, and finally, Section 6 concludes the paper.

11. Notation

We use the notation of using boldface for vectors \(\mathbf{a}\) (lower case), and matrices \(\mathbf{A}\) (upper case). The transpose and the conjugate transpose operators are denoted by the symbols \((\cdot)^{\top}\) and \((\cdot)^{\dagger}\), respectively. tr(\·) is the trace of the square matrix argument and \(\cdot\) represents its determinant. \(\text{Diag}(\cdot)\) indicates the diagonal matrix formed by the components of the vector \(\mathbf{a}\), whereas \(\text{blkDiag}(\cdot)\) denotes the block diagonal matrix formed by its arguments. The symbol \(\odot\) stands for the Hadamard (element-wise) product of matrices, and, finally, \(E\{\cdot\}\) denotes the statistical expectation.

2. Signal model

Consider a widely separated MIMO (WS-MIMO) radar system with \(N_T\) transmitters and \(N_R\) receivers. In such distributed systems, the transmitters and receivers are distributed widely in the region. The locations of the transmitters and receivers are denoted by \(\mathbf{d}_i^t\), \(1 \leq i \leq N_T\) and \(\mathbf{d}_r^c\), \(1 \leq r \leq N_R\), respectively. In addition, we assume that \(N_T\) targets exist in the region and the target’s parameters vector is defined as below:

\[
\theta_k = \begin{bmatrix} \mathbf{d}_k \vspace{1mm} \\ \mathbf{v}_k \end{bmatrix}, \quad k = 1, \ldots, N_T,
\]

where \(\mathbf{d}_k\) consists of the \(k\)’th target’s position and \(\mathbf{v}_k\) represents its velocity vector. Consequently, the distances to the \(k\)'th target to \(i\)'th transmitter and to the \(r\)'th receiver are

\[
\ell_{ik} = |\mathbf{d}_k - \mathbf{d}_i^t|, \quad \ell_{rk} = |\mathbf{d}_k - \mathbf{d}_r^c|
\]

Therefore the delay and Doppler shifts of the reflected signal of the \(i\)'th transmitter from the \(r\)'th target at the \(r\)'th receiver can be obtained as

\[
\tau_{ikr} = \frac{|\ell_{rk}| + |\ell_{ik}|}{c}
\]

\[
f_{ikr} = \frac{f_{\ell_{ik}}(\mathbf{v}_k) + f_{\ell_{rk}}(\mathbf{v}_k)}{|\ell_{ik}| + |\ell_{rk}|},
\]

The target’s RCS\(^2\) is assumed to have Swerling I model, i.e.

\[
f_{\sigma_0}(\sigma_k) = \frac{\sigma_k}{\sigma_0^2} e^{-\sigma_k^2/2\sigma_0^2},
\]

where \(\sigma_0^2\) is the average RCS of the \(k\)'th target. In addition, RCSs of different targets are assumed to be independent. The received signal at the \(r\)'th receiver is

\[
Z_r(t) = \sum_{k=1}^{N_T} \sum_{i=1}^{N_T} \sigma_{ki} e^{j\phi_{ikr}} S_k(t - \tau_{ikr}) e^{j2\pi \ell_{ik}} + n_r(t), \quad r = 1, \ldots, N_R,
\]

where \(S_k(t)\) and \(\phi_{ikr}\) are the \(i\)'th transmitted signal and its phase shift at the \(r\)'th receiver, respectively. Also,

\[
\sigma_{ikr}^2 = \frac{P_{i} G_r^i G_t^k \lambda_r^2 \sigma_0^2}{(4\pi)^3 I_{ikr}},
\]

\(^1\) Signal to Interference plus Noise Ratio.

\(^2\) Radar Cross Section.
where \( \lambda_i \) is the wavelength of the \( i \)th transmitted signal, \( G_i^T \) and \( G_i^R \) are the \( i \)th transmitter’s and \( r \)th receiver’s antenna gains, \( P_i \) is the \( i \)th transmitter’s power and \( L_{\text{int}} \) represents other losses that may exist.

The distribution of \( \phi_{\text{int}} \) is assumed to be uniform on \([0, 2\pi]\). Thus,

\[
f_{\phi_{\text{int}}}(\phi_{\text{int}}) = \frac{1}{2\pi}
\]

Now, let us define

\[
\alpha_{\text{int}} \triangleq \sigma_{\text{int}} e^{i\phi_{\text{int}}}
\]

The probability characteristic of the received signal depends on the probability density function (PDF) of \( \alpha_{\text{int}} \).

As the amplitude and phase of the \( \alpha_{\text{int}} \) are random variables with Rayleigh and uniform distributions respectively, \( \alpha_{\text{int}} \) has a Gaussian distribution,

\[
f_{\alpha_{\text{int}}}(\alpha_{\text{int}}) \sim \mathcal{CN}(0, \sigma_{\text{int}}^2),
\]

where

\[
\sigma_{\text{int}}^2 = \frac{P_i G_i^T G_i^R \lambda_i^2 \sigma_0^2}{(4\pi)^2 L_{\text{int}}}
\]

By sampling \( N \) samples from the received signal, we have

\[
z = \sum_{k=1}^{N} \sum_{\kappa=1}^{N_s} \frac{1}{|f_{\text{int}}^T f_{\kappa}|} y_{\text{int}}(\theta_\kappa) + n_r, \quad r = 1, \ldots, N_R
\]

where

\[
y_{\text{int}} \triangleq s_i[n - r_{\text{int}}]e^{i2\pi f_{\text{int}} n / N},
\]

This equation can be shown in a matrix form as

\[
z(\theta) = \sum_{k=1}^{N} Y(\theta_\kappa) \alpha(\theta_\kappa) + n,
\]

where

\[
\theta = \begin{bmatrix}
\theta_1 \\
\theta_2 \\
\vdots \\
\theta_{N_s}
\end{bmatrix}, \quad z(\theta) = \begin{bmatrix}
z_1(\theta) \\
z_2(\theta) \\
\vdots \\
z_{N_s}(\theta)
\end{bmatrix}, \quad n = \begin{bmatrix}
n_1 \\
n_2 \\
\vdots \\
n_{N_s}
\end{bmatrix},
\]

and

\[
Y(\theta_\kappa) = \begin{bmatrix}
y_{1,1}(\theta_\kappa) & y_{2,1}(\theta_\kappa) & \cdots & y_{N_s,1}(\theta_\kappa) \\
0 & y_{1,2}(\theta_\kappa) & \cdots & y_{N_s,2}(\theta_\kappa) \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & y_{1,\kappa}(\theta_\kappa) & y_{2,\kappa}(\theta_\kappa) & \cdots & y_{N_s,\kappa}(\theta_\kappa)
\end{bmatrix}_{N_s \times N_s}.
\]

where \( y_{\text{int}}(\theta_\kappa) \) is the signal emitted by the \( i \)th transmitter, reflected by the \( \kappa \)th target and received at the \( r \)th receiver. Besides,

\[
\alpha(\theta_\kappa) = \begin{bmatrix}
\alpha_1 \kappa & \alpha_2 \kappa & \cdots & \alpha_{N_s} \kappa & \alpha_{N_s} \kappa \end{bmatrix}_{N_s}^T \quad \alpha(\theta_\kappa) \in \mathbb{C}^{N_s \times 1}
\]

