Optimal Production and Setup Control of a Dynamic Two-Product Manufacturing System: Analytical Solution

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Abstract—This paper deals with the optimal control of a one-machine two-product manufacturing system with setup changes, operating in a continuous time dynamic environment. The system is deterministic. A known constant setup time is incurred when production is switched from one product to the other. Each product or part has specified constant processing time and constant demand rate, as well as an infinite supply of raw material. The problem is formulated as a feedback control problem. The objective is to minimize the sum total of backlog and inventory costs incurred over a finite horizon. The optimal solution provides the optimal production rate and setup switching epochs as a function of the state of the system (backlog and inventory levels). For the steady state, the optimal cyclic schedule is determined. To solve the transient case, the system's state space is partitioned into mutually exclusive regions, such that, with each region the optimal control policy is determined analytically.

Keywords—Production control, Dynamic setups, Scheduling

1. INTRODUCTION

The setup scheduling problem has been extensively investigated by many researchers due to its importance, since almost no manufacturing system is perfectly flexible. A special version of this problem is known as the Economic Lot Scheduling Problem (ELSP). The goal of the ELSP is the scheduling of the production of several products on one or more similar machines. Each product has a known constant demand rate and a fixed production rate when it is being produced. When production is switched from one product to the next, a sequence independent constant setup time as well as a fixed setup cost are incurred. The system is in steady state and no backlog is allowed. The objective is to determine lot sizes that minimize the average setup and inventory holding costs per unit time. A thorough review of the models used for the ELSP can be found in [1]. Recent research in the field includes the work of Dobson [2], Goyal [3], and Carreno [4]. Gallego [5] extended the ELSP problem to allow backlog.

The more general setup scheduling problem can be formulated as a feedback control problem. The control must respond to various disruptions so as to minimize a certain criterion. This kind of formulation is usually classified under production flow control models. The first production flow control model was introduced by Kinemia and Gershwin [6] in the early '80s, where they
modeled the movement of parts as a continuous flow and suggested a feedback control of the flow rates of parts through a flexible manufacturing system, to respond to machine failures and to closely track the demand of all parts.

Using the formalism of Kimemia and Gershwin [6], Sharifnia et al. [7] investigated a single machine setup scheduling problem. They proposed a feedback setup scheduling policy which uses corridors in the surplus/backlog space to determine the epochs of setup changes. The corridors are chosen so as to guide the surplus trajectory to a target cycle which is referred to as the Limit Cycle. Srivatsan and Gershwin [8] extended the ideas of Sharifnia et al. and developed methods for choosing the parameters of the corridors when the setup frequencies are not all the same. Perkins and Kumar [9] and Kumar and Seidman [10] studied the performance of distributed real-time setup scheduling policies and investigated the conditions under which the system remains stable. Connolly [11] proposes a heuristic for a two-part-type, one-machine setup system, based on known results of a perfectly flexible system. The approach is based on a local optimization that maximizes the progress toward a target Limit Cycle. Bai and Elhafsi [12] studied the real-time scheduling of an unreliable one-machine, two-part-type, nonresumable setup system. They provide a continuous dynamic programming formulation of the problem which they discretize and solve numerically. Based on the numerical solution they provide two heuristics to solve the stochastic problem. Gallego [5] studied the ELSP problem in the case of a machine subject to disruptions of small magnitude. He shows that the optimal policy selects the production lot sizes as a linear function of the current inventory levels.

In this paper, we study the production and setup control of a deterministic one-machine, two-part-type system within a feedback control framework (Kimemia and Gershwin's framework). This work is based on a previous work by the authors (Bai and Elhafsi [13]), where the same system was studied. In [13], the solution was obtained by means of a novel algorithm which we referred to as the Direction Sweeping Algorithm (DSA). In this paper, we determine the optimal solution analytically. One of the advantages of an analytical solution is that it allows us to study different extreme cases which are of practical or theoretical importance.

The remainder of the paper is organized as follows. In Section 2, we present an optimal control formulation of the dynamic setup problem. In Section 3, we provide some previous results from [13] which are relevant to this paper. In Section 4, the optimal policy is obtained analytically. In Section 5, we study some extreme cases which are much easier to deal with when an analytical solution is available. We conclude our study with Section 6.

2. PROBLEM FORMULATION

We consider a manufacturing system which has a single machine that produces two distinct parts (or products). There is a constant demand rate \( d_i \) \( (i = 1, 2) \) for each part. The machine incurs a given constant setup time \( S_i \) \( (i = 1, 2) \) when switching from Part Type \( j \) to Part Type \( i \) \( (j \neq i) \). Let \( x_i(t) \) be the production surplus (positive or negative) of Part Type \( i \) \( (i = 1, 2) \) at time \( t \); a positive value of \( x_i(t) \) represents inventory while a negative value represents backlog. Here, we follow the general framework introduced by Kimemia and Gershwin [6] and model the production flow as continuous rather than discrete. Let \( u_i(t) \) be the controlled production rate of the machine producing Type \( i \) parts at time \( t \). Denote by \( \sigma(t) \) the setup state vector of the machine at time \( t \), which is defined as

\[
\sigma(t) = (\sigma_1(t), \sigma_2(t), \sigma_{12}(t), \sigma_{21}(t)),
\]

where, \( \sigma_i(t), \sigma_{ij}(t) \) \( (j \neq i, i = 1, 2, j = 1, 2) \) are right continuous binary functions of \( t \), such that \( \sigma_i(t) = 1 \), when the machine is ready to produce Type \( i \) parts and \( \sigma_i(t) = 0 \) otherwise; \( \sigma_{ij}(t) = 1 \), when the machine is undergoing a setup change from Part Type \( i \) to Part Type \( j \) and \( \sigma_{ij}(t) = 0 \) otherwise. Let \( s(t) \) be a right continuous function of \( t \) that takes on the value \( \delta_i \) at the beginning
of each setup change to Part Type \( i (i = 1, 2) \) and decreases with time. It will indicate whether a setup is completed or not.

We assume, initially the machine is not set up to either part type.

**SYSTEM DYNAMICS AND CONSTRAINTS.** The dynamics of the system can be described by:

\[
\begin{align*}
\frac{d}{dt} x_i(t) &= u_i(t) - d_i, \quad i = 1, 2, \\
0 \leq u_i(t) &\leq U_i \sigma_i(t), \quad i = 1, 2,
\end{align*}
\]

where \( U_i \) is the maximum production rate of the machine when it is producing Type \( i \) parts. The setup states of the machine obey the following set of constraints:

\[
\sigma_1(t) + \sigma_2(t) + \sigma_{1,2}(t) + \sigma_{2,1}(t) = 1;
\]

for \( i = 1, 2, j = 1, 2, i \neq j \).

The above equations (3)–(6) merely state that if \( \sigma_i(t) = 1 \), we can either continue producing Part Type \( i \), or decide to switch production to Part Type \( j \). In the latter case we must spend exactly \( \delta_j \) amount of time setting up the machine for Part Type \( j \). That is, \( \sigma_{ij}(t) = 1 \) for exactly \( \delta_j \) amount of time \((0 \leq s(t) \leq \delta_j)\). After the setup change, \( \sigma_j(t) = 1 \), and the machine is ready to produce Part Type \( j \).

**STATE VARIABLES AND CONTROL VARIABLES.** The state variable of the system is given by \( x(t) = (x_1(t), x_2(t)) \). The variables \( u(t) = (u_1(t), u_2(t)) \) and \( a(t) = (a_1(t), a_2(t), a_{11}(t), a_{22}(t)) \) are the control variables. We denote by \( (a, u) \) the complete control vector.

**CAPACITY SET.** The capacity set represents the set of feasible production rates. When the setup state is \( \sigma(t) \), at time \( t \), it is given by:

\[
\Omega(\sigma(t)) = \{ u(t) \mid 0 \leq u_i(t) \leq U_i \sigma_i(t), \quad i = 1, 2 \}.
\]

**SETUP CONSTRAINTS SET.** The setup constraints set is the set of all possible setup vectors \( \sigma(t) = (\sigma_1(t), \sigma_2(t), \sigma_{12}(t), \sigma_{21}(t)) \) satisfying constraints (3)–(6). Let \( \Phi \) denote this set.

**ADMISSIBLE CONTROL POLICIES.** We denote by \( \Xi(\Phi, \Omega) \) the set of feasible controls, which depends on \( \Phi \) and \( \Omega \). The set of admissible control policies, \( \mathcal{A} \), is the set of all mappings \( \mu : \mathbb{R}^2 \rightarrow \Xi(\Omega, \Phi) \), which satisfy \( \mu(x) = (\sigma, u) \), and are piecewise continuously differentiable. These admissible control policies are feedback controls that specify the control actions (setup and production level of the machine) to be taken, given the state of the system.

