Prey-predator model under fuzzy uncertanties

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Abstract. This work studies the influence of fuzzy uncertainties on the asymptotic behavior of the solution of a prey-predator model. Here, initial conditions and parameters are interpreted as fuzzy variable. The population densities at a specific time are also interpreted as a fuzzy variable in which the possibility distribution function depends on the possibility distribution functions of the parameters. We provide closed formulas for expected values of some equilibrium points. We also compare the expected value of the fuzzy solution with the deterministic solution providing computational simulations in order see the difference between theses approaches.

Keywords: fuzzy dynamical systems; expected value; fuzzy variables; possibility theory; fuzzy solutions

1 Introduction

It is not always possible to know exactly the initial number of individuals or the carrying capacity in a given environment in applied problems of population dynamics. In general, one gets information by means of linguistic statements such as *the initial condition is approximately* x_0 or *the carrying capacity is about* k_0. To the extent that the label *approximately* is imprecise, it can be modeled as a fuzzy set. Thus, linguistic statements like these can be seen as fuzzy restrictions on the values taken by the variable of interest [1].

Zadeh proposed a fuzzy restriction as a possibility distribution with its membership function playing the role of a possibility distribution function. In the context of population dynamics, let us suppose that the label *approximately* x_0 is modeled by a fuzzy set $\boldsymbol{x}_0$ with membership $\mu_{\boldsymbol{x}_0}(x)$. Thus, given a specific numerical value $x = u_0$, the value $\mu_{\boldsymbol{x}_0}(u_0)$ is the degree of possibility that the actual initial condition of the dynamical system assumes the value u_0 given the proposition *the initial condition is approximately* x_0. Thus, the membership function $\mu_{\boldsymbol{x}_0}(x)$ is the distribution of the possibility associated with the variable *initial condition.*

Once we do not have precise information about the actual value of initial condition or parameters we can not require a precise description of the state of the population on a fixed time $t > 0$. It is reasonable to look for a description of the state of the system by means of a fuzzy restriction as *the state of the system at time* τ *is approximately* u_0. This, in turn, defines a possibility distribution on the values assumed by the state of the system.

Several approaches have been presented in order to consider fuzzy uncertainties on differential equations. Some authors use the $H-$ derivative to obtain solutions with fuzzy uncertainties [2–11]. Others construct the fuzzy solutions of differential equations by means of a family of differential inclusions [12–15]. A third approach consists in applying the Zadeh extension principle on the initial conditions of deterministic solutions to obtain fuzzy solutions of differential equations [16–24].

However, when dealing with fuzzy uncertainties as fuzzy restrictions on the values taken by the variable of interest, or possibility distribution function, one faces the problem of describing how the possibility distribution function evolves over time. One gets a similar problem when looking at the initial condition as a random variable described by probability distribution functions [25–28]. Thus, here we follow a similar approach used in probability theory to deal with fuzzy uncertainties on initial conditions and parameters. As we will see, interpreting in this way, we end up with the approach of applying the Zadeh extension principle on the initial condition and parameters of deterministic solutions. Therefore, considering parameters and initial conditions as possibility distribution functions of fuzzy variables lead us to define fuzzy solutions by taking Zadeh extension of deterministic solutions which is similar to the third approach previously described.

A naive approach to handle deterministic differential equations with uncertainties, fuzzy or probabilistic, on parameters of the dynamical systems would be to obtain a representative value of these parameters by means of some statistical procedure [31]. These representative values, in turn, are inserted in the equation and the analysis is carried on. That is, by this approach, we deal with uncertainties prior to the analysis of the dynamical systems. Here we are going to think in another direction. First, we are interested in describing how the possibility distribution function evolves over time and, after that, we calculate a representative value of such a fuzzy variable. As we will see, these two approaches may lead to distinct results.

Thus, in order to measure the effects of the fuzzy uncertainties on the dynamics we wonder about the expected value of the values assumed by the state of the system. We do this by comparing the expected value with the deterministic solution defined by the expected value of the initial condition and parameters. We provide closed-form expressions for the expected value of the fuzzy variable described by the logistic equation and for the fuzzy variable that represents the maximum growth time.