\( \odot \) \( W(\theta_\kappa) \)

\[
\begin{array}{c} \theta_1 \\
\theta_2 \\
\vdots \\
\theta_{N_s}
\end{array} = \begin{array}{c} n_1 \\
n_2 \\
\vdots \\
n_{N_s}
\end{array},
\]

where

\[
w(\theta) = \begin{bmatrix}
1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}_{N_s \times N_s}^T
\]

(18)

Assuming the antennas separated widely enough, different target’s RCSs from different angles are independent, which results in the independency of \( \alpha_{\text{int}} \)’s [1]. Thus,

\[
E(\alpha(\theta_\kappa)\alpha^T(\theta_\kappa)) = \sigma_{\text{int}}^2 \mathbf{C}(\theta_\kappa),
\]

where \( \mathbf{C}(\theta_\kappa) = \text{Diag}(\mathbf{W}(\theta_\kappa)) \). In addition, the noise vector is assumed to have a Gaussian distribution of \( \mathcal{CN}(\mathbf{0}, \sigma_n^2 \mathbf{I}) \).

Next, we explore the WS-MIMO radar’s ambiguity function.

### 3. Problem formulation and solution

The definition, used here for the MIMO ambiguity function, is the expected value of the log-likelihood function of the received signal with respect to \( p(z | \theta) \) [32,33]:

\[
A(\theta, \theta') \triangleq -E_{p(z | \theta)}[\ln p(\theta | z)] = -E_{p(z | \theta')}[\ln p(\theta' | z)].
\]

Here, \( z \), is the observed received signal with the delays and Dopplers of the targets at \( \theta \). It can be inferred from (20) that \( A(\theta, \theta') \) is the ambiguity about the existence of the targets at \( \theta' \), assuming that they are at \( \theta \).

Using the signal model explained in the previous section, the ambiguity function can be written as [34]

\[
A(\theta, \theta') = \frac{N^2_s}{2} \ln(2\pi) + \frac{1}{2} \ln Q(\theta) + \frac{1}{2} \text{tr}[\mathbf{Q}^{-1}(\theta) \mathbf{Q}(\theta)].
\]

(21)

where

\[
Q(\theta) = \sum_{k=1}^{N^2_s} \mathbf{M}(\theta_\kappa) + \sigma_n^2 \mathbf{I}_{N^2_s} = \sum_{k=1}^{N_s} \sigma_0^2 Y(\theta_\kappa) \mathbf{C}(\theta_\kappa) Y^T(\theta_\kappa)
\]

\[
+ \sigma_n^2 \mathbf{I}_{N^2_s}, \quad \kappa = 1, \ldots, N_s.
\]

(22)

In addition, we can conclude that

\[
A(\theta, \theta) = \frac{N^2_s}{2} \ln(2\pi) + \frac{1}{2} \ln Q(\theta) + \frac{1}{2} \text{tr}[\mathbf{I}_{N^2_s \times N^2_s}]
\]

\[
= \frac{N^2_s}{2} \ln(2\pi) + \frac{1}{2} \text{tr}[\mathbf{I}_{N^2_s \times N^2_s}]
\]

(23)

where \( A(\theta, \theta) \) is the maximum of \( A(\theta, \theta') \), i.e.,

\[
A(\theta, \theta) \geq A(\theta, \theta').
\]

Next, we will design the transmit signals (i.e. \( s_i(t), i = 1, \ldots, N_s \)) in order to maximize the AF’s peak (i.e. \( A(\theta, \theta) \)). This goal will be traced in the case of a single target and multiple targets. Then, we will optimize the weighted linear combination of single target AFs (sum of prioritized
3.1. Optimization of single target AF

For single target AF, \( N_T = 1 \) in (22), (23). Assuming \( N_T = 1 \), we can rewrite AF as

\[
A(\theta, \omega) = \frac{NN_c}{2} (1 + \ln(2\pi)) + \frac{1}{2} \ln |1 + Y(\theta) (\sigma_n^2 C(\theta)) Y' (\theta)|.
\]

Our goal is to design the transmit signals of (24), in both MISO and MIMO configurations, in order to maximize \( A(\theta, \omega) \).

3.1.1. MISO configuration

Assuming \( N_R = 1 \) in (24), and neglecting the constant terms, the optimization problem will be turned to

\[
\mathcal{P}: \max_{\theta, \omega} |1 + Y(\theta) (\sigma_n^2 C(\theta)) Y' (\theta)|.
\]

Our strategy is to find the solution of \( \mathcal{P} \) (i.e. \( Y^*(\theta) \)) first, and then, find \( \{s_i\}_{i=1}^{N_T} \) according to \( Y^*(\theta) \).

\[
\mathcal{P}_1 \left\{ \max_{\theta, \omega} \ln |1 + \frac{\sigma_n^2}{\sigma_n^2} Y(\theta) C(\theta) Y(\omega)| \right. \\
\text{s.t.} \quad \text{tr}(Y(\theta) Y'(\theta)) = N_T.
\]

Here, we have enforced \( \text{tr}(Y(\theta) Y'(\theta)) = N_T \) constraint, to confine the total transmit power. In the Appendix it is shown that \( \mathcal{P}_1 \) results in

\[
\begin{align*}
\begin{bmatrix} s_i' s_i \end{bmatrix} &= \left( \begin{smallmatrix} \lambda^{-1} - 1 \\ \frac{1}{C_{ii}} \end{smallmatrix} \right)^+, \quad i = 1, \ldots, N_T \\
\sum_{i=1}^{N_T} s_i' s_i &= N_T \\
s_j' f(\tau_i) s_i &= 0, \quad i \neq j, \quad i, j = 1, \ldots, N_T
\end{align*}
\]

(25)

where \( (\cdot)^+ \) is an indicator of the waterfilling concept and means that, for any \( i \), if \( \left( \lambda^{-1} - \frac{1}{C_{ii}} \right) \) is negative, we set \( s_i s_i = 0 \), and solve the problem for the remaining transmitters. Such a power allocation is seen in MIMO communication systems too [35]. In these systems, in order to maximize the bit rate \( \lambda \) the channel loss, the optimum power of each channel is set equal to the difference between a constant term and a term proportional to 1/SNR. Therefore, the channel with more SNR will be dedicated more transmit power. Such allocation scheme is called waterfilling due its behavior [35]. Thus in the MISO single target case, optimum power allocation is similar to the waterfilling power allocation in MIMO communication.

The number of equations in (25) is \( N_T + \frac{N_T(N_T - 1)}{2} \) and the number of unknown parameters is \( N_T \times N \). Therefore, \( \{s_i\}_{i=1}^{N_T} \) are not determined uniquely.

It can be inferred from (25) that the transmit signals, after experiencing the delays from the transmitters to the target, should be orthogonal to each other. Besides, it determines the power of each signal. In order to find the solution of this problem, i.e., the transmit signals, we first find a set of orthonormal \( \{\hat{s}_i\}_{i=1}^{N_T} \). Assuming \( J(\tau_i) \hat{s}_i = \sqrt{P_i} \hat{s}_i \) (where \( P_i = \lambda^{-1} - \frac{1}{C_{ii}} \)), we can find \( s_i \), by shifting each \( \hat{s}_i \) by \( -\tau_i \), and then multiplying its amplitude by \( \sqrt{P_i} \). In this way, the resulting signals will satisfy the equations of (25). We call the first step “waveform design” and the second step “power allocation”. Indeed, the first step determines the shape of each transmit signal and the second step determines its power. The procedure to obtain \( \{s_i\} \) from (25) is illustrated in Algorithm 1.