**OBJECTIVE FUNCTION.** The objective is to find an optimal control policy \( \mu^* \in \mathcal{A} \), corresponding to a setup control \( \sigma^* = (\sigma_1^*, \sigma_2^*, \sigma_{12}, \sigma_{21}) \) and a production flow rate control \( u^* = (u_1^*, u_2^*) \), that minimizes for each initial state \( x(t) \) the following cost functional:

\[
J_\mu(x(t), t) = \int_t^{t_f} g(x(s)) \, ds,
\]

where \( g(x) \) is the penalty function.
where the minimization is over all functions $\mu(x(\tau)) = (\sigma(\tau), u(\tau))$, such that $x(\tau)$, $\sigma(\tau)$, and $u(\tau)$ satisfy constraint (1) and $(\sigma(\tau), u(\tau)) \in \Xi(\Psi, \Omega)$ for $t \leq \tau \leq t_f$, where $t_f$ is assumed to be sufficiently large.

In the next section, we summarize some results that have been obtained in [13], which will be needed in this paper.

3. BRIEF SUMMARY OF PREVIOUS RESULTS

In [13], it was shown that for a finite horizon formulation, the inventory/backlog state vector $x(t) = (x_1(t), x_2(t))$ reaches a cyclic schedule (also referred to as the Limit Cycle) in finite time. We referred to this cyclic solution as the steady state solution (Figure 3.1), which is characterized by its optimal location in $x$-space. It was also shown in [13], that the optimal production rate of the machine at the steady state is to produce at maximum speed when the machine is producing either part type. The location of the cyclic schedule is obtained by minimizing the average inventory and backlog costs incurred during one complete cycle and is given by the following points in $x$-space:

\[
A = \left( \begin{array}{c} A_1 \\ A_2 \end{array} \right), \quad B = \left( \begin{array}{c} B_1 \\ B_2 \end{array} \right), \quad C = \left( \begin{array}{c} C_1 \\ C_2 \end{array} \right), \quad D = \left( \begin{array}{c} D_1 \\ D_2 \end{array} \right), \quad I = \left( \begin{array}{c} I_1 \\ I_2 \end{array} \right),
\]

where,

\[
A_1 = \frac{c_1^-}{(c_1^+ + c_1^-)} \frac{1 - \rho_1}{(1 - \rho)} \delta d_1, \quad A_2 = \delta_2 d_2 - \frac{c_2^+}{(c_2^+ + c_2^-)} \frac{1 - \rho_2}{(1 - \rho)} \delta d_2, \quad (10)
\]

\[
B_1 = \frac{c_1^-}{(c_1^+ + c_1^-)} \frac{1 - \rho_1}{(1 - \rho)} \delta d_1 - \delta_2 d_1, \quad B_2 = -\frac{c_2^+}{(c_2^+ + c_2^-)} \frac{1 - \rho_2}{(1 - \rho)} \delta d_2, \quad (11)
\]

\[
C_1 = \delta_1 d_1 - \frac{c_1^+}{(c_1^+ + c_1^-)} \frac{1 - \rho_1}{(1 - \rho)} \delta d_1, \quad C_2 = -\frac{c_2^-}{(c_2^+ + c_2^-)} \frac{1 - \rho_2}{(1 - \rho)} \delta d_2, \quad (12)
\]

\[
D_1 = -\frac{c_1^+}{(c_1^+ + c_1^-)} \frac{1 - \rho_1}{(1 - \rho)} \delta d_1, \quad D_2 = \frac{c_2^-}{(c_2^+ + c_2^-)} \frac{1 - \rho_2}{(1 - \rho)} \delta d_2 - \delta_1 d_2, \quad (13)
\]

\[
I_1 = \left( \frac{c_1^-}{(c_1^+ + c_1^-)} \frac{1 - \rho_1}{(1 - \rho)} \right) d_1 \delta_1 - \left( \frac{1 - \rho_1}{(1 - \rho)} \right) d_1 \delta_2, \quad (14)
\]

\[
I_2 = \left( \frac{c_2^-}{(c_2^+ + c_2^-)} \frac{1 - \rho_2}{(1 - \rho)} \right) d_2 \delta_2 - \left( \frac{1 - \rho_2}{(1 - \rho)} \right) d_2 \delta_1.
\]

![Figure 3.1. Location of the cyclic schedule in x-space.](image)
Here and elsewhere in the paper, \( \rho_i = d_i/U_i \) (\( i = 1,2 \)) denotes the utilization factor of the machine by Type \( i \) parts, and \( \rho = \rho_1 + \rho_2 \) is the utilization factor of the machine. For the problem to be feasible, we need to have \( \rho < 1 \). Also, \( \delta = \delta_1 + \delta_2 \) which is the sum of the setup times.

The total average cost over one cyclic schedule is given by

\[
\text{Average Cost} = \frac{1}{2} \frac{c_1^r c_1^+}{(c_1^r + c_1^+)} \frac{(1 - \rho_1)}{(1 - \rho)} d_1 \delta + \frac{1}{2} \frac{c_2^r c_2^+}{(c_2^r + c_2^+)} \frac{(1 - \rho_2)}{(1 - \rho)} d_2 \delta.
\]

To obtain the optimal solution of the transient case, the \( x \)-space is divided into two mutually exclusive major regions (\( \mathbb{R}^u \) and \( \mathbb{R}^o \) in Figure 3.2) which are defined as follows:

\[ \mathbb{R}^u = \{(x_1, x_2) \mid d_1(1 - \rho_2)(x_1 - A_1) + d_1 \rho_2(x_2 - A_2) < 0;\ d_2 \rho_1(x_1 - C_1) + d_1(1 - \rho_1)(x_2 - C_2) < 0 \}, \]

\[ \mathbb{R}^o = \{(x_1, x_2) \mid d_2(1 - \rho_2)(x_1 - A_1) + d_1 \rho_2(x_2 - A_2) \geq 0;\ d_2 \rho_1(x_1 - C_1) + d_1(1 - \rho_1)(x_2 - C_2) \geq 0 \}. \]

In Figure 3.2, lines L1, L2, L12 and L2 are defined as follows:

Line L1: \( d_2 \rho_1(x_1 - A_1) + d_1(1 - \rho_1)(x_2 - A_2) = 0 \), \hspace{1cm} (15)

Line L12: \( d_2(1 - \rho_2)(x_1 - A_1) + d_1 \rho_2(x_2 - A_2) = 0 \), \hspace{1cm} (16)

Line L21: \( d_2 \rho_1(x_1 - C_1) + d_1(1 - \rho_1)(x_2 - C_2) = 0 \), \hspace{1cm} (17)

Line L2: \( d_2(1 - \rho_2)(x_1 - C_1) + d_1 \rho_2(x_2 - C_2) = 0 \). \hspace{1cm} (18)

Without loss of generality, we index the parts such that Part Type 1 is the part type with the larger setup time (i.e., \( \delta_1 \geq \delta_2 \)).

The optimal solution for initial surplus levels (inventory/backlog) in Region \( \mathbb{R}^o \) was obtained by inspection. We will state this solution later on in the paper. The optimal solution for initial surplus levels in Region \( \mathbb{R}^u \) is not obvious and is more involved mathematically. The following results were established in [13].

**FACT 3.1.** For initial surplus levels in Region \( \mathbb{R}^u \), the optimal way to progress toward the cyclic schedule is by producing at maximum machine speed whenever it is possible. That is, \( u^* = (U_1, 0) \) or \( (0, U_2) \) if the machine is producing and \( u^* = (0, 0) \) if the machine is undergoing a setup change.

Therefore, given the current setup state, we know the direction of the surplus trajectory in \( \mathbb{R}^u \).
DEFINITION 3.1. We say that a trajectory is following Direction $D_i$, if it moves parallel to Line $L_i$ in the direction of increasing $z_i$. That is, the machine is producing Part Type $i$ ($i = 1, 2$) at maximum machine speed. If the machine is undergoing a setup change, then the trajectory follows Direction $D_0$, where both surplus levels deplete (Direction $(-d_1, -d_2)$).

REMARK. Since for surplus levels in Region $\mathbb{R}^n$ the machine is either producing at its maximum rate or being set up, the trajectories will move either along Direction $D_1$, along Direction $D_2$ or along Direction $D_0$.

DEFINITION 3.2. We call a $D_i - n$-step trajectory ($i = 1, 2$; $n > 1$), a trajectory that performs alternately $m_i$, setup change-production runs of Part Type $i$ and $m_j$ setup change-production runs of Part Type $j$ ($j \neq i$); with the initial setup change to Part Type $i$ and the last segment touching the Limit Cycle at point B or D. If $n$ is even then $m_i = m_j = n/2$. If $n$ is odd then $m_i = (n + 1)/2$ and $m_j = (n - 1)/2$. A $D_1 - 3$-step trajectory is shown in Figure 3.5.