The organization of this article is as follows: in Section 2 we discuss some basic concepts on fuzzy sets and fuzzy variables; in Section 3 we present the prey-predator model we are considering in this work; in Section 4 we discuss about the expected value of the fuzzy variable that describes the state of the system at time $t > 0$; in Section 5 we discuss about the expected value of the equilibrium points; in Section 6 we provide numerical simulations to illustrate our main results.

2 Some basic concepts

2.1 Fuzzy sets

As it is well known, given a set U, a fuzzy subset of U is characterized by a function defined on U taking values on $[0, 1]$ ([33]). This function is called *membership function.* Given $\alpha \in [0, 1]$, an α-cut or α-level of a fuzzy set is defined as the set of points of U where the membership function is greater than or equal to α. Precisely, if $\boldsymbol{u}$ is a fuzzy set of U with membership function $\mu_{\boldsymbol{u}} : U \to [0, 1]$ then, for $0 < \alpha \leq 1$, the α-cut of $\boldsymbol{u}$ is a subset of U given by

$$[\boldsymbol{u}]^\alpha = \{x \in U : \mu_{\boldsymbol{u}}(x) \geq \alpha\}$$

and, for $\alpha = 0$,

$$[\boldsymbol{u}]^0 = \overline{\{x \in U : \mu_{\boldsymbol{u}}(x) > 0\}}$$

is the *support* of $\boldsymbol{u}$ ([4]).

Let us denote by $\mathcal{F}(U)$ the set of fuzzy subsets of $U \subset \mathbb{R}$, in which the α-cuts are non-empty, compact and (simply-) connected for every $\alpha \in [0, 1]$. We can measure the distance between two fuzzy sets in the following way: given two points $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{F}(U)$, the distance between $\boldsymbol{u}$ and $\boldsymbol{v}$ is defined by

$$d_\infty(\boldsymbol{u}, \boldsymbol{v}) = \sup_{\alpha \in [0,1]} d_H([\boldsymbol{u}]^\alpha, [\boldsymbol{v}]^\alpha), \tag{1}$$

where d_H is the Hausdorff distance for compact sets. We also denote by $\chi_{\{A\}}$ the characteristic function of the set A.

In this work, we are interested in *fuzzy variables* taking values on $U = [0, +\infty)$, the set of non-negative real numbers. However, we describe the following concepts for a general set $U \subset \mathbb{R}^n$.

A fuzzy subset $\boldsymbol{u}$ of U, defined by a membership function $\mu_{\boldsymbol{u}} : U \to [0, 1]$, induces, according to Zadeh [1], a *possibility distribution function* on the set of values of a variable of interest ξ. That is, if ξ is a fuzzy variable then $\mu_{\boldsymbol{u}}(x)$ is the degree of possibility that ξ assumes the particular value x. In the context of possibility theory, $\mu_{\boldsymbol{u}}(x) = 0$ means that it is impossible that the variable ξ assumes the value x. The quantity $\mu_{\boldsymbol{u}}(x)$ represents the degree of possibility of the assignment $\xi = x$, where some values x being more possible than others. The closer the value $\mu_{\boldsymbol{u}}(x)$ is to 1, the more possible it is that x is the actual value of the variable.

It is well known ([1]) that given a subset $A \subset U$, the *possibility measure* of A is defined by

$$\mathrm{Pos}_\mu(A) = \sup_{x \in A} \mu_{\boldsymbol{u}}(x),$$

and the *necessity measure* of A is defined by

$$\mathrm{Nec}_\mu(A) = 1 - \mathrm{Pos}_\mu(A^c),$$

where A^c stands for the complement set of A in U. The *credibility measure* of A, $A \subset U$, according to [32], is defined as

$$\mathrm{Cr}_\mu(A) = \frac{1}{2}\left(\mathrm{Pos}_\mu(A) + \mathrm{Nec}_\mu(A)\right).$$

We point out that $\mathrm{Pos}_\mu(A)$ is a measure of the possibility of the fuzzy variable ξ to assume values in A. Note that $\mathrm{Pos}_\mu(\emptyset) = 0$ and $\mathrm{Pos}_\mu(U) = 1$. On the other hand, $\mathrm{Nec}_\mu(A)$ can be seen as a measure of the fuzzy variable ξ not assume values in A^c. Thus, both numbers are measures for the question if an event A either occur or not ([36]). Thus, the $\mathrm{Cr}_\mu(A)$ is the average of these two answers to the occurrence of an event A.