Algorithm 1. Algorithm to obtain the transmit signal in the case of MISO single target AF.

\[
\text{Require: } C_i (r_i' r_i), \\
\text{Ensure: } \text{A signal set } \{s_i\} \text{ as a solution to problem (25).}
\]

1. Waveform design: Find an arbitrary set of orthonormal vectors \( \{\hat{s}_i\}_{i=1}^{N_T} \);
2. Power allocation: Set \( P_i = \left( \lambda^{-1} - \frac{1}{C_{ii}} \right)^+ \) s.t. \( \sum_{i=1}^{N_T} P_i = N_T \);
3. Output \( s_i \), as the shifted vector of \( \sqrt{P_i} \hat{s}_i \) by \( -\tau_i \).

3.1.2. MIMO configuration

Assuming \( N_R \) receivers, we can write B as

\[
B = Y'(\theta) Y(\theta) = \text{blkDiag}(Y_1', \ldots, Y_N') \text{blkDiag}(Y_1, \ldots, Y_N)
\]

\[
= \text{blkDiag}(Y_1^*, Y_1^* Y_2, \ldots, Y_N^* Y_N)
\]

(26)

where \( Y_r, r = 1, \ldots, N_R \) is the received signal at the \( r \)th receiver. Note that for each \( r \), the diagonal elements of matrix \( Y_r^* Y_r \) are, in fact, the powers of the transmitters. Therefore

\[
\text{diag}(Y_r^* Y_r) = \text{diag}(Y_r^* Y_r), \quad \forall i, j \in \{1, \ldots, N_R\}
\]

(27)

Regarding (26) and (27), the optimization problem of the AF’s peak, for the case of MIMO configuration, can be written as

\[
\mathcal{P}_2 \left\{ \max_{\theta, \omega} \ln |1 + B C(\theta)| \\
\text{s.t.} \quad \text{tr}(B_r) = N_T, \quad r = 1, \ldots, N_R \\
\text{diag}(B_r) = \text{diag}(B_r), \quad \forall r, r = 1, \ldots, N_R.
\]

The following lemma [36] helps us solve this optimization problem.

Lemma 1. For each positive definite matrix \( M \) with dimensions \( K \times K \) and elements of \( m_{ij} \), we have

\[
det(M) \leq \prod_{i=1}^{K} |m_{ii}|
\]

(29)

and the equality occurs when \( M \) is diagonal.

Proof. See [36].

As \( \ln(\cdot) \) is a strictly increasing function, and \( C_i I \) are diagonal matrices, if the solution to \( \mathcal{P}_2 \), \( B^* \), were a nondiagonal matrix, according to lemma 1, we could increase the maximization goal function by setting all nondiagonal elements of \( B^* \) to zero. Also, as the constraint of \( P_i \) is enforced only on the diagonal of \( B \), it can be concluded that its solution, \( B^* \) (or equivalently any \( B_r \)), should be diagonal. Similar to the previous section, this means that the transmit signals, after experiencing the delays from the transmitters to the target,
are orthogonal to each other. Next, we should find the power allocated to each transmitter.

Defining \( \tilde{b}_{ir} \) as the \((ir, ir)\)th element of \( \mathbf{B} \), according to (27), we have

\[
\tilde{b}_{ir} = \frac{b_{ir}}{\sqrt{r}}; \quad \forall r, r = 1, \ldots, R, \quad \forall i = 1, \ldots, N_r
\]  

(30)

So, we define

\[
b_i = \tilde{b}_{ir}, \quad \forall r = 1, \ldots, R_r
\]

and \( \mathcal{P}_3 \) is recast to

\[
\mathcal{P}_4 \quad \begin{cases} 
\max_{b_i} \quad \lambda = \sum_{i=1}^{N_c} \ln(1 + b_i C_{ir}) \\
\text{s.t.} \quad \sum_{i=1}^{N_c} b_i - N_F = 0,
\end{cases}
\]  

(31)

where \( C_{ir} \)'s are the diagonal elements of \( \mathbf{C} \). The solution of \( \mathcal{P}_4 \) can be obtained using the Lagrangian method:

\[
\mathcal{L} = \lambda \left( \sum_{i=1}^{N_c} b_i - N_F \right)
\]

\[
\frac{\partial \mathcal{L}}{\partial b_i} = 0, \quad i = 1, \ldots, N_F
\]

\[
\sum_{i=1}^{N_c} b_i - N_F = 0,
\]

\[
b_i \geq 0
\]

(32)

As

\[
\frac{\partial \mathcal{L}}{\partial b_i} = \lambda - \sum_{i=1}^{N_c} \frac{C_{ir}}{1 + b_i C_{ir}} = \lambda - \sum_{i=1}^{N_c} \frac{1}{b_i + C_{ir}} = 0.
\]

we have

\[
\sum_{r=1}^{N_c} \frac{1}{b_i + C_{ir}} = \lambda, \quad i = 1, \ldots, N_F
\]

\[
\sum_{i=1}^{N_c} b_i = N_F,
\]

\[
b_i \geq 0
\]

On the other hand, we can rewrite \( C_{ir} \) as

\[
C_{ir} = \frac{\sigma_i^2}{\sigma_{ii}^2} = \frac{1}{\|e_r^{ri} \|^2} C_{ii}.
\]

Thus

\[
\sum_{r=1}^{N_c} \frac{1}{b_i + \|e_r^{ri} \|^2 C_{ii}} = \lambda, \quad i = 1, \ldots, N_F
\]

\[
\sum_{i=1}^{N_c} b_i = N_F,
\]

\[
b_i \geq 0
\]

(35)

If we define \( f_i(x) \) as

\[
f_i(x) = \sum_{r=1}^{N_c} \frac{1}{\|e_r^{ri} \|^2 C_{ii}}
\]

(36)

the problem of finding \( b_i \quad i = 1, \ldots, N_F \) leads to the following problem:

\[
\begin{cases} 
\max_{b_i} \quad \lambda = f_i(x), \\
\text{s.t.} \quad \sum_{i=1}^{N_c} b_i = N_F,
\end{cases}
\]

\[
\Rightarrow \quad \begin{cases} 
\sum_{i=1}^{N_c} b_i = N_F, \\
\lambda \geq 0
\end{cases}
\]

(37)

It is obvious that \( f_i(x) \) is a strictly decreasing function. Therefore, the equation of \( f_i(x) = \lambda \) cannot have more than one solution, if any solution exists. In addition, as \( b_i \geq 0 \), the condition of having a solution will be

\[
f_i(0) \geq \lambda, \quad \lambda \geq 0
\]

(38)

In the case of not satisfying this condition, we set \( b_i = 0 \). Besides, the values of \( b_i \quad i = 1, \ldots, N_c \) can be calculated numerically. Finally, the procedure to obtain \( \mathbf{s} \) from (37) is illustrated in Algorithm 2.