FACT 3.2. To reach the cyclic schedule in finite time, starting with initial surplus levels in Region $\mathbb{R}^n$, the trajectory leading to the cyclic schedule must touch the boundary $L_{ij}$ ($i, j = 1, 2$, and $i \neq j$) of Region $\mathbb{R}^0$, just before switching to Part Type $j$ and reaching the cyclic schedule at one of the points B or D. That is the last setup change, before reaching the cyclic schedule, initiated on the boundary $L_{12}$ or $L_{21}$.

According to Fact 3.2, to reach the cyclic schedule in finite time, we need to perform a setup change on one of the two boundaries $L_{12}$ or $L_{21}$. However, starting from a point in $\mathbb{R}^n$, we may need more than one setup change to bring the surplus trajectory to the cyclic schedule. For instance, as illustrated in Figure 3.3, starting at the point $0$ in $\mathbb{R}^n$ with an initial setup change to Part Type 2, we have a choice to change the setup at the point 1 on the boundary $L_{21}$ and reach the cyclic schedule at D, or have setup changes at 2 and then at 3 on $L_{12}$ and reach the cyclic schedule at B. If the cost can be reduced, we may want to initiate a setup change on the segment $[2',3]$ and swing back to boundary $L_{21}$. As a consequence, the optimal trajectory leading to the cyclic schedule in finite time will be at least $D_i - 2$-step since we need a setup change at the initial point and one more, once the boundary of Region $\mathbb{R}^0$ is reached.

Figure 3.3.

An algorithm (Direction Sweeping Algorithm (DSA)) was developed in [13] to obtain the optimal trajectory leading to the cyclic schedule in finite time. Figure 3.4 shows the application of the DSA algorithm to an example where optimal trajectories starting at different initial points
are shown. Notice that the setup switching policy is a special corridor policy (see [7]) with two windows. Based on numerical results, we observe that the general optimal setup switching policy is as illustrated in Figure 3.5. In this case, the surplus trajectory will bounce back and forth between the two walls of the corridor ($B_1$ and $B_2$ in Figure 3.5) until the trajectory passes through one of the windows of the corridor. When this happens, the trajectory will encounter either Line $L_{12}$ or Line $L_{21}$, then bounces back and follows either Line $L_2$ or Line $L_1$, and reaches the cyclic schedule at Point D or Point B (see Figure 3.5). Therefore, the optimal setup switching policy is completely characterized when the corridor walls $B_1$ and $B_2$, and the corridor windows defined by the two points $W_1$ and $W_2$ in $x$ space, are determined.

Figure 3.4. Optimal setup switching policy.

Figure 3.5. Illustration of the setup switching policy.
The purpose of this paper is to obtain the optimal setup switching policy in Region $\mathbb{R}^n$ analytically. This is achieved by determining the equations of the boundaries $B_1$ and $B_2$ as well as the points $W_1$ and $W_2$ analytically. One of the advantages of the analytical solution, is that once we have the equations of Boundaries $B_1$ and $B_2$, we eliminate the line search procedure in the DSA algorithm and substitute it with an intersection point calculation, which is clearly much faster and much simpler to implement (as a formula). Another advantage of the analytical solution, is that it usually involves the use of a formula, which is an excellent tool for studying extreme cases, where one is interested in the behavior of the solution with respect to one or many parameters of the problem.

4. ANALYTICAL SOLUTION

Based on Facts 3.1 and 3.2, the optimal trajectory emanating at a point in Region $\mathbb{R}^n$, and leading to the cyclic schedule in finite time can be obtained as follows. Given an initial surplus point in Region $\mathbb{R}^n$, we choose the first setup and calculate the cost of the trajectory leading to the cyclic schedule with two setup changes only. At this point, we have a $D_i - 2$-step trajectory, where $i$ is the initial setup for Part Type $i$. The next step is to try to lower the cost of the current trajectory by introducing a setup change to the other part type, so that the cyclic schedule is reached at the opposite side. If the cost can be reduced, then the obtained, new trajectory is a $D_i - 3$-step trajectory. We keep trying to reduce the cost of the current trajectory by introducing, each time, a setup change before the cyclic schedule is reached, until we cannot lower the cost anymore. At this point we have an optimal $D_i - n$-step trajectory (provided we start with a setup change to Part Type $i$) emanating in Region $\mathbb{R}^n$, and reaching the cyclic schedule in finite time. In the same manner, we obtain the optimal $D_j - n$-step ($j \neq i$) trajectory starting with a setup to Part Type $j$ first. The optimal trajectory would be the one with the lowest cost.

The above procedure suggests that we first determine the optimal location of the switching points in $z$-space, where the switch to the other part type allows to reach the cyclic schedule with a lower cost. In other words, we determine the boundaries $B_1$ and $B_2$ analytically (see Figure 3.5). This, we do in Section 4.2. In Section 4.3, we determine up to what point the switch to the other part type is optimal (i.e., results in a lower cost of the trajectory leading to the cyclic schedule). That is, we determine the points $W_1$ and $W_2$ which will indicate where exactly the corridor windows are located in $z$-space (see Figure 3.5). In Section 4.4, we give the complete optimal production and setup policy analytically.

4.1. Preliminaries

In this section, we state some preliminary results that are needed in the subsequent analysis.

We denote by Line $L(Y)$ the line going through a point $Y = (y_1, y_2)$ in $z$-space and parallel to Line $L_2$. We denote by Line $L'(Y)$ the line going through a point $Y = (y_1, y_2)$ in $z$-space and parallel to Line $L_1$. The equations of $L(Y)$ and $L'(Y)$ are given by:

$$\text{Line } L(Y) : d_2(1 - \rho_2)(x_1 - y_1) + d_1\rho_2(x_2 - y_2) = 0,$$

$$\text{Line } L'(Y) : d_2\rho_1(x_1 - y_1) + d_1(1 - \rho_1)(x_2 - y_2) = 0.$$  \hspace{1cm} (19)

Let $Z_1 = (z_{11}, z_{12})$ be the intersection point of $L(Y)$ with $L_2$, and $Z_2 = (z_{21}, z_{22})$ be the intersection point of $L'(Y)$ and $L_1$. Then, $Z_1$ and $Z_2$ are given as follows:

$$z_{11} = \frac{1}{(1 - \rho)} \left\{ (1 - \rho_1)(1 - \rho_2) y_1 + \left( \frac{d_1}{d_2} \right)(1 - \rho_1) \rho_2 y_2 - \rho_1 \rho_2 C_1 - \left( \frac{d_1}{d_2} \right)(1 - \rho_1) \rho_2 C_2 \right\},$$

$$z_{12} = \frac{1}{(1 - \rho)} \left\{ - \left( \frac{d_2}{d_1} \right) \rho_1 (1 - \rho_2) y_1 - \rho_1 \rho_2 y_2 + \left( \frac{d_2}{d_1} \right) \rho_1 (1 - \rho_2) C_1 + (1 - \rho)(1 - \rho_2) C_2 \right\},$$  \hspace{1cm} (21a)

$$z_{21} = \frac{1}{(1 - \rho)} \left\{ \left( \frac{d_2}{d_1} \right) \rho_1 (1 - \rho_2) y_1 + \left( \frac{d_1}{d_2} \right)(1 - \rho_1) \rho_2 y_2 - \rho_1 \rho_2 C_1 - \left( \frac{d_1}{d_2} \right)(1 - \rho_1) \rho_2 C_2 \right\},$$

$$z_{22} = \frac{1}{(1 - \rho)} \left\{ - \left( \frac{d_1}{d_2} \right) \rho_1 (1 - \rho_2) y_1 - \rho_1 \rho_2 y_2 + \left( \frac{d_1}{d_2} \right) \rho_1 (1 - \rho_2) C_1 + (1 - \rho)(1 - \rho_2) C_2 \right\},$$  \hspace{1cm} (21b)
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\[ z_{21} = \frac{1}{(1-\rho)} \left\{ -\rho_1 \rho_2 y_1 - \left( \frac{d_1}{d_2} \right) (1-\rho_1) \rho_2 y_2 + (1-\rho_1)(1-\rho_2) A_1 + \left( \frac{d_1}{d_2} \right) (1-\rho_1) \rho_2 A_2 \right\}, \]  
(22a)

\[ z_{22} = \frac{1}{(1-\rho)} \left\{ \left( \frac{d_2}{d_1} \right) \rho_1 (1-\rho_2) y_1 + (1-\rho_1)(1-\rho_2) y_2 - \left( \frac{d_2}{d_1} \right) \rho_1 (1-\rho_2) A_1 - \rho_1 \rho_2 A_2 \right\}. \]  
(22b)

We denote by \( C_{X,Y} \) the cost of the segment \([X, Y]\) in x-space, when \( X \) and \( Y \) are within the same quadrant in the x-space. For \( X = (x_1, x_2) \) and \( Y = (y_1, y_2) \), we have

\[ C_{X,Y} = \sum_{i=1}^{2} \frac{h_i}{2\left(\sigma_i - d_i\right)} \left\{ y_i^2 - x_i^2 \right\}, \]  
(23)

where \( h_i = c_i^+ \) if \( x_i \) and \( y_i \) are positive, and \( h_i = -c_i^- \) if \( x_i \) and \( y_i \) are negative for \( i = 1, 2 \).