In order to get a representative value of a fuzzy variable ξ, it is important to define the concept of its expected value ([1]). The *expected value* of ξ is defined as

$$E[\xi] = \int_0^\infty \mathrm{Cr}_\mu([r, +\infty))dr - \int_{-\infty}^0 \mathrm{Cr}_\mu((-\infty, r))dr,$$

provided that at least one of these integrals is finite ([32]).

Now, when U is the set of non-negative real numbers, if we define the quantities

$$\xi'_\alpha = \inf\{x : \mu_{\boldsymbol{u}}(x) \geq \alpha\} \quad \text{and} \quad \xi''_\alpha = \sup\{x : \mu_{\boldsymbol{u}}(x) \geq \alpha\},$$

for all $\alpha > 0$, then the previous formula becomes

$$E[\xi] = \frac{1}{2}\int_0^1 (\xi'_\alpha + \xi''_\alpha)\, d\alpha, \tag{2}$$

provided that ξ'_α and ξ''_α are finite ([32]). We emphasize that the definition of $\mathrm{Pos}(A)$, $\mathrm{Nec}(A)$ and $\mathrm{Cr}(A)$ depends on the possibility distribution function $\mu_{\boldsymbol{u}}$ of the fuzzy variable ξ.

2.2 Transformations of fuzzy variables

Consider now a continuous function g defined on some subset of the real numbers. If ξ is a fuzzy variable then so is $\eta = g(\xi)$, and there is a natural way to define a possibility distribution function $\mu_{g(\boldsymbol{u})}(x)$ to $g(\xi)$ from the distribution $\mu_{\boldsymbol{u}}(x)$ of ξ as it follows: given $A \subset U$, then $g(\xi)$ assumes values on A if and only if ξ assumes values on $g^{-1}(A)$. Thus, by definition, the possibility of $g(\xi)$ to assume values in A is the same as the possibility of ξ to assume values on $g^{-1}(A)$. Thus, following [1], it turns out that

$$\mathrm{Pos}_\eta(A) = \sup_{y \in A} \mu_{\hat{g}(\boldsymbol{u})}(y) = \sup_{x \in g^{-1}(A)} \mu_{\boldsymbol{u}}(x) = \mathrm{Pos}_\xi\left(g^{-1}(A)\right),$$

and, as a consequence, we obtain the possibility distribution function of the fuzzy variable $\eta = g(\xi)$ by taking

$$\mu_{\hat{g}(\boldsymbol{u})}(y) = \sup_{x \in g^{-1}(y)} \mu_{\boldsymbol{u}}(x). \tag{3}$$

We remark that expression (3) is the Zadeh extension of g as given in [33]. This is the reason why we denote the possibility distribution function of $g(\xi)$ by $\mu_{\hat{g}(\boldsymbol{u})}(y)$. We also remark that this approach is similar to that one followed in the context of transformation of random variables in probability theory (see, for instance, [25]).

According to [34], if g is monotone (increasing or decreasing) then the expected value of the fuzzy variable $\eta = g(\xi)$ can be computed by

$$E\left[g\left(\xi\right)\right] = \frac{1}{2}\int_0^1 \left(g(\xi'_\alpha) + g(\xi''_\alpha)\right) d\alpha. \tag{4}$$

This formula will be useful in the following sections.

2.3 Several fuzzy variables

One faces the problem of uncertainties on several variables in population dynamics and other applications. Before proceeding to the next section, let us present the main ideas on this subject.