\section{Algorithm 2. Algorithm to obtain the transmit signal in the case of MIMO single target AF}

\textbf{Require:} \( \mathbf{C}, (\tau r)^{N_c} \).

\textbf{Ensure:} A signal set \( \mathbf{s} \) as a solution to problem [28].

1: Waveform design: Find an arbitrary set of orthonormal vectors \( \mathbf{b}_i^{N_c} \).

2: Power allocation: Set \( \mathbf{P}_i = (f_i^{-1}(\lambda))^{+} \) according to (36);

3: Output \( \mathbf{s} \) as the shifted vector of \( \sqrt{\mathbf{P}_i} \mathbf{b}_i \) by \( -\tau r \).

\subsection{3.2. Optimization of multi-target MIMO AF}

Defining \( \mathbf{Y} \) as

\[
\mathbf{Y} = [\mathbf{Y}(\theta_1)\mathbf{Y}(\theta_2)\ldots\mathbf{Y}(\theta_{N_T})]_{NN_k \times NN_k N_T},
\]

(39)

where \( \mathbf{Y}(\theta_k) \) is the received signal from the \( k \)th target, and defining \( \mathbf{C} \) as

\[
\mathbf{C}_{NN_k \times NN_k N_T} = \text{blkDiag}(\mathbf{C}(\theta_1), \mathbf{C}(\theta_2), \ldots, \mathbf{C}(\theta_{N_T})),
\]

(40)

we can write the multitarget AF (i.e. the ambiguity of multiple targets existing at \( \theta \)) as

\[
A(\theta, \theta) = \ln |\mathbf{Y} \mathbf{C} \mathbf{Y}^H| + \frac{NN_k}{2(1 + \ln(2\pi))} = \ln |\mathbf{Y} \mathbf{C} \mathbf{Y}^H| + \ln |\mathbf{Y} \mathbf{C} \mathbf{Y}^H| + \ln |\mathbf{I} + \mathbf{B}|, cte,
\]

(41)

where \( \theta \) is defined in (15) and

\[
\mathbf{B} = \mathbf{Y}^H \mathbf{Y} = \begin{bmatrix} \mathbf{Y}(\theta_1)^H \mathbf{Y}(\theta_1) & \mathbf{Y}(\theta_1)^H \mathbf{Y}(\theta_2) & \ldots & \mathbf{Y}(\theta_1)^H \mathbf{Y}(\theta_{N_T}) \\ \mathbf{Y}(\theta_2)^H \mathbf{Y}(\theta_1) & \mathbf{Y}(\theta_2)^H \mathbf{Y}(\theta_2) & \ldots & \mathbf{Y}(\theta_2)^H \mathbf{Y}(\theta_{N_T}) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{Y}(\theta_{N_T})^H \mathbf{Y}(\theta_1) & \mathbf{Y}(\theta_{N_T})^H \mathbf{Y}(\theta_2) & \ldots & \mathbf{Y}(\theta_{N_T})^H \mathbf{Y}(\theta_{N_T}) \end{bmatrix} \in \mathbb{R}^{NN_k \times NN_k N_T N_T}
\]

(42)

Similar to the previous section, the optimization problem can be formulated as

\[
\mathcal{P}_6 \quad \max_{\mathbf{B}} \quad \ln |\mathbf{I} + \mathbf{B}| \quad \text{s.t.} \quad \text{tr}(\mathbf{B}) = NN_k N_T.
\]

(43)

Again, using Lemma 1, the solution will be a diagonal matrix and can be written in the following form:

\[
\mathbf{B} = \text{blkDiag}(\mathbf{b}_1^1, \mathbf{b}_1^2, \ldots, \mathbf{b}_1^{N_T}), \quad \mathbf{B} = \text{blkDiag}(\mathbf{b}_2^1, \mathbf{b}_2^2, \ldots, \mathbf{b}_2^{N_T}), \quad \mathbf{B} = \text{Diag}(\mathbf{b}_1^1, \mathbf{b}_2^1, \ldots, \mathbf{b}_N^1).
\]

(44)
Note that, here, being $\mathbf{B}$ diagonal implies that the transmit signals, after experiencing the delays from the transmitters to each of the targets, are orthogonal to each other.

Now, the goal function can be expressed as

$$A = \ln(\mathbf{I} + \mathbf{BC}) = \sum_{i=1}^{N_R} \sum_{r=1}^{N_T} \sum_{k=1}^{N_c} \ln(1 + b_i c_k^r),$$

(45)

where $c_k^r = \frac{1}{d_k^r(C_k^r)^{1/2}}$ is the path loss from the $i$th transmitter to the $k$th target at the $r$th receiver.

Using the Lagrangian method, $P_6$ can be solved:

$$\mathcal{L}_3 = A - \lambda (\text{tr}(\mathbf{B}) - N_c N_R N_T).$$

(46)

Differentiating from the Lagrangian with respect to the problem’s parameters, we have

$$\frac{\partial \mathcal{L}_3}{\partial b_i} = 0, \quad i = 1, \ldots, N_T.$$

(47)

Hence,

$$\frac{\partial A}{\partial b_i} = \lambda N_c N_T,$$

(48)

and

$$\sum_{r=1}^{N_T} \sum_{k=1}^{N_c} c_k^r b_i c_k^r = \lambda N_c N_T,$$

(49)

Defining $h(x)$ as

$$h_i(x) = \frac{1}{N_c N_T} \sum_{r=1}^{N_T} \sum_{k=1}^{N_c} \frac{1}{x + (c_k^r)^{-1}}.$$

(50)

$P_6$ can be reformulated as

$$\mathcal{L}_3 = \sum_{i=1}^{N_T} \left( \frac{1}{N_c N_T} \sum_{r=1}^{N_T} \sum_{k=1}^{N_c} \frac{1}{x + (c_k^r)^{-1}} \right) b_i.$$

(51)

Similar to the previous optimization problems, (51) is solved numerically. The overall procedure is explained in Algorithm 3.

The problem of code design for finding signals with specific correlation characteristics has been studied in various fields, such as optical code design, wireless communications and radar signal design [e.g. see 37–39]. Therefore, our waveform design step, which includes finding a set of waveforms orthogonal to each other at specific delays, can be solved using the traditional methods. It should be noted that such waveforms satisfy the implementation constraints.

In addition, as the waveforms should be orthogonal to each other at the targets’ positions, there is no need to all delays but only the delays corresponding to one receiver, e. g. $\{\tau_{\kappa i}\}, \quad i = 1, \ldots, N_T, \quad \kappa = 1, \ldots, N_T.$

**Algorithm 3.** Algorithm to obtain the transmit signal in the case of MIMO multi-target AF.

**Require:** $\mathbf{C}, \{\tau_{\kappa i}\}, \quad i = 1, \ldots, N_T, \quad \kappa = 1, \ldots, N_T.$

**Ensure:** A signal set $\{\mathbf{s}_i\}_{i=1}^{N_T}$ as a solution to problem (43).

1: Waveform design: Find a set of $\{\mathbf{s}_i\}_{i=1}^{N_T}$ that are orthonormal after experiencing the delays from the TXs to each target;

2: Power allocation: Set $P_i = (h_i^{-1} (\lambda))^+$ according to (50);

3: Output $\mathbf{s}_i$ as $\sqrt{P_i} \mathbf{s}_i$.  