REMARK. If \( X \) and \( Y \) are not in the same quadrant, we can always break the segment into two or three parts, each contained within one quadrant only in x-space. Then the total \([X, Y]\) segment cost is the sum of the two or three segment costs.

**Figure 4.1. Corridor wall.**

### 4.2. Equations of Boundaries \( B_1 \) and \( B_2 \)

In this section, we will determine the locus of the points where it might be optimal to switch setup to the other part type. In other words, how far should we go with the production of the current part type before we switch to the other. Notice that, at this point, we do not know whether this switch will lower the cost of the current trajectory or not. This will be decided later on, when we determine the points \( W_1 \) and \( W_2 \) which will indicate whether this switch is optimal or not. Indeed, if the switch happens to be within the corridor windows, then it is optimal not to switch and continue producing the current part type until either Boundary \( L_{12} \) or Boundary \( L_{21} \) is encountered, where it is optimal to switch to the other part type and reach the cyclic schedule at one of the Points B or D.

To determine the equations representing the corridor walls in x-space, we will start with an initial surplus point in the third quadrant of the x-space. It will be shown that the equations of the corridor walls (Boundaries \( B_1 \) and \( B_2 \)) do not depend on the initial surplus. To be able to follow the derivation, the reader is referred to Figure 4.1. First, we derive the equation of
Before we start the derivation, we need to determine the coordinates of the points $X = (x_1, x_2)$ and $V = (v_1, v_2)$, given $X_0 = (x_1^0 + \delta_2 d_1, x_2^0 + \delta_2 d_2)$ (without loss of generality). In this case, $X$ belongs to Line $L(X_0)$, and $V$ is the intersection of Line $L_{12}$ and Line $L'(X')$, where $X'_0 = (x_1^0, x_2^0)$. Then, using (19), (22a) and (22b), we have:

$$x_2 = \left( \frac{d_2}{d_1} \right) \frac{(1 - \rho_2)}{\rho_2} (x_1 - x_1^0) + x_2^0,$$

$$v_1 = \frac{1}{(1 - \rho)} \left\{ -\rho_1 \rho_2 (x_1 - \delta_1 d_1) - \left( \frac{d_1}{d_2} \right) (1 - \rho_1) \rho_2 (x_2 - \delta_1 d_2) + (1 - \rho_1) (1 - \rho_2) A_1 
+ \left( \frac{d_1}{d_2} \right) (1 - \rho_1) \rho_2 A_2 \right\},$$

$$v_2 = \frac{1}{(1 - \rho)} \left\{ \left( \frac{d_2}{d_1} \right) \rho_1 (1 - \rho_2) (x_1 - \delta_1 d_1) + (1 - \rho_1) (1 - \rho_2) (x_2 - \delta_1 d_2) 
- \left( \frac{d_2}{d_1} \right) \rho_1 (1 - \rho_2) A_1 - \rho_1 \rho_2 A_2 \right\}.$$  

As mentioned before, the objective is to determine the location in $x$-space of the switching point $X$ that minimizes the cost of the trajectory emanating at the point $X_0$ in $x$-space and reaching the cyclic schedule at the point $B$ (see Figure 4.1). Notice here that the current trajectory in Figure 4.1 is a $D_3 - 2$-step trajectory given by $\{X_0, X'_0, X, X'_1, V, V'_1, B, I\}$. Now if we introduce a setup switch at point $X$, the new trajectory will be a $D_3 - 3$-step trajectory given $\{X_0, X'_0, X, X', V, V', B, I\}$. At this stage, we have no guarantee that the new trajectory will have a lower cost than the current one. This will be decided later when we determine $W_1$ and $W_2$. All we are saying at this point, is that if switching to the other part type would lower the cost of reaching the cyclic schedule, then the optimal switch must happen at the point $X$ in $x$ space.

Let $C^2(X_0)$ be the cost of Trajectory $\{X_0, X'_0, X, X', V, V', B, I\}$. Using (23), i.e., calculating the sum of costs of segments of the Trajectory $\{X_0, X'_0, X, X', V, V', B, I\}$ within the same quadrant, we obtain:

$$C^2(X_0) = \frac{1}{2d_1(1 - \rho_1)} \{ (x_1 - \delta_1 d_1)^2 c_1 + v_1^2 c_1^+ \} + \frac{1}{2d_2(1 - \rho_2)} \{ x_2^2 c_2^- + (v_2 - \delta_2 d_2)^2 c_2^+ \}$$

$$- \frac{1}{2d_1} \{ (x_1^0 + \delta_2 d_1)^2 c_1^- + I_1^2 c_1^+ \} + \frac{(x_2^0)^2 c_2^-}{2d_2(1 - \rho_2)} - \frac{1}{2d_2} (x_2^0 + \delta_2 d_2)^2 c_2^-$$

$$+ \frac{\rho_2}{2d_2(1 - \rho_2)} I_2^2 c_2^+.$$
Notice that, if we substitute $x_2, v_1, v_2$ by their expressions as a function of $x_1$, $C^2_1(X_0)$ will be a quadratic function of $x_1$. Hence, $C^2_1(X_0)$ may have a unique minimum. A necessary condition for $x_1$ to be a minimum of $C^2_1(X_0)$ (and sufficient in this case) is given by $\frac{dc^2_1(X_0)}{dx_1} = 0$. Differentiating $x_2, v_1, v_2$ with respect to $x_1$, we get: $\frac{dx_2}{dx_1} = -\left(\frac{d^2}{dx_1^2}(1 - \rho_2)/\rho_2\right); \frac{dv_1}{dx_1} = 1; \frac{dv_2}{dx_1} = -\left(\frac{d^2}{dx_1^2}(1 - \rho_2)/\rho_2\right)$. Now differentiating $C^2_1(X_0)$ with respect to $x_1$ and expressing $v_1$ and $v_2$ as a function of $x_1$ and $x_2$ and setting the result to 0, gives the following equation of a line in $x$-space:

$$B1 : M_1x_1 + M_2x_2 + M_3 = 0,$$

where,

$$M_1 = \frac{1}{(1 - \rho_1)}c_1^- - \frac{\rho_1\rho_2}{(1 - \rho_1)(1 - \rho)}c_1^+ - \left(\frac{d_1}{d_2}\right)\frac{\rho_1}{\rho_2}(1 - \rho)c_2^-, \quad (25a)$$

$$M_2 = -\left(\frac{d_1}{d_2}\right)\frac{\rho_2}{(1 - \rho)}c_1^+ - \frac{1}{\rho_2}c_2^+ - \frac{1 - \rho_1}{\rho_2(1 - \rho)}c_2^-, \quad (25b)$$

$$M_3 = -(M_1C_1 + M_2C_2). \quad (25c)$$

Notice that $B1$ contains the point $C$ on the cyclic schedule. This result is not a surprise since if we started with an initial surplus $X_0$ belonging to the line $L_{12}$, the point $C$ on the cyclic schedule would realize the minimum for $C^2_1(X_0)$, and therefore is a candidate for a setup change.

With a similar procedure, we obtain the equation of the other corridor wall (Boundary $B2$ in Figure 3.5), where it is possible to switch setup from Part Type 1 to Part Type 2.

$$B2 : N_1x_1 + N_2x_2 + N_3 = 0,$$

where,

$$N_1 = -\left(\frac{d_2}{d_1}\right)\frac{\rho_1}{(1 - \rho)}c_2^+ - \frac{1}{\rho_1}c_1^+ - \frac{1 - \rho_1}{\rho_1(1 - \rho)}c_2^-; \quad (27a)$$

$$N_2 = \frac{1}{(1 - \rho_2)}c_2^+ - \frac{\rho_1\rho_2}{(1 - \rho_2)(1 - \rho)}c_2^+ - \left(\frac{d_2}{d_1}\right)\frac{\rho_2}{\rho_1}(1 - \rho)c_1^+; \quad (27b)$$

$$N_3 = -(N_1A_1 + N_2A_2). \quad (27c)$$

It is worth mentioning that in [13], we observed from numerical solutions that the optimal setup switching boundaries in $x$-space are linear. It is verified here by the equations of the corridor walls. In the following subsection, we determine the corridor windows.

4.4. Corridor Windows

In this section, we determine the corridor windows, which are characterized by the coordinates of the points $W1 = (w_{11}, w_{12})$ and $W2 = (w_{21}, w_{22})$ in $x$-space (see Figure 3.5). $W1$ and $W2$ are the points on the corridor boundaries where an additional setup change does not reduce the cost of reaching the cyclic schedule. Geometrically, this means: beyond these two points $W1$ and $W2$ on the boundaries $B1$ and $B2$, it would not be optimal to add a new setup change to the current trajectory (that is the cost of the current trajectory cannot be lowered anymore).