Let ξ_1 and ξ_2 be fuzzy variables with possibility distribution functions $\mu_{\boldsymbol{u}_1}$ and $\mu_{\boldsymbol{u}_2}$, respectively, both defined on U. Following [1], these variables define a fuzzy variable, namely $\eta = (\xi_1, \xi_2)$, on $U \times U$, in which its joint possibility distribution function $\mu_{\boldsymbol{u}} : U \times U \to [0, 1]$ is given by

$$\mu_{\boldsymbol{u}}(x, y) = \min\{\mu_{\boldsymbol{u}_1}(x), \mu_{\boldsymbol{u}_2}(y)\}. \tag{5}$$

We are assuming that the variables ξ_1 and ξ_2 are unrelated, or *non-interactive*, in the sense that a specific value of ξ_1 gives no information about the possible values that ξ_2 can assume.

3 A prey-predator model

We are considering the prey-predator model given by the system of differential equations

$$\begin{cases} \dfrac{dx}{dt} = a_1x - b_1x^2 - c_1xy, \quad x(0) = x_o > 0, \\ \\ \dfrac{dy}{dt} = a_2y - b_2y^2 + c_2xy, \quad y(0) = y_o > 0, \end{cases} \tag{6}$$

in which the parameters are all non negative except possibly a_2. Let $\varphi_t(x_o, y_o, p)$ be the solution of Eq. (6) at (x_o, y_o) and a vector of parameters p. As is well known, the application $\varphi_t : \mathbb{R}^2 \to \mathbb{R}^2$ is the flow acting on the phase space $\mathbb{R}^2$. Thus, for every initial condition $(x_o, y_o) \in \mathbb{R}^2$ we have a deterministic solution $\varphi_t(x_o, y_o, p) = (x(t, x_o, y_o, p), y(t, x_o, y_o, p))$.

Eq.(6) has Jacobian matrix at $(\bar{x}, \bar{y})$ given by

$$J(\bar{x}, \bar{y}) = \begin{pmatrix} a_1 - 2b_1\bar{x} - c_1\bar{y} & -c_1 \\ c_2 & a_2 - 2b_2\bar{y} + c_2\bar{x} \end{pmatrix} \tag{7}$$

and thus it turns out that:

a) The equilibrium point $q_1 = (0, 0)$ is unstable;
b) The equilibrium point $q_2 = (a_1/b_1, 0)$ is unstable;
c) The equilibrium point $q_3 = (0, a_2/b_2)$ is unstable provided that $c_1a_2 < a_1b_2$;
d) The equilibrium point

$$q_4 = \left(\frac{a_1b_2 - a_2c_1}{b_1b_2 + c_1c_2}, \frac{a_2b_1 + a_1c_2}{b_1b_2 + c_1c_2} \right)$$

is unstable provided that $a_1b_2 < a_2c_1$.

The following analyzes the behavior of the solution $x(t)$ and $y(t)$ under fuzzy uncertainties.

4 Fuzzy uncertainties on the model

Due to the lack of complete information or error of measurements, more often than not, one needs to deal with imprecision on the parameters. A naive approach to deal with uncertainties in models like the previous one defined by Eq. (6), it could be to compute the average values of the parameters and then analyzing the dynamics by means of its deterministic solution using these average values for the parameters.

Thus, let us assume that parameters and initial conditions are under restriction given by fuzzy label as *approximately*, for instance. That is, we are assuming that these variables satisfy a statement like *the variable* ξ *is approximately* ξ_o. Thus, according to Zadeh, the membership function of the fuzzy label *approximately* is the possibility distribution function of ξ. In works like [24] and [16], in case of having fuzzy uncertainties on the initial conditions, the authors define the *fuzzy solution* of Eq. (6) as the Zadeh's extension of the deterministic flow $\varphi_t : \mathbb{R}^2 \to \mathbb{R}^2$. In case of having fuzzy uncertainties on other parameters of Eq. (6), we consider those parameters as initial conditions of a differential equation with zero derivative and proceed as before, taking the Zadeh's extension on the initial condition of the deterministic flow.