3.3. Optimization of the prioritized MIMO AF

Here, we assume that different positions of the radar surveillance region have different importance (or equivalently priority), from the surveillance point of view. Indeed, we assume that multiple $(N_T)$ targets are present.

For each one, the single target AF is written, and, then, the prioritized AF is defined as the sum of these AFs $(N_T)$ number of AFs). However, in practice, each target may have different importance from the others. Therefore, the summation is done considering an importance coefficient for each single AF. In other words the prioritized AF will be a weighted sum of the single target AFs:

$$A_{\omega}(\theta) = \sum_{k=1}^{N_T} \alpha_k A_{\omega}(\theta_k).$$

(52)

where $\theta = [\theta_1^T \theta_2^T \ldots \theta_k^T]^T$. Using the single target AF defined in (71), we can write

$$A_{\omega}(\theta) = \sum_{k=1}^{N_T} \alpha_k \ln |\mathbf{I} + \mathbf{B}^\kappa \mathbf{C}^\kappa| + \frac{N_N}{2}(1 - \ln(2\pi))$$

(53)

where $\mathbf{B}^\kappa = \mathbf{Y}^\kappa(\theta_k) \mathbf{Y}(\theta_k)$,

$$\mathbf{C}^\kappa = \text{Diag} \left[ \frac{1}{|\mathbf{C}_\kappa^\kappa|^2} \right], \quad \kappa = 1, \ldots, N_T,$$

(54)

$$\mathbf{b}_{ij}^\kappa = \text{The}(i, j)\text{th element of } \mathbf{B}^\kappa$$

(55)

$$\mathbf{P}_{\text{WAM}} = \max_{\mathbf{b}_{ij}^\kappa} \left\{ \mathbf{A}_{\omega}(\theta) \right\}$$

(56)

where each $\mathbf{B}^\kappa = \mathbf{Y}^\kappa(\theta_k) \mathbf{Y}(\theta_k)$ consists of $N_c$ matrices:

$$\mathbf{B}^\kappa = \text{blkDiag} \left( \mathbf{B}_1^\kappa, \mathbf{B}_2^\kappa, \ldots, \mathbf{B}_N^\kappa \right), \quad \kappa = 1, \ldots, N_T,$$

(57)

and $\mathbf{B}_\kappa = \mathbf{Y}^\kappa(\theta_k) \mathbf{Y}(\theta_k)$ refers to the signal, received at the $r$th receiver, due to the $\kappa$th target. Furthermore, $\mathbf{C}^\kappa$ is defined as below:

$$\mathbf{C}^\kappa = \text{blkDiag} \left( \mathbf{C}_1^\kappa, \mathbf{C}_2^\kappa, \ldots, \mathbf{C}_N^\kappa \right),$$

(58)

$$\mathbf{C}_\kappa = \text{Diag}(c_k^r)$$

Thus,

$$A_{\omega}(\theta) = \sum_{k=1}^{N_T} \sum_{r=1}^{N_T} \alpha_k \ln |\mathbf{I} + \mathbf{B}_r^\kappa \mathbf{C}_r^\kappa|.$$
\( \mathbf{B}_r^* \) represent the transmit powers, we have \( \mathbf{B}_r^* = \mathbf{B}_r^* \), for \( \forall i, j = 1, \ldots, N_T \). Thus, \( \mathbf{B}_r^{N_T} \) can be represented as

\[
\hat{\mathbf{B}} = \mathbf{B}_r^* = \text{Diag}(b_i), \quad i = 1, \ldots, N_T
\]

\( r = 1, \ldots, N_R \)
\( \kappa = 1, \ldots, N_T \)

In addition, the constraint in (55) is recast to \( \text{tr}(\hat{\mathbf{B}}) = N_T \). Therefore,

\[
A_w(\theta) = \sum_{\kappa = 1}^{N_R} \sum_{\tau = 1}^{N_T} \sum_{i = 1}^{N_i} \alpha_i \ln(1 + b_i c_i^n). \tag{60}
\]

Solving \( P_{w_M} \) with the Lagrangian method, we have

\[
L_w(\theta) = A_w(\theta) - \lambda \left( \text{tr}(\hat{\mathbf{B}}) - N_T \right)
\]

\[
= \sum_{\kappa = 1}^{N_R} \sum_{\tau = 1}^{N_T} \sum_{i = 1}^{N_i} \alpha_i \ln(1 + b_i c_i^n) - \lambda \left( \sum_{\tau = 1}^{N_T} b_i - N_T \right)
\]

\[
\frac{\partial L_w(\theta)}{\partial b_i} = \sum_{\kappa = 1}^{N_R} \sum_{\tau = 1}^{N_T} \alpha_i \frac{c_i^n}{1 + b_i c_i^n} - \lambda = \sum_{\kappa = 1}^{N_R} \sum_{\tau = 1}^{N_T} \alpha_i \frac{c_i^n}{1 + \left( c_i^n \right)^{-1}} - \lambda
\]

So,

\[
\begin{cases}
\sum_{i = 1}^{N_i} \sum_{\tau = 1}^{N_T} \alpha_i \frac{c_i^n}{1 + b_i c_i^n} = \lambda \\
\sum_{i = 1}^{N_i} b_i = N_T \\
b_i \geq 0 \quad \forall i = 1, \ldots, N_T
\end{cases}
\tag{62}
\]

Now, we can define \( g_w(\alpha) \) as

\[
g_w(\alpha) = \sum_{i = 1}^{N_i} \sum_{\tau = 1}^{N_T} \alpha_i \frac{c_i^n}{1 + \left( c_i^n \right)^{-1}}
\]

and reformulate the problem as

\[
\begin{cases}
g_w(b_i) = \lambda \\
\sum_{i = 1}^{N_i} b_i = N_T \\
b_i \geq 0, \quad i = 1, \ldots, N_T
\end{cases}
\tag{64}
\]

Finally, (64) can be solved numerically.

**Algorithm 4.** Algorithm to obtain the transmit signal in the case of prioritized MIMO AF.

**Require:** \( \mathbf{C}, \{\mathbf{r}_\kappa\} \), \( i = 1, \ldots, N_i \), \( \kappa = 1, \ldots, N_T \).

**Ensure:** A signal set \( \mathbf{s}_\kappa \) as a solution to problem (55).

1: Waveform design: Find a set of \( \mathbf{s}_\kappa \), that are orthonormal after experiencing the delays from the TXs to each target;
2: Power allocation: Set \( P_i = (g_w^{-1}(\lambda))^+ \) according to (63);
3: Output \( \mathbf{s}_\kappa \) as \( \sqrt{P_i} \cdot \mathbf{s}_\kappa \).

**4. MIMO gain of signal design**

In this section, we explore if the MIMO radar signal design provides any MIMO gain, similar to the spatial diversity gain provided by increasing the number of antennas, in a MIMO radar [40].