We will start by deriving the coordinates of $W1$ first. To be able to follow the derivation the reader is referred to Figure 4.2. As we mentioned above, $W1$ is the point in $x$-space that realizes the equilibrium between the cost of trajectory $\{W1, V1, V1', I\}$ and the cost of trajectory $\{W1, W1', Z1, Z1', I\}$. $W1$ must be a point of $B1$ since we know that $B1$ is the set of optimal locations where we switch to Part Type 2 (previous section). Figure 4.3 shows an example of three trajectories $\tau_1, \tau_2$ and $\tau_3$. For trajectory $\tau_1$, it is optimal to switch setup to Part Type 1. For
trajectory \( \tau_2 \), we are indifferent between switching to Part Type 1 or continuing the production of Part Type 2, switching setup at the point \( V_1 \) and then reaching the cyclic schedule at the point \( D \). In this case, the two trajectories have the same cost. For trajectory \( \tau_3 \), it is not optimal to switch to Part Type 1, and it is optimal to continue the production of Part Type 2 and reach the cyclic schedule at point \( D \).

Before we start the derivation, we need the coordinates of the points \( V_1 = (v_{11}, v_{12}) \) and \( Z_1 = (z_{11}, z_{12}) \) shown in Figure 4.2. In this case, \( V_1 \) belongs to \( L(W_1) \cap L_{21} \) and \( Z_1 \) belongs to \( L'(W_1') \cap L_{12} \). Once we have \( V_1 \) and \( Z_1 \), \( V_1' \) and \( Z_1' \) are easily obtained. \( V_1 \) and \( Z_1 \) are defined as follows:

\[
\begin{align*}
x_{11} &= \eta_{11} w_{11} + \nu_{11}, \\
x_{12} &= \eta_{12} w_{12} + \nu_{12}, \\
v_{11} &= \gamma_{11} w_{11} + \mu_{11}, \\
v_{12} &= \gamma_{12} w_{12} + \mu_{12},
\end{align*}
\]

where,

\[
\begin{align*}
\eta_{11} &= \frac{\rho_2}{1 - \rho} \left\{ \left( \frac{d_1}{d_2} \right) \left( \frac{M_1}{M_2} \right) (1 - \rho_1) \right\}, \\
\eta_{12} &= \frac{\rho_2}{1 - \rho} \left\{ \left( \frac{d_1}{d_2} \right) \rho_1 - \left( \frac{M_1}{M_2} \right) (1 - \rho_1) \right\}, \\
\nu_{11} &= \frac{1 - \rho_1}{1 - \rho} (1 - \rho_2) (1 - \rho_1) A_1 + \frac{1 - \rho_1}{1 - \rho} \rho_2 \left( \frac{d_1}{d_2} \right) \left( A_2 + \left( \frac{M_3}{M_2} \right) \right) + \frac{\rho_2}{1 - \rho} \delta_1 d_1, \\
\nu_{12} &= -\frac{\rho_1 (1 - \rho_2)}{1 - \rho} \left( \frac{d_2}{d_1} \right) A_1 - \frac{\rho_1 \rho_2}{1 - \rho} (1 - \rho_1) (1 - \rho_2) \left( \frac{M_3}{M_2} \right) \\
&\quad - \frac{1 - \rho_1}{1 - \rho} \left( \frac{d_2}{d_1} \right) \delta_1 d_1, \\
\gamma_{11} &= \frac{1 - \rho_1}{1 - \rho} \left\{ (1 - \rho_2) - \rho_2 \left( \frac{d_1}{d_2} \right) \left( \frac{M_1}{M_2} \right) \right\}, \\
\gamma_{12} &= \frac{\rho_1}{1 - \rho} \left\{ \rho_2 \left( \frac{M_1}{M_2} \right) - (1 - \rho_2) \left( \frac{d_2}{d_1} \right) \right\}, \\
\mu_{11} &= -\frac{\rho_1 \rho_2}{1 - \rho} C_1 - \frac{\rho_2 (1 - \rho_1)}{1 - \rho} \left( \frac{d_1}{d_2} \right) \left( C_2 + \left( \frac{M_3}{M_2} \right) \right), \\
\mu_{12} &= \frac{\rho_1 (1 - \rho_2)}{1 - \rho} \left( \frac{d_2}{d_1} \right) C_1 + \frac{1 - \rho_1 (1 - \rho_2)}{1 - \rho} C_2 + \frac{\rho_1 \rho_2}{1 - \rho} \left( \frac{M_3}{M_2} \right) .
\end{align*}
\]

Denote by \( TC_{11} \), the cost of trajectory \( \{W_1, V_1, V_1', I\} \) and by \( TC_{12} \) the cost of trajectory \( \{W_1, W_1', Z_1, Z_1', I\} \). Using (23), it is not difficult to show that:

\[
\begin{align*}
TC_{11} &= \frac{1}{2} \left\{ \frac{-w_{11}^2 c_1^2}{d_1} - \frac{I_2^2 c_1^2}{d_2} + \frac{\left( v_{11} - \delta_1 d_1 \right)^2 c_1^2 + \rho_1 I_1^2 c_1^2}{d_1 (1 - \rho_1)} \right\} + \frac{w_{12}^2 c_2^2 - \rho_2 w_{22}^2 c_2^2}{d_2 (1 - \rho_2)}, \\
TC_{12} &= \frac{1}{2} \left\{ \frac{-w_{11}^2 c_1^2 + I_2^2 c_1^2}{d_1} + \frac{w_{12}^2 c_2^2}{d_2} + \frac{\left( v_{11} - \delta_1 d_1 \right)^2 c_1^2 + z_{11}^2 c_1^2}{d_1 (1 - \rho_1)} \right\} + \frac{\left( z_{12} - \delta_2 d_2 \right)^2 c_2^2 + \rho_2 I_2^2 c_2^2}{d_2 (1 - \rho_2)} .
\end{align*}
\]
As we mentioned above, \( W_1 = (w_{11}, w_{12}) \) is the point in \( x\)-space for which we have \( TC_{11} = TC_{12} \). Furthermore, \( W_1 \) belongs to the boundary \( B_1 \). Therefore, \( W_1 \) should satisfy the following two conditions:

\[
TC_{11}(w_{11}) - TC_{12}(w_{11}) = 0,
\]

\[
w_{12} = \left( \frac{M_1}{M_2} \right) w_{11} - \left( \frac{M_3}{M_2} \right).
\]  

(36)

Notice that, substituting \( v_{11}, v_{12}, z_{11}, z_{12} \) and \( w_{12} \) with their expressions as a function of \( w_{11} \) makes \( TC_{11} \) and \( TC_{12} \) depend on \( w_{11} \) only. After manipulation, we get the following system of equations:

\[
\alpha_1 w_{11}^2 + \alpha_2 w_{11} + \alpha_3 = 0,
\]

\[
w_{12} = -\left( \frac{M_1}{M_2} \right) w_{11} - \left( \frac{M_3}{M_2} \right),
\]  

(37)

where,

\[
\alpha_1 = \frac{c_1^+ - \eta_{11} c_1^-}{2d_1 (1 - \rho_1)} + \frac{c_2^+ - \eta_{12} c_2^-}{2d_2 (1 - \rho_2)},
\]  

(38a)

\[
\alpha_2 = \frac{\eta_{11}(\mu_{11} - \delta_1 d_1) + \delta_1 d_1 c_1^- - \eta_{11} \nu_{11} c_1^+}{d_1 (1 - \rho_1)} + \frac{\eta_{12}(\mu_{12} - \delta_2 d_2) + \delta_2 d_2 c_2^- - \eta_{12} \nu_{12} c_2^+}{d_2 (1 - \rho_2)},
\]  

(38b)

\[
\alpha_3 = \frac{\mu_{11}(\nu_{11} - 2 \delta_1 d_1) c_1^- - (\nu_{11} - \nu_{12} d_2) c_2^-}{2d_1 (1 - \rho_1)} + \frac{\mu_{12}(\nu_{12} - 2 \delta_2 d_2) c_2^-}{2d_2 (1 - \rho_2)}.
\]  

(38c)

For most practical situations (finite \( c_1^+ \) and \( c_2^+ \) \((i = 1, 2))\), it can be shown that \( W_1 \) is located in the second quadrant of the \( x\)-space. Hence, \( w_{11} \) in (37) must be the negative root.