Here, however, we are going to take another direction. Since we are assuming that x_o, y_o and some parameter, or vector of parameters, p in Eq. (6) are fuzzy variables, these quantities $x(t, x_o, y_o, p)$ and $y(t, x_o, y_o, p)$, for a fixed $t > 0$, are fuzzy variables as well. Following the recipe described in the previous sections, for a fixed $t > 0$, we can obtain the possibility distribution function of $\varphi_t(x_o, y_o, p)$ by means of the Zadeh's extension on the parameters x_o, y_o and p of the functions $x(t, x_o, y_o, p)$ and $y(t, x_o, y_o, p)$. That is, the number of pray and predators at a fixed time $t \geq 0$ are fuzzy variables whose possibility distribution are $\hat{x}(t, \boldsymbol{x}_o, \boldsymbol{y}_o, \boldsymbol{p})$ and $\hat{y}(t, \boldsymbol{x}_o, \boldsymbol{y}_o, \boldsymbol{p})$.

Although these two approaches seem different, the relationship between them is as it follows ([17]).

Theorem 1 ([17]). *Let the applications $\hat{\pi}_x$ and $\hat{\pi}_y$ be the Zadeh's extensions of the orthogonal projections $\pi_x : \mathbb{R}^2 \to \mathbb{R}$ and $\pi_y : \mathbb{R}^2 \to \mathbb{R}$ on the x and y axis, respectively. Then it follows that:*

$$\hat{x} = \hat{\pi}_x \circ \hat{\varphi}_t \qquad \hat{y} = \hat{\pi}_y \circ \hat{\varphi}_t.$$

4.1 Fuzzy uncertainties on equilibrium points

We consider fuzzy uncertainties on the parameters so that the equilibrium points are also fuzzy variables whose the possibility distribution function are the Zadeh's extension of the expressions that define such equilibrium points. In [17] the authors have proved if an equilibrium point is asymptotically stable then $\hat{x}$ and $\hat{y}$ converge to the Zadeh's extension of the x and y coordinates of the expressions that define such equilibrium point. In other words, we have that $\hat{q}$ is the Zadeh's extension of an equilibrium point then

$$\hat{x} = \hat{\pi}_x \circ \hat{\varphi}_t \to \hat{\pi}_x \circ \hat{q}$$

$$\hat{y} = \hat{\pi}_y \circ \hat{\varphi}_t \to \hat{\pi}_y \circ \hat{q}$$

Theorem 2. *Suppose that the fuzzy set $\boldsymbol{c}_1$ is possibility distribution function of the fuzzy variables c_1 and furthermore suppose that*

$$\eta'_\alpha = \inf\{x : \mu_{\boldsymbol{c}_1}(x) \geq \alpha\}, \qquad \eta''_\alpha = \sup\{x : \mu_{\boldsymbol{c}_1}(x) \geq \alpha\}.$$

Then it turns out that:

a) The α - cuts of the fuzzy set $\hat{\pi}_x \circ \hat{q}$ are

$$\left[\frac{a_1 b_2 - a_2 \eta''_\alpha}{b_1 b_2 + c_2 \eta''_\alpha}, \frac{a_1 b_2 - a_2 \eta'_\alpha}{b_1 b_2 + c_2 \eta'_\alpha}\right].$$

b) The α - cuts of the fuzzy set $\hat{\pi}_y \circ \hat{q}$ are

$$\left[\frac{a_2 b_1 + a_1 c_2}{b_1 b_2 + c_2 \eta''_\alpha}, \frac{a_2 b_1 + a_1 c_2}{b_1 b_2 + c_2 \eta'_\alpha}\right].$$

Proof. Since π_x, π_y and each coordinate of the equilibrium point q_4 are continuous functions for $c_1 > 0$ then we have for a continuous function f that $[\hat{f}(\boldsymbol{u})]^\alpha = f([\boldsymbol{u}]^\alpha)$. In both cases, the x and y coordinates are decreasing functions with respect to c_1 and this proves the statement.

Next we look at the expected values of fuzzy solutions and equilibrium points.