Generally, in our ambiguity function’s optimizations, the goal function can be written as

\[
\ln \left| \mathbf{I} + \frac{\sigma_0}{\sigma_n} \mathbf{Y} \mathbf{Y}^T \right| = \ln \left| \text{SNR} \mathbf{Y} \mathbf{Y}^T \right|
\tag{65}
\]

In addition, the diversity gain of a MIMO system is defined as the slope of the log plot at high SNRs (i.e., SNR \( \rightarrow \infty \)) [40]. Thus, we should consider the high SNR regime, where our goal function can be recast to

\[
\ln \left| \mathbf{I} + \text{SNR} \mathbf{Y} \mathbf{Y}^T \right| = \ln \text{SNR} \mathbf{Y} \mathbf{Y}^T - \ln \left| \mathbf{I} + \left( \text{SNR} \mathbf{Y} \mathbf{Y}^T \right)^{-1} \right|
\]

\[
\approx \ln \text{SNR} N_s N_r + \ln \left| \mathbf{Y} \mathbf{Y}^T \right| \approx N_s N_r \ln \text{SNR}
\tag{66}
\]

From (66), it can be seen that the diversity gain is, indeed, independent of the transmit signals. Therefore, we expect that, by signal design, no MIMO gain is obtained. We will examine this theoretical result through simulations in the subsequent section.

**5. Simulations**

In this section, we show the performance improvement made by our proposed waveform design and power allocation. First, the peak of the ambiguity function is chosen as a goal function. It is computed and compared for different waveforms. This comparison is done for the case of a single target, multiple targets and prioritized targets. Next, this peak is depicted as a function of SNR, and its slope at high SNRs will be considered as a performance gauge. Then, the resulted Cramer–Rao bound is compared for different waveforms, including the proposed optimum signals. Finally, the detection and surveillance performance of the proposed optimization procedures are assayed.
It should be noted that, as mentioned before, the solutions of (37), (51), (64) are, in fact, extensions of the waterfilling concept of MIMO communications. In other words, these solutions imply that more power is assigned to the transmitters with higher resulting SNR, and for the transmitters with SNR below a specific threshold, the power is set to zero. As an example, consider four transmitters, one receiver \((R \times 1)\) and the target as shown in Fig. 1 (MISO case). The optimum power allocation of the channels and their losses are shown in Fig. 2(a). The “waterfilling” concept can be seen in the powers allocated to the transmitters. In addition, the first transmit power is equal to zero because of its high channel loss.

Now consider all transmitters and receivers of Fig. 1 (MIMO case). For each transmitter, the parameter of "channel loss" is considered as the inverse of the sum of the resulting SNRs at the three receivers. This parameter and the optimum assigned powers are shown in Fig. 2(b), for each transmitter. It can be seen that lower channel loss leads to higher assigned power, which is, indeed, the aforementioned waterfilling concept.

### 5.1. Comparison of the AF’s peak

In this section, the scenario configuration is the same as Fig. 1.

#### 5.1.1. Single target AF

The AF’s peak is computed for three different transmit waveforms and powers. These waveforms are the FM signal, the proposed optimum waveforms with uniform powers, and the proposed optimum waveforms with optimum powers of (37). It is worth noting that the application of the FM signal can be considered in a MIMO PCL\(^3\) [41], where multiple FM radio stations are used as the noncooperative transmitters. In addition, we assume that all the FM radio stations are transmitting the same FM signal. The target’s position is changed over the whole region of Fig. 1, and subsequently for each position, the AF’s peak is computed for the three cases of transmit signals. Finally, the average of these peaks is calculated. This average is demonstrated in Table 1. It can be seen that an improvement of about 4 dB is gained by optimum waveform design and further 5 dB improvement can be achieved by optimum power allocation. Fig. 3(a) shows the improvement of the AF’s peak, for different target’s positions, in the case of using optimum waveforms with uniform powers, wrt the case of using FM signals. In other words, this plot shows how much the AF’s peak is increased when we replace the FM signals of equal powers with the optimum waveforms of equal powers. The same improvement diagram is shown in Fig. 3(b), for the case of using optimum waveforms with optimum powers, wrt the case of using optimum waveforms with uniform powers. Indeed, Fig. 3(b) shows the improvement due to power allocation. It can be seen, from Fig. 3(a), that the proposed waveform design leads to considerable improvement for the target’s positions near the receivers. The reason can be explained as follows.

Consider the goal function of (28), i.e., \(\ln|I+BC|\). The proposed waveform design leads to having a diagonal \(B\). For a diagonal \(B\) we have (see (31))

\[
\ln|I+BC| = \sum_{r=1}^{N_t} \sum_{i=1}^{N_r} \ln(1+b_iC_{ir})
\]

From (67), it can be inferred that more \(C_{ir}\)’s result in more improvement in the goal function. Besides, it can be seen from (34) that high \(C_{ir}\)’s are achieved for the targets near the receivers. Although for a target near a transmitter, the \(C_{ir}\) corresponding to that transmitter gets high, the \(C_{ir}\) resulted from other transmitters are low. Therefore, only for the targets near the receivers, all \(C_{ir}\)’s get high.

On the other hand, it can be inferred from Fig. 3(b) that power allocation results in significant improvement near the transmitters. The reason is that, when the target is near a transmitter the power allocation results in
allocation of high power to that transmitter. Therefore, the ambiguity of the target decreases considerably wrt the case of uniform transmit powers, in which there is no difference between the transmitters far from the target and the transmitters near the target.

Another MIMO power allocation method is derived in [27]. In Fig. 3(c), the difference (or equivalently improvement) of the AF’s peak when using the optimum powers of (37) instead of the optimum powers of [27] is shown. It can be seen that the proposed method results in the improvement of AF’s peak.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>FM, equal powers</th>
<th>Designed waveforms, equal powers</th>
<th>Designed waveforms, optimum powers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average AP’s peak (dB)</td>
<td>21.88</td>
<td>25.75</td>
<td>30.63</td>
</tr>
</tbody>
</table>

Finally, in order to assess the robustness of the design wrt the errors in estimating the target’s position, consider the scenario of Fig. 1. After optimizing the transmit signals for the target’s position depicted in Fig. 1, the

![Fig. 4. Allocated powers for the multi-target AF scenario.](image)

Fig. 4. Allocated powers for the multi-target AF scenario.

![Fig. 3. Improvement of AF’s peak for (a) optimum waveforms with equal powers wrt FM, (b) optimum waveforms with optimum powers wrt optimum waveforms with equal powers, (c) optimum waveforms with optimum powers wrt optimum waveforms with the power allocation scheme proposed in [27].](image)
improvement of the AF's peak wrt the case of using FM signals is 7.17. Now, assume that although the signal design is done for the scenario of Fig. 1, the actual target’s position is 300 m away from the point shown in Fig. 1: 300 m to the north or 300 m to the south or 300 m to the east or 300 m to the west of the region. For each of these four points the improvement of the AF's peak wrt the case of using FM signals is calculated again. The average of the results is 6.74. As can be seen this value has not changed considerably wrt the improvement corresponding to the estimated target's position (i.e. 7.17), inferring that the signal design works satisfactorily even if a little error exists in estimating the target's position.

5.1.2. Multi-target AF

Next, we compare the multi-target AF's peak of (41) for the same three cases of transmit signals of the previous section, using the solution of Algorithm 3. The scenario configuration and the resulting optimum powers are depicted in Fig. 4. In addition, the multi-target AF’s peak is shown in Table 2. Obviously, the peak has improved significantly by the proposed waveform design and power allocation.