In a similar fashion, we find the coordinates of the point \( W_2 \) in \( x\)-space, by solving the following system of two equations:

\[
\beta_1 w_{22}^2 + \beta_2 w_{22} + \beta_3 = 0,
\]

\[
w_{21} = -\left( \frac{N_2}{N_1} \right) w_{22} - \left( \frac{N_3}{N_1} \right),
\]  

(39)

where,

\[
\beta_1 = \frac{c_1^+ - \eta_{21} c_1^-}{2d_1 (1 - \rho_1)} + \frac{c_2^+ - \eta_{22} c_2^-}{2d_2 (1 - \rho_2)},
\]  

(40a)

\[
\beta_2 = \frac{\eta_{21}(\mu_{21} - \delta_2 d_2) c_1^- - \eta_{22}(\nu_{21} - \delta_1 d_1) c_1^-}{d_1 (1 - \rho_1)} + \frac{\eta_{22}(\nu_{22} - \delta_2 d_2) c_2^- - \eta_{22}(\nu_{22} - \delta_2 d_2) c_2^-}{d_2 (1 - \rho_2)},
\]  

(40b)

\[
\beta_3 = \frac{\mu_{21}(\nu_{21} - \nu_{22} d_2) c_2^-}{2d_1 (1 - \rho_1)} + \frac{\mu_{22}(\nu_{22} - \delta_2 d_2) c_2^-}{2d_2 (1 - \rho_2)},
\]  

(40c)

\[\eta_{21} = \frac{1 - \rho_1}{1 - \rho_1} \left\{ \left( \frac{d_1}{d_2} \right) \rho_2 - \left( \frac{N_2}{N_1} \right) \right\}.
\]  

(41a)
\[
\eta_2 = \frac{\rho_1}{(1 - \rho)} \left\{ \left( \frac{d_2}{d_1} \right) \left( \frac{N_2}{N_1} \right) (1 - \rho_2) - \rho_2 \right\}, \\
\nu_21 = -\frac{\rho_1 \rho_2}{(1 - \rho)} - \frac{\rho_1 (1 - \rho_1)}{(1 - \rho)} \left( \frac{d_1}{d_2} \right) C_2 - \frac{(1 - \rho_1)(1 - \rho_2)}{(1 - \rho)} \left( \frac{N_3}{N_1} \right) \\
\nu_22 = \frac{\rho_1 (1 - \rho_2)}{(1 - \rho)} \left( \frac{d_2}{d_1} \right) \left( C_1 + \left( \frac{N_3}{N_1} \right) \right) + \frac{(1 - \rho_1)(1 - \rho_2)}{(1 - \rho)} C_2 + \frac{\rho_1}{(1 - \rho)} \delta_2 d_2, \\

\gamma_21 = \rho_1 \left( \frac{N_2}{N_1} \right) - (1 - \rho_1) \left( \frac{d_1}{d_2} \right), \\
\gamma_22 = \frac{\rho_1}{(1 - \rho)} \left\{ (1 - \rho_1) - \rho_2 \left( \frac{d_2}{d_1} \right) \left( \frac{N_2}{N_1} \right) \right\}, \\
\mu_21 = \frac{(1 - \rho_1)(1 - \rho_2)}{(1 - \rho)} A_1 + \rho_2 (1 - \rho_1) \left( \frac{d_1}{d_2} \right) A_2 + \frac{\rho_1 \rho_2}{(1 - \rho)} \left( \frac{N_3}{N_1} \right), \\
\mu_22 = -\frac{\rho_1 (1 - \rho_2)}{(1 - \rho)} \left( \frac{d_2}{d_1} \right) \left( A_1 + \left( \frac{N_3}{N_1} \right) \right) - \frac{\rho_1 \rho_2}{(1 - \rho)} A_2.
\]

Similarly, for practical situations it can be shown that \( W_2 \) is located in the fourth quadrant of the \( z \)-space. Hence, \( \nu_{22} \) in (39) must be the negative root.

4.5. The Complete Solution

At this stage, we have the complete analytical solution of the optimal production and setup control policy. But before we state the solution of this problem, we partition Region \( \mathbb{R}^6 \) into the following mutually exclusive sets (readers are referred to [13] for details):

\[ G_{11} = \{(x_1, x_2) \mid x_1 - \delta d_1 \geq 0; -d_2(x_1 - C_1) + d_1(x_2 - C_2) \geq 0\} \]
\[ G_{12} = \{(x_1, x_2) \mid d_2(x_1 - C_1) - d_1(x_2 - C_2) \geq 0; -d_2(x_1 - I_1) + d_1(x_2 - I_2) \geq 0; d_2(1 - \rho_2)(x_1 - A_1) + d_1 \rho_2(x_2 - A_2) \geq 0\} \]
\[ G_{21} = \{(x_1, x_2) \mid -d_2(x_1 - A_1) + d_1(x_2 - A_2) \geq 0; d_2(x_1 - I_1) - d_1(x_2 - I_2) \geq 0; d_2 \rho_1(x_1 - C_1) + d_1(1 - \rho_2)(x_2 - C_2) \geq 0\} \]
\[ G_{22} = \{(x_1, x_2) \mid x_2 - \delta d_2 \geq 0; d_2(x_1 - A_2) - d_1(x_2 - A_2) > 0\} \]
\[ H_1 = \{(x_1, x_2) \mid x_1 - \delta d_1 \geq 0; d_2(x_1 - C_1) - d_1(x_2 - C_2) > 0; d_2 \rho_1(x_1 - C_1) + d_1(1 - \rho_2)(x_2 - C_2) \geq 0; -d_2 (1 - \rho_2)(x_1 - A_1) - d_1 \rho_2(x_2 - A_2) \geq 0\} \]
\[ H_{11} = \{(x_1, x_2) \mid x_1 - \delta d_1 \geq 0; d_2(x_1 - C_1) - d_1(x_2 - C_2) > 0; d_2(x_1 - C_1) + d_1 (1 - \rho_2)(x_2 - C_2) \geq 0; d_2(1 - \rho_2)(x_1 - A_1) - d_1 \rho_2(x_2 - A_2) \geq 0\} \]
\[ H_{12} = \{(x_1, x_2) \mid d_2 \rho_1(x_1 - C_1) - d_1 (1 - \rho_2)(x_2 - C_2) > 0; -d_2(x_1 - g_11) + d_1(x_2 - g_12) \geq 0; -d_2(1 - \rho_2)(x_1 - A_1) - d_1 \rho_2(x_2 - A_2) \geq 0\} \]
\[ H_2 = \{(x_1, x_2) \mid x_2 - \delta d_2 \geq 0; -d_2(x_1 - I_1) - d_1(x_2 - I_2) \geq 0; d_2(1 - \rho_2)(x_1 - A_1) - d_1 \rho_2(x_2 - A_2) \geq 0; -d_2 \rho_1(x_1 - C_1) - d_1(1 - \rho_2)(x_2 - C_2) \geq 0; -d_2(x_1 - g_21) + d_1(x_2 - g_22) \geq 0\} \]
\[ H_{21} = \{(x_1, x_2) \mid d_2(x_1 - I_1) - d_1(x_2 - I_2) > 0; d_2(1 - \rho_2)(x_1 - A_1) - d_1 \rho_2(x_2 - A_2) \geq 0; -d_2 \rho_1(x_1 - C_1) - d_1(1 - \rho_2)(x_2 - C_2) \geq 0; -d_2(x_1 - g_21) + d_1(x_2 - g_22) \geq 0\} \]
\[ H_{22} = \{(x_1, x_2) \mid x_2 - \delta d_2 \geq 0; -d_2(x_1 - A_1) - d_1(x_2 - A_2) > 0; -d_2 \rho_1(x_1 - C_1) - d_1(1 - \rho_1)(x_2 - C_2) \geq 0; -d_2(x_1 - g_21) + d_1(x_2 - g_22) > 0\} \]
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\( G_1 = \{ (x_1, x_2) \mid -x_1 + \delta_1 d_1 > 0; \ d_2 \rho_1 (x_1 - C_1) + d_1 (1 - \rho_1)(x_2 - C_2) \geq 0 \}; \)

\( G_2 = \{ (x_1, x_2) \mid -x_2 + \delta_2 d_2 > 0; \ d_2 (1 - \rho_2)(x_1 - A_1) + d_1 \rho_2 (x_2 - A_2) \geq 0 \}; \)

where \( g_1 = (g_{11}, g_{12}) \) is the point in \( x \)-space given by:

\[
\begin{align*}
g_{11} &= \delta_1 d_1, \\
g_{12} &= C_2 + \frac{(C_1 - \delta_1 d_1) d_2 \rho_1}{d_1 (1 - \rho_1)},
\end{align*}
\]

and \( g_2 = (g_{21}, g_{22}) \) is the point in \( x \)-space given by:

\[
\begin{align*}
g_{21} &= A_1 + \frac{(A_2 - \delta_2 d_2) d_1 \rho_2}{d_2 (1 - \rho_2)}, \\
g_{22} &= \delta_2 d_2.
\end{align*}
\]

To summarize the control actions in Region \( \mathcal{R}^a \), let \( x = (a, b) \) be the vector of initial surplus levels in Region \( \mathcal{R}^o \) (for more details the reader is referred to [13]).