5 Expected values of fuzzy solutions and equilibrium points

We are interested, in this section, in the behavior of the expected values of the fuzzy solution defined by considering x_o, y_o and c_1 as fuzzy variables whose the possibility distribution functions are $\boldsymbol{x}_o$, $\boldsymbol{y}_o$, $\boldsymbol{c}_1$, respectively. To this end, we have the following theorem.

Theorem 3. *Let $q_x(c_1)$ and $q_y(c_1)$ the fuzzy variables defined by the x and y coordinates of q_4 respectively. If c_1 is a fuzzy variable with triangular possibility distribution function $\mu_{\boldsymbol{c}_1}(x) = (c-\varepsilon/c/c+\varepsilon)$ then we have that:*

$$E\left[q_x\right] = -\frac{a_2}{c_2} + \frac{b_2 A}{2c_2^2\varepsilon}\ln\left(\frac{B+c_2\varepsilon}{B-c_2\varepsilon}\right)$$

$$E\left[q_y\right] = \frac{A}{2c_2\varepsilon}\ln\left(\frac{B+c_2\varepsilon}{B-c_2\varepsilon}\right)$$

in which $A = a_2b_1 + a_1c_2$ and $B = b_1b_2 + c_2c$.

Proof. To prove the first statement we must observe that the α - cuts of $\mu_{\boldsymbol{c}_1}(x)$ are the intervals $[\eta'_\alpha, \eta''_\alpha]$ where

$$\eta'_\alpha = c - (1-\alpha)\varepsilon \quad \text{and} \qquad \eta''_\alpha = c + (1-\alpha)\varepsilon.$$

Since the expected value of the fuzzy variable $q_x c$ is given by

$$E[q_x(c_1)] = \frac{1}{2}\int_0^1 \left[q_x(\eta'_\alpha) + q_x(\eta''_\alpha)\right] d\alpha,$$

integrating we obtain the desired result.

On the other hand, the second statement can be prove similarly taking into account the expression that defines $q_y(c_1)$.

We have also the following theorem.

Theorem 4. *Suppose that the fuzzy sets $\boldsymbol{x}_o$, $\boldsymbol{y}_o$ and $\boldsymbol{c}_1$ are possibility distribution functions of the fuzzy variables x_o, y_o and c_1, respectively, and let q_x and q_y be the fuzzy variables defined by the x and y coordinates of q_4. If the equilibrium point $q_4(c_1)$ is asymptotically stable for all $c_1 \in [\boldsymbol{c}_1]^0$ then we have that:*

a) The expected value of the fuzzy variable $x(t)$ converges to the expected value of q_x. That is, $E\left[x(t)\right] \to E\left[q_x\right]$ as $t \to \infty$.
b) The expected value of the fuzzy variable $y(t)$ converges to the expected value of q_y. That is, $E\left[y(t)\right] \to E\left[q_y\right]$ as $t \to \infty$.

Proof. Since $q_4(c_1)$ is asymptotically stable for all $c_1 \in [\boldsymbol{c}_1]^0$, according to [35] (Corollary 14, p. 12), the family of function indexed by t, $x(t) : K \to \mathbb{R}$, $K = [\boldsymbol{x}_o]^0 \times [\boldsymbol{y}_o]^0 \times [\boldsymbol{c}_1]^0$, converges uniformly to $f : K \to \mathbb{R}$, defined by $f(x_o, y_o, c_1) =$

q_x, as $t \to \infty$. That is, given $\varepsilon > 0$, there is a $T > 0$ such that for all $t > T$ we have $|x(t, x_o, y_o, c_1) - q_x(c_1)| < \varepsilon$ for all $(x_o, y_o, c_1) \in K$. Thus,

$$\begin{aligned}|E\left[x(t,x_o,y_o,c_1)\right] - E\left[q_x(c_1)\right]| &= \left|\frac{1}{2}\int_0^1 \left[x(t,h'_\alpha) - q_x(\zeta''_\alpha) + x(t,h''_\alpha) - q_x(\zeta'_\alpha)\right] d\alpha\right| \\ &\leq \frac{1}{2}\int_0^1 |x(t,h'_\alpha) - q_x(\zeta''_\alpha)|\, d\alpha + \frac{1}{2}\int_0^1 |x(t,h''_\alpha) - q_x(\zeta'_\alpha)|\, d\alpha \\ &< \varepsilon\end{aligned}$$

in which $h'_\alpha = (\xi'_\alpha, \eta'_\alpha, \zeta'_\alpha)$ and $h''_\alpha = (\xi''_\alpha, \eta''_\alpha, \zeta''_\alpha)$. This inequality proves the first statement.