5.1.3. Prioritized AF

In the last scenario for the AF's peak comparison, the new goal function is the weighted sum of the AF's peaks, i.e. $A_w(\theta)$ of (52), for the targets of the previous scenario. Targets’ weights and the resulting optimum powers are depicted in Fig. 5. As can be seen, the weight assigned to a target has direct effect on the power allocated to its near transmitters. In addition, the resulting peak of $A_w(\theta)$, using the solution of Algorithm 4 for the three aforementioned cases of transmit signals, is shown in Table 3. The improvement resulted from the proposed procedure is obvious.

5.2. MIMO gain of signal design

In this subsection, we investigate the result of Section 4, by studying different waveforms including the proposed optimum signals, to see whether the signal design provides any MIMO gain.

Consider the scenario of Fig. 1. The AF’s peak versus SNR is plotted in Fig. 6, for three signals of: simple pulse with duration of 10 µs, FM radio and optimum designed signals of Algorithm 2. It should be noted that, in the case of a pulse or FM radio, all transmitters are transmitting the same signal. As expected, the peak is maximum for the designed signals. Besides, the slopes of these curves at high SNRs are equal, which justifies our claim that no diversity gain is obtained through signal design. Note that the number of transmit and receive antennas is equal for these three cases.

In another scenario, the number of antennas is increased while the same signal (optimum designed signal) is used, in order to test the dependency of the diversity gain on the number of antennas. The result is shown in Fig. 6(b). As can be seen, the slope of the plot, which corresponds to the diversity gain, increases by increasing the number of antennas.

5.3. Cramer–Rao bound

For another performance assessment, we study the Cramer–Rao bound. The antennas’ positions are shown in Table 4. The optimum signals are obtained using Algorithm 3. For the three different sets of transmit signals of the previous section, the corresponding CRBs of $\theta$ versus the number of present targets are shown in Fig. 7. It can be seen that the maximum number of unambiguously detectable targets is independent of the transmit signals. However, the CRB of the designed signals is better than the other signals' CRBs. The reason lies in the fact that the signal design acts as if we have increased the SNR, resulting in no MIMO gain, as we claimed earlier. Indeed, it is mentioned in [1] that the number of targets in the field of view of the MIMO radar serves as an indicator of the MIMO multiplexing gain.

5.4. Detection performance

In this part, we study the detection performance of the optimized waveforms. To this end, we use the MIMO detector designed in [27]. The antennas configuration is the same as Fig. 1. The target is placed at (7,0) km and the transmit waveforms are optimized to maximize the AF's peak at this point. In addition, $P_{fa}$ is set to $10^{-4}$. Using Monte Carlo method, this simulation is run 1000 times and

<table>
<thead>
<tr>
<th>Parameter</th>
<th>FM, equal powers</th>
<th>Designed waveforms, equal powers</th>
<th>Designed waveforms, optimum powers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average AF's peak (dB)</td>
<td>50.92</td>
<td>74.88</td>
<td>77.93</td>
</tr>
</tbody>
</table>

Fig. 5. Targets' priorities and allocated powers for the prioritized AF scenario.

---

4 Cramer–Rao bound.
the probability of detection ($P_d$) of the aforementioned detector at the CUT$^5$ is computed for the two cases of transmit signals:

- signals optimized using Algorithm 2;
- FM signals with the characteristics of Section 5.1.1.

Moreover, the probability of appearing a false target at a cell other than CUT ($P_{ft}$) is calculated. The results are shown in Table 5. As expected, optimizing the ambiguity function leads to better detection performance.

### 5.5. Surveillance application

Here, we intend to show the application of the proposed optimization procedure in the surveillance operation. To this end, we consider a surveillance region with each point having a priority. This priority as a function of the region’s points is depicted in Fig. 8, and represents the importance of each point from the surveillance point of view. To determine the transmit signals of the MIMO radar for such purpose, six candidate points are chosen from the region as shown in Fig. 9. Then the transmit signals are optimized through the Algorithm 4 regarding the priorities of these points. Here, the antennas configuration is the same as Fig. 1. In order to study the performance of the proposed optimization from the surveillance perspective over the whole region, we compare the AF’s peak resulted from the optimized signals with the AF’s peak resulted

---

**Table 3**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>FM, equal powers</th>
<th>Designed waveforms, equal powers</th>
<th>Designed waveforms, optimum powers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average AF’s peak (dB)</td>
<td>15.97</td>
<td>18.36</td>
<td>20.15</td>
</tr>
</tbody>
</table>

---

**Table 4**


<table>
<thead>
<tr>
<th>Antenna</th>
<th>Position (x,y) km</th>
</tr>
</thead>
<tbody>
<tr>
<td>TX1</td>
<td>(-10,10)</td>
</tr>
<tr>
<td>TX2</td>
<td>(12,2)</td>
</tr>
<tr>
<td>RX1</td>
<td>(10,10)</td>
</tr>
<tr>
<td>RX2</td>
<td>(-5,-8)</td>
</tr>
<tr>
<td>RX3</td>
<td>(-1,11)</td>
</tr>
<tr>
<td>RX4</td>
<td>(-10,2)</td>
</tr>
</tbody>
</table>

---

**Fig. 6.** Ambiguity function’s peak versus SNR: (a) different signals, (b) optimum signals with different number of antennas.

**Fig. 7.** Cramer–Rao bound for different signals.

---

$^5$ Cell Under Test.
from the FM signals introduced in Section 5.1.1. To this end, for each point of the region, the AF’s peak is calculated. Then, by changing the target’s position over the whole region, the AF’s peaks are averaged over such region while weighted by the priorities of Fig. 8. Indeed, here the weighted average AF’s peak is computed. The result is listed in Table 6. As can be seen, the weighted average AF’s peak of the optimized signals is higher than FM signals, which implies that better surveillance performance can be achieved using the proposed optimization of prioritized AF (i.e., Algorithm 4).

6. Conclusion

In this paper, the problem of MIMO radar signal design (including the waveform design and power allocation) was considered, in order to maximize the MIMO ambiguity function’s peak. It is known that ambiguity or energy (volume under the surface) can be moved around in the delay-Doppler plane but not be removed [30,31]. So, maximizing the AF’s peak is expected to decrease the AF’s sidelobes too.

To assess the proposed optimization schemes, three cases were assumed for the ambiguity function: ambiguity of a single target, ambiguity of multiple targets and sum of the prioritized ambiguities of multiple targets. For each case optimum signals were obtained theoretically and justified through simulations. In addition, we showed that signal design does not provide a MIMO gain, but acts as if the SNR of the system is increased. However, as shown before, the traditional MIMO spatial gain is achieved through increasing the number of antennas. For future work, it would be interesting to consider the optimization of the antennas’ geometry of a WS-MIMO radar, and explore whether the MIMO gain can be achieved through geometry optimization or it acts like signal optimization, i.e. it is similar to the case of increasing the SNR. The basic equations for such optimization, with the goal of ambiguity function’s improvement, are the same as what we used in this paper. In addition, we considered only the ambiguity function’s peak as the optimization goal function. For another future work, designing proper signals, wrt other characteristics of this function, such as SLL6, seems a challenging but inevitable problem.