- **If** \( x \in G_1 \):
  1. Set up the machine for Part Type 1;
  2. After the setup change, produce Part Type 1 at the rate \( U_1 \);
  3. When the surplus level of Part Type 1 becomes 0, change the production rate to \( d_1 \);
  4. When the surplus level of Part Type 2 becomes \( x_2 = A_2 + A_1 d_2 \rho_1 / d_1 (1 - \rho_1) \), switch to control actions of the cyclic schedule.

- **If** \( x \in G_{11} \cup H_{11} \):
  1. Do not produce either part type;
  2. When the surplus level of Part Type 1 becomes \( \delta_1 d_1 \), start a setup change for Part Type 1;
  3. After the setup change, produce Part Type 1 at the demand rate \( d_1 \);
  4. When the surplus level of Part Type 2 becomes \( x_2 = A_2 + A_1 d_2 \rho_1 / d_1 (1 - \rho_1) \), switch to the control actions of the cyclic schedule.

- **If** \( x \in G_{12} \cup H_{21} \):
  1. Do not produce either part type;
  2. When the surplus level of Part Type 2 reaches level \( l_2 \), immediately start a setup change for Part Type 2. Level \( l_2 \) is given by

\[
l_2 = (1 - \rho_2) \left( b - \left( \frac{d_2}{d_1} \right) a \right) + \left( \frac{d_2}{d_1} \right) (1 - \rho_2) A_1 + \rho_2 A_2;
\]

  3. At the end of the setup change switch to the cyclic schedule control actions.

- **If** \( x \in G_{21} \cup H_{12} \):
  1. Do not produce either part type;
  2. When the surplus level of Part Type 1 reaches level \( l_1 \), immediately start a setup change for Part Type 1. Level \( l_1 \) is given by

\[
l_1 = \rho_1 \left( b - \left( \frac{d_2}{d_1} \right) a \right) + \left( \frac{d_2}{d_1} \right) \rho_1 C_1 + (1 - \rho_1) C_2;
\]

  3. At the end of the setup change switch to the cyclic schedule control actions.

- **If** \( x \in G_{22} \cup H_{22} \):
  1. Do not produce either part type;
  2. When the surplus level of Part Type 2 becomes \( \delta_2 d_2 \), start a setup change for Part Type 2;
3. After the setup change, produce Part Type 2 at the demand rate $d_2$;
4. When the surplus level of Part Type 1 becomes $x_1 = C_1 + C_2 d_1 \rho_2 / d_2 (1 - \rho_2)$, switch to the control actions of the cyclic schedule.

- If $x \in G_2$:
  1. Set up the machine for Part Type 2;
  2. After the setup change, produce Part Type 2 at the rate $U_2$;
  3. When the surplus level of Part Type 2 becomes 0, change the production rate to $d_2$;
  4. When the surplus level of Part Type 1 becomes $x_1 = C_1 + C_2 d_1 \rho_2 / d_2 (1 - \rho_2)$, switch to the control actions of the cyclic schedule.

For initial surplus $x = (a, b)$ in Region $\mathbb{R}^n$, the following procedure gives the optimal trajectory of the surplus levels in $x$-space.

**PROCEDURE**

**STEP 0:** $X_1(0) := a, X_2(0) := b$.

$k := 1$;

IF $C_1^2(X(0)) \geq C_2^2(X(0))$ THEN

$X_1(k) := X_1(0) - \delta_2 d_1$;

$X_2(k) := X_2(0) - \delta_2 d_2$;

GOTO STEP 1;

ELSE

$X_1(k) := X_1(0) - \delta_1 d_1$;

$X_2(k) := X_2(0) - \delta_1 d_2$;

GOTO STEP 2;

ENDIF;

**STEP 1:** $k := k + 1$;

$X_1(k) := (d_1 \rho_2 (M_3 + X_2(k-1)M_2) + d_2 (1 - \rho_2) X_1(k-1)M_2) / (d_2 (1 - \rho_2) M_2 - d_1 \rho_2 M_1)$

$X_2(k) := - \left( \frac{d_2}{d_1} \right) X_1(k) - \left( \frac{d_1}{d_2} \right)$;

IF $X_1(k) \geq w_{11}$ THEN

$X_1(k) := \gamma_{11} X_1(k-1) + \mu_{11}$;

$X_2(k) := \gamma_{12} X_2(k-1) + \mu_{12}$;

$k := k + 1$;

$X_1(k) := X_1(k-1) - \delta_1 d_1$;

$X_2(k) := X_2(k-1) - \delta_1 d_2$;

$k := k + 1$;

$X(k) := I$;

GOTO STEP 3;

ELSE

GOTO STEP 2;

ENDIF;

**STEP 2:** $k := k + 1$;

$X_2(k) := (d_2 \rho_1 (N_3 + X_1(k-1)N_1) + d_1 (1 - \rho_1) X_1(k-1)N_1) / (d_1 (1 - \rho_1) N_1 - d_2 \rho_1 N_2)$

$X_1(k) := - \left( \frac{N_3}{N_1} \right) X_2(k) - \left( \frac{N_2}{N_1} \right)$;

IF $X_2(k) \geq w_{22}$ THEN

$X_1(k) := \gamma_{21} X_1(k-1) + \mu_{21}$;

$X_2(k) := \gamma_{22} X_2(k-1) + \mu_{22}$;

$k := k + 1$;

$X_1(k) := X_1(k-1) - \delta_2 d_1$;

$X_2(k) := X_2(k-1) - \delta_2 d_2$;

$k := k + 1$;

$X(k) := I$;
GOTO STEP 3;
ELSE
  GOTO STEP 1;
ENDIF;

STEP 3: OUTPUT: OPTIMAL TRAJECTORY = \{X(0), X(1), \ldots, X(k-1), X(k)\};

Given the optimal trajectory of the surplus levels in x-space, it is not difficult to translate it into control actions.

The above procedure is based on the following Conjecture.

**Conjecture 4.1.** The initial setup of the optimal trajectory is given by \(i^* = \arg\min_{1,2} \{C_2^T(x)\} \).

Conjecture 4.1 states that the initial setup is optimal. That is, once we obtain the initial setup, we do not need to go back and check whether starting with the other setup is optimal (i.e., belongs to the optimal trajectory). Although we do not have a mathematical proof for this observation, the application of the above procedure with all the cases we tried always provided the optimal trajectory. Note, that if we do not use Conjecture 4.1, we use the above procedure to calculate the cost of the trajectory starting with initial setup to Part Type 1 and the cost of the trajectory starting with initial setup to Part Type 2. The optimal trajectory is the one with the lower cost.

In the following section, we study some extreme cases which are of either practical or theoretical importance. Since we have the analytical solution, it is relatively easy to study such cases.

## 5. SPECIAL CASES

In this section, four special cases are investigated.

- **Case 1:** No backlog is allowed for Part Type 2 (without loss of generality).
- **Case 2:** No backlog is allowed for both parts.
- **Case 3:** No inventory is allowed for Part Type 2.
- **Case 4:** No inventory is allowed for both part types.

### 5.1. No Backlog for Part Type 2

This case can be encountered in practice, where a backlog for a part type is undesirable either because of customer requirements or because of a very high penalty incurred when the products are not delivered in time. To obtain the optimal policy in this case, we simply let the backlog instantaneous cost of Part Type 2 go to infinity. That is \(c_2^- \to \infty\).

As before, we determine the corridor wall first, then we determine the windows of the corridors (if they exist).

**Corridor Wall \(B1\).** Dividing the equation of \(B1\) by \(c_2^-\) in (24) and taking the limit when \(c_2^- \to \infty\), we get:

\[
\lim_{c_2^- \to \infty} \frac{M_1}{c_2^-} = - \left( \frac{d_1}{d_2} \right) \left( \frac{\rho_1}{\rho_2} \right) \frac{(1 - \rho_2)}{(1 - \rho)},
\]

\[
\lim_{c_2^- \to \infty} \frac{M_2}{c_2^-} = - \frac{(1 - \rho_1)(1 - \rho_2)}{\rho_2(1 - \rho)}.
\]

\[
\lim_{c_2^- \to \infty} \frac{M_3}{c_2^-} = \left( \frac{d_1}{d_2} \right) \left( \frac{\rho_1}{\rho_2} \right) \left( \frac{(1 - \rho_2)}{(1 - \rho)} \right) C_1 + \left( \frac{(1 - \rho_1)(1 - \rho_2)}{\rho_2(1 - \rho)} \right) C_2,
\]

rearranging terms in the expression of \(B1\), gives the following line equation:

\[d_2 \rho_1(x_1 - C_1) + d_1(1 - \rho_1)(x_2 - C_2) = 0,\]
which is exactly the equation of Line L21 (see (17)). Hence, when backlog is not allowed for Part Type 2, the boundary B1 becomes Line L21.