We can prove the second statement analogously.

Once that $q_x(E(c_1))$ is not necessarily equal to $E(q_x(c_1))$ then we can conclude from last theorem that the expected value a the fuzzy solution are not necessarily equal a deterministic solution, at least not near an equilibrium point. Thus, although we are not able to find a closed-formula for the expected value of fuzzy solutions of Eq. (6) from last statement we can conclude that the two approaches of dealing with uncertainties discussed here provide different numerical values.

6 Worked example

In order to illustrate the results obtained in previous sections, let us consider x_o, y_o and c_1 as fuzzy variables given by the triangular fuzzy possibility distribution functions $\mu_{\boldsymbol{x}_o}(x) = (5/6/7)$, $\mu_{\boldsymbol{y}_o}(x) = (0.01/0.51/1.01)$ and $\mu_{\boldsymbol{c}_1}(x) = (0.0150/0.0250/0.0400)$. The others parameters of Eq. (6) are: $a_1 = 0.1$, $b_1 = 0.01$, $a_2 = -0.02$, $b_2 = 0.01$ and $c_2 = 0.005$.

The expected value of the fuzzy variable c_1 is 0.0250 and so the equilibrium point $q_4 = (6.6667, 1.3333)$ is asymptotically stable. However, since c_1 is a fuzzy variable thus q_4 is also a fuzzy variable and, as predicted by Theorem 3, the expected value of this fuzzy variable is $E(q_4) = (6.7119, 1.3560)$. By Theorem 4, the expected value of the projections of the fuzzy solution of Eq. (6), the red curves in Figure 1, converges to $E(q_4)$ as the time evolves.

7 Conclusion

In this work we have interpreted the initial condition and parameters of a prey-predator model as fuzzy variables in which the possibility distribution function is given by a membership function of fuzzy sets. These fuzzy sets represent a label acting as a restriction on the values taken by the variables of interest. As we have shown, this approach leads to different results than the standard approach in which the uncertainties are handled apart from the dynamical system. Finally, we would like to point out that if we see parameters and initial

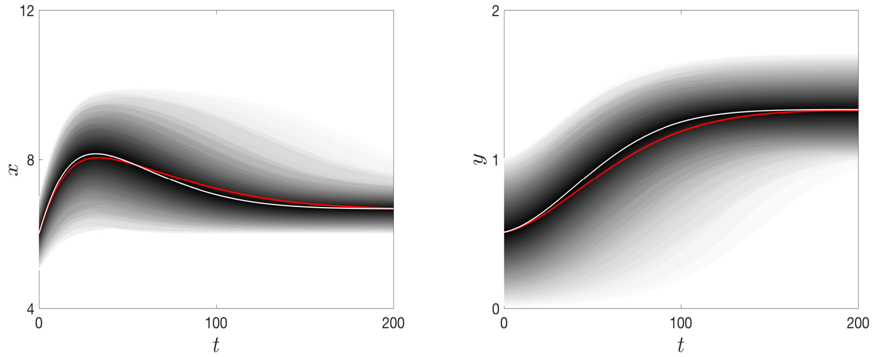

Fig. 1. Projections of the fuzzy solutions of Eq. (6). The white curve represents the deterministic solution calculated using the expected values of the fuzzy variables x_o, y_o and c_1. The red curve represents represents the expected value of the fuzzy projections.

conditions as possibility distributions functions of fuzzy variables then Zadeh's extension of deterministic solutions is the natural way to define fuzzy solutions for autonomous differential equations.

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