Moreover, other constraints such as PAR7 of the transmit waveforms’ amplitudes are important from the radar design perspective. In practice, in order to maximize the efficiency in a radar system, power amplifiers typically have to operate in the saturation mode. As a result, it is important that the probing signals for a radar have constant modulus, i.e. PAR = 1. Thus, solving the waveform optimization problem with such controlled characteristics should be considered for the continuation of the current work. Finally, testing the optimized signals in a real radar system, with real data, should be considered as a future important work.

Appendix A. Proof of Eq. (25)

\( P_1 \) can be rewritten as

\[
\begin{align*}
\mathcal{P}_1 & = \max_{\mathbf{Y}(\theta)} \ln \left| \mathbf{I} + \frac{\sigma_r^2}{\sigma_n^2} \mathbf{Y}(\theta) \mathbf{Y}(\theta) ^\dagger \right| ^{\frac{1}{2}} \\
\text{s.t.} & \quad \text{tr} \left( \mathbf{Y}(\theta) \mathbf{Y}(\theta) ^\dagger \right) = N_E,
\end{align*}
\]

Table 5

Detection performance.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>FM signals</th>
<th>Optimized signals</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P_d )</td>
<td>0.69</td>
<td>0.96</td>
</tr>
<tr>
<td>( P_f )</td>
<td>0.06</td>
<td>Less than 0.001</td>
</tr>
</tbody>
</table>

Table 6

Surveillance performance.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>FM signals</th>
<th>Optimized signals</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weighted average AF’s peak</td>
<td>31.24</td>
<td>37.51</td>
</tr>
</tbody>
</table>

Fig. 8. The priority function.

Fig. 9. The surveillance region and its six main points chosen to be used for signal design.

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6 Side Lobe Level.
7 Peak to Average Ratio.
since
\[ | \mathbf{1} + \mathbf{A}\mathbf{B}^T | = | \mathbf{1} + \mathbf{B}\mathbf{A}^T |, \quad \text{tr} \left( \mathbf{A}\mathbf{B}^T \right) = \text{tr} \left( \mathbf{B}\mathbf{A}^T \right) \] (69)

Now, we define the matrix \( \mathbf{B} \) as follows:
\[ \mathbf{B}_{N_x\times N_y} = \mathbf{Y}_{N_x\times N_y}(\theta) \mathbf{Y}^T_{N_x\times N_y}(\theta) \] (70)

From here on, as \( \theta \) is known, \( \mathbf{C}(\theta) \) will be shown by \( \mathbf{C} \). Therefore, the optimization problem is recast to

\[ \begin{align*}
\max_{\mathbf{B}} & \quad \text{tr} \left( \mathbf{A} \right) = \ln | 1 + \mathbf{B}\mathbf{C} | \\
\text{s.t.} & \quad \text{tr} \left( \mathbf{B} \right) = N_x.
\end{align*} \] (71)

In order to solve it, we use the Lagrangian method:

\[ \begin{align*}
\nabla \mathbf{B} \text{tr} \left( \mathbf{B} \right) - \mathbf{B} \text{tr} \left( \mathbf{B} \right) - N_x = 0 \\
\text{tr} \left( \mathbf{B} \right) = N_x,
\end{align*} \] (72)

where \( \nabla \mathbf{B} \text{tr} \left( \mathbf{B} \right) \) means the derivative of the scalar \( \text{tr} \left( \mathbf{B} \right) \) wrt all elements of \( \mathbf{B} \). It can be shown that

\[ \begin{align*}
\nabla \mathbf{B} \text{tr} \left( \mathbf{B} \right) - \mathbf{B} \text{tr} \left( \mathbf{B} \right) - N_x = 0,
\end{align*} \] (74)

So (72) can be rewritten as

\[ \begin{align*}
\left( \mathbf{C}(\mathbf{I} + \mathbf{B}\mathbf{C})^{-1} \right)^T - \lambda \mathbf{I} = 0, \\
\text{tr} \left( \mathbf{B} \right) = N_x,
\end{align*} \] (75)

which results in

\[ \left( \mathbf{C}(\mathbf{I} + \mathbf{B}\mathbf{C})^{-1} \right)^T = \lambda \mathbf{I} \Rightarrow \mathbf{B}\mathbf{C} = \lambda^{-1} \mathbf{C} - \mathbf{I}. \] (76)

Therefore,

\[ \begin{align*}
\mathbf{B} = \lambda^{-1} \mathbf{I} - \mathbf{C}^{-1} \\
\text{tr} \left( \mathbf{B} \right) = N_x
\end{align*} \] (77)

or,

\[ \begin{align*}
\mathbf{Y}(\theta)\mathbf{Y}(\theta)^T = \lambda^{-1} \mathbf{I} - \mathbf{C}^{-1} \\
\text{tr} \left( \mathbf{Y}(\theta)\mathbf{Y}(\theta)^T \right) = N_x
\end{align*} \] (77)

We can deduce two results from (77). First, it is inferred that \( \mathbf{Y}(\theta)\mathbf{Y}(\theta)^T \) is diagonal (since \( \mathbf{C} \) is diagonal), which results in the orthogonality between the received signals. Second, the diagonal elements of \( \lambda^{-1} \mathbf{I} - \mathbf{C}^{-1} \) determine the transmit powers.

Next, using (77), we will synthesize the transmit signals \( \langle \mathbf{s}_i \rangle_{i=1}^{N_x} \). To do this end, we should first, obtain the relation between \( \mathbf{Y}(\theta) \) and \( \langle \mathbf{s}_i \rangle_{i=1}^{N_x} \). Let us define the matrix operator \( \mathbf{J}(\tau_i) \) as

\[ \mathbf{J}(\tau_i) = \begin{bmatrix}
\mathbf{0}_{N_x 	imes L} & \mathbf{I}_{L 	imes N_x} & \mathbf{0}_{N_x 	imes L} \\
\end{bmatrix} \] (78)

Then, using (13) and (15), we can write

\[ \mathbf{Y}(\theta) = \left[ \mathbf{J}(\tau_1) \mathbf{s}_1 \right] \left[ \mathbf{J}(\tau_2) \mathbf{s}_2 \right] \cdots \left[ \mathbf{J}(\tau_{N_x}) \mathbf{s}_{N_x} \right] \] (79)

The \((ij)\)th element of \( \mathbf{Y}^T(\theta)\mathbf{Y}(\theta) \) can be expressed as

\[ (ij)_{\mathbf{Y}^T(\theta)\mathbf{Y}(\theta)} = s_i^T \mathbf{J}(\tau_i) \mathbf{J}(\tau_i) s_j = 0, \quad i \neq j \] (80)

Subsequently, (77) can be written as the following equations:

\[ \begin{align*}
\mathbf{s}_i^T \mathbf{s}_i &= \left( \lambda^{-1} - \frac{1}{\lambda} \right)_{ii}, \\
\sum_{i=1}^{N_x} \mathbf{s}_i^T \mathbf{s}_i &= N_x, \\
\mathbf{s}_i^T \mathbf{J}(\tau_i) \mathbf{J}(\tau_i) \mathbf{s}_j &= 0, \quad \forall i \neq j, \quad i, j = 1, \ldots, N_x
\end{align*} \] (81)

References