**CORRIDOR WALL B2.** Dividing the equation of B2 by $c_2^-$ in (26) and taking the limit when $c_2^- \to \infty$, we get:

\[
\lim_{c_2^- \to \infty} \frac{N_1}{c_2^-} = 0,
\]

\[
\lim_{c_2^- \to \infty} \frac{N_2}{c_2^-} = \frac{1}{(1 - \rho_2)},
\]

\[
\lim_{c_2^- \to \infty} \frac{N_3}{c_2^-} = -\frac{\delta_2 d_2}{(1 - \rho_2)},
\]

which gives the following line equation for B2:

\[
x_2 = \delta_2 d_2.
\]

This result was intuitively expected. Indeed, $x_2$ cannot be negative, hence the switch to Part Type 2 must occur exactly when its surplus reaches the level $\delta_2 d_2$, so that after the setup change, the surplus level of Part Type 2 is not negative (which is not allowed).

**CORRIDOR WINDOWS.** To determine the corridor window, we divide expression (38a), (38b), and (38c) by $c_2^-$ then take the limit when $c_2^- \to \infty$ and get:

\[
\lim_{c_2^- \to \infty} \frac{\alpha_1}{c_2^-} = \alpha_1^\infty = -\frac{((M_1/M_2)^2 c_2^+ + \eta_{12})}{2d_2(1 - \rho_2)},
\]

\[
\lim_{c_2^- \to \infty} \frac{\alpha_2}{c_2^-} = \alpha_2^\infty = -\frac{((M_1/M_2) (M_3/M_2) c_2^+ + \eta_{12} (\nu_{12} - \delta_2 d_2))}{d_2(1 - \rho_2)},
\]

\[
\lim_{c_2^- \to \infty} \frac{\alpha_3}{c_2^-} = \alpha_3^\infty = -\frac{((M_3/M_2)^2 c_2^+ + (\nu_{12} - \delta_2 d_2)^2)}{2d_2(1 - \rho_2)}.
\]

Now, if we compute the discriminant in the quadratic $\alpha_1^\infty w_1^2 + \alpha_2^\infty w_1 + \alpha_3^\infty = 0$, we get:

\[(\alpha_2^\infty)^2 - 4(\alpha_1^\infty)\alpha_3^\infty = -4c_2^+ [(M_1/M_2) (\nu_{12} - \delta_2 d_2) + (M_3/M_2) \eta_{12}]^2,
\]

which is obviously negative. This means that the cost of reaching the cyclic schedule from the Point D, is always less than that of reaching the limit cycle from the Point B. Hence, the corridor window is the whole corridor wall given by the equation of Line L21. Figure 5.1 illustrates this policy.
5.2. No Backlog for Either Part Type

In this case, we have $c_1^- \to \infty$ and $c_2^- \to \infty$. In a similar fashion as the previous case, we can show that: $B1: x_1 = \delta_1 d_1$ and $B2: x_2 = \delta_2 d_2$. This means that Region $\mathcal{R}^u$ becomes “forbidden”. That is, all initial surplus levels must be in Region $\mathcal{R}^0$ (otherwise the cost would be infinite), and therefore, the optimal control actions of Region $\mathcal{R}^0$ apply (see Figure 5.2).

5.3. No Inventory for Part Type 2

This case happens in practice when there is no space available for a product and the latter has to be delivered to the customer at once after being produced. Another situation is when the inventory cost for the product is much higher than its backlog cost. In this situation it is better to deliver the product late than store it for a while before being delivered to the customer. This is usually known as “produce to order system” in the literature. To obtain the optimal policy in this case, we simply let the inventory instantaneous cost of Part Type 2 go to infinity. That is $c_2^+ \to \infty$.

\[ \lim_{c_2^+ \to \infty} \frac{M_1}{c_2^+} = 0, \]
\[ \lim_{c_2^+ \to \infty} \frac{M_2}{c_2^+} = -\frac{1}{\rho_2}, \]
\[ \lim_{c_2^+ \to \infty} \frac{M_3}{c_2^+} = 0, \]

which gives:

\[ x_2 = 0. \]

This result was intuitively expected. Indeed, $x_2$ cannot be positive, hence the switch to Part Type 2 must occur when the backlog of that part type is completely eliminated.
CORRIDOR WALL B2. Dividing the equation of B2 by $c_2^+$ in (26) and taking the limit when $c_2^+ \to \infty$, gives:

$$
\lim_{c_2^+ \to \infty} N_1 \frac{c_2^+}{d_2} = -\left(\frac{d_1}{d_1} \frac{\rho_1}{(1-\rho)}\right),
$$

$$
\lim_{c_2^+ \to \infty} N_2 \frac{c_2^+}{d_2} = -\frac{\rho_1 \rho_2}{(1-\rho_2)(1-\rho)},
$$

$$
\lim_{c_2^+ \to \infty} N_3 \frac{c_2^+}{d_2} = \left(\frac{d_1}{d_1} \frac{\rho_1}{(1-\rho)}\right) A_1 + \left(\frac{\rho_1 \rho_2}{(1-\rho_2)(1-\rho)}\right) A_2,
$$

rearranging terms in the expression of B2, gives the following line equation:

$$
d_2(1-\rho_2)(x_1 - A_1) + d_1 \rho_2(x_2 - A_2) = 0,
$$

which is exactly the equation of Line L12 (see (16)). Hence, when inventory is not allowed for Part Type 2, the boundary B2 becomes Line L12.

CORRIDOR WINDOWS. To determine the corridor window, we divide expressions (40a), (40b) and (40c) by $c_2^+$ then take the limit when $c_2^+ \to \infty$ and get:

$$
\lim_{c_2^+ \to \infty} \beta_1 = \beta_1^\infty = 0,
$$

$$
\lim_{c_2^+ \to \infty} \beta_2 = \beta_2^\infty = 0,
$$

$$
\lim_{c_2^+ \to \infty} \beta_3 = \beta_3^\infty = \left(\frac{1-\rho_2}{2(1-\rho)}\right) d_2 \rho_2^2,
$$

which implies that $\beta_1^\infty w_{22}^2 + \beta_2^\infty w_{22}^2 + \beta_3^\infty = \beta_3^\infty > 0$. Hence, the cost of reaching the cyclic schedule from the point B, is always less than that of reaching the cyclic schedule from the point D. Therefore, the corridor window is the whole corridor wall given by the equation of Line L12. Figure 5.3 illustrates this policy.

Figure 5.3. Case 3: $c_2^+ \to \infty$. 
5.4. No Inventory for Either Part Type

In this case, we have \( c_1^+ \to \infty \) and \( c_2^+ \to \infty \). In a similar fashion as in the previous case, we can show that: \( B1 : x_1 = 0 \) and \( B2 : x_2 = 0 \). This means that all activities take place in the third quadrant of the \( x \)-space. Also, it is not difficult to show that when \( c_1^+ \to \infty \) and \( c_2^+ \to \infty \),

\[
W1 \to C^\infty = \left( \delta_1 d_1 - \frac{(1 - \rho_1)}{\rho} \delta d_1, 0 \right) \quad \text{and} \quad W2 \to A^\infty = \left( 0, \delta_2 d_2 - \frac{(1 - \rho_2)}{\rho} \delta d_2 \right),
\]

which implies that the boundaries \( B1 \) and \( B2 \) have no windows. Therefore, the optimal surplus trajectory will keep bouncing back and forth between the two boundaries and never reaches the cyclic schedule in finite time. In other words, the cyclic schedule can only be reached asymptotically when \( c_1^+ \to \infty \) and \( c_2^+ \to \infty \) (Figure 5.4 illustrates this case). This result is intuitively expected. Indeed, the optimal Limit Cycle touches the \( x_1 \) and \( x_2 \) axes of the third quadrant of the \( x \)-space at Points \( A \) and \( C \). Therefore, any corridor window will allow the optimal trajectory to step in the positive surplus region of either part type, thus incurring an instantaneous infinite cost. Also, the optimal trajectory keeps eliminating backlog as much as possible without incurring any inventory while progressing towards the Limit Cycle (which is the most economical path that the optimal trajectory can follow). Therefore, there should be no windows in the corridor.

![Figure 5.4. Case 4: \( c_1^+ \to \infty \), \( c_2^+ \to \infty \).](image)

6. CONCLUSION

In this paper, we studied a deterministic one-machine two-product dynamic manufacturing system with setup changes. We provided a feedback control formulation of the problem. Based on the results of a previous paper [13], we derived the complete optimal solution (expressed as a feedback control law) of the problem analytically. Given the analytical optimal solution, we studied some extreme cases where we dealt with the effects of eliminating either backlog or inventory for either or both products. The results obtained were as expected intuitively.

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