Stable numerical methods for conservation laws with discontinuous flux function

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Abstract

We develop numerical methods for solving nonlinear equations of conservation laws with flux function that depends on discontinuous coefficients. Using a relaxation approximation, the nonlinear equation is transformed to a semilinear diagonalizable problem with linear characteristic variables. Eulerian and Lagrangian methods are used for the advection stage while an implicit–explicit scheme solves the relaxation stage. The main advantages of this approach are neither Riemann problem solvers nor linear iterations are required during the solution process. Moreover, the characteristic-based relaxation method is unconditionally stable such that no CFL conditions are imposed on the selection of time steps. Numerical results are shown for models on traffic flows and two-phase flows.
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1. Introduction

In this paper, we are interested in numerically solving the Cauchy problem associated with the following scalar conservation law:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(k(x), u) = 0, \quad x \in \mathbb{R}, \quad t > 0,$$

$$u(0, x) = u_0(x), \quad x \in \mathbb{R},$$

where $u \in \mathbb{R}$ is the scalar unknown, the flux function $f(k(x), u) : \mathbb{R} \rightarrow \mathbb{R}$ is non-linear and $k(x)$ is a given function that can depend on time as well. We assume that the Jacobian $f_u(k(x), u) = \partial f(k(x), u)/\partial u$ is diagonalizable with real eigenvalues. To illustrate the numerical techniques discussed in this paper, we use the multiplicative flux function

$$f(k(x), u) = k(x)g(u), \quad k(x) = \begin{cases} k_L, & \text{if } x < 0, \\ k_R, & \text{if } x > 0, \end{cases}$$

where $k_L$ and $k_R$ are positive constants with $k_L \neq k_R$. Hyperbolic equations of conservation laws (1)–(2) occur in many physical applications, for example in porous media flows [4], sedimentation phenomena [3], resonant models [6] and vehicular traffic flows [12,5]. Most practical applications of these problems cannot be solved analytically and hence require numerical methods. One of the main difficulties in the analysis of problem (1)–(2) is the correct definition of a solution. It is well known that after a finite time the problem (1)–(2) does not in general possess a continuous solution even if the initial data $u_0$ is sufficiently smooth. Hence a solution of (1)–(2) has to be understood in the weak sense. Moreover, the solution of (1)–(2) involves abrupt changes in flow variables at the interface result into the notion of Riemann problems. Among the difficulties that arose when approximating solutions of (1)–(2) are numerical instability, poor shock and rarefaction resolutions, and even spurious numerical solutions.

Many numerical methods are available in the literature to solve conservation laws with discontinuous flux function. The most popular techniques are finite difference schemes which are based on exact or approximate solver such as Lax-Friedrich, Godunov and Engquist-Osher among others. In fact, the Godunov-type methods use approximate Riemann solvers to compute the numerical flux instead of exact Riemann solvers. These procedures are mathematically hard to treat and computationally demanding, particularly for flux functions changing the derivative signs in more than one interface or depending on discontinuous time-dependent coefficients. In addition, the application of Godunov method to solve the problem (1)–(2) requires a discretization of the discontinuous coefficient $k$ staggered with respect to that of the solution $u$, compare [17] and further details are therein.

Recently, relaxation schemes [7] have become a very interesting numerical tools for solving hyperbolic equations of conservation laws. The main advan-
tage is the simplicity of the schemes since neither Riemann problems nor linear systems of algebraic equations have to be solved. The only information necessary to know from the problem at hand is a rough estimate of the spectral radius of the flux functions. Therefore, the schemes are attractive for problems where the Riemann problems are too difficult to approximate or to implement as those which arise in conservation laws with discontinuous flux function. The relaxation methods have been studied in [15] for the two-dimensional Riemann problems in gas dynamics, in [16] for the shallow water equations, and in [2] for compressible and incompressible flows. In [14] we have proposed a general frame to reconstruct relaxation methods of higher order. A third-order relaxation scheme was also studied in [1] for systems of conservation laws. We should emphasis that the relaxation schemes are strongly related to the central difference methods such that one of their common properties is that both methods are Riemann solvers free and thus they combine high accuracy with simplicity.

The goal of this paper is to present a class of numerical methods that are simple, easy to implement, and accurately solves conservation laws with discontinuous flux function without relying on a Riemann solver. This goal is reached by transforming the nonlinear problem to a semilinear system using relaxation method. In characteristic variables, the relaxation system reduces to a diagonalizable $2 \times 2$ system which is trivial to discretize using upwinding without resolving Riemann problem at each interface of the mesh. The stiff source term in the relaxation system is discretized implicitly. In this paper, we also develop a characteristic-based relaxation method that avoids the usual CFL condition on the time steps used in Eulerian relaxation methods.

The paper begins with a brief description of the relaxation approximation for conservation laws. For simplicity of presentation we will consider the discontinuous flux function given by (2), but methods herein presented extend directly to other general flux functions and are also applicable to the systems of conservation laws. Next, we formulate relaxation methods in both Eulerian and Lagrangian frameworks. Finally, numerical results are presented for various test examples on scalar conservations appeared in modeling traffic flows and two-phase flow in porous media.

2. Relaxation approximation of conservation laws

Following the work in [7] a relaxation approximation of (1) reads

\[
\begin{align*}
\frac{\partial u}{\partial t} + \frac{\partial v}{\partial x} &= 0, \\
\frac{\partial v}{\partial t} + \lambda^2 \frac{\partial u}{\partial x} &= -\frac{1}{\varepsilon} (v - f(k(x), u)), \\
u(0, x) &= u_0(x), v(0, x) = f(k(x), u_0(x)),
\end{align*}
\]
where $v \in \mathbb{R}$ is the relaxation variable, $\varepsilon \in [0, 1)$ is the parameter that measures the relaxation rate, and $\lambda$ is the characteristic speed. We use $\lambda^2$ to denote the variable $\lambda$ in [7] to avoid square roots in formulas below. By letting $\varepsilon \to 0^+$, one expects that solution of the system (3) converges to the solution of the conservation law (1). Indeed, from the second equation in (3) we can write

$$
v = f(k(x), u) - \varepsilon \left( \frac{\partial v}{\partial t} + \lambda^2 \frac{\partial u}{\partial x} \right).$$

A Chapman–Enskog expansion in (3) for small $\varepsilon$ gives

$$
\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left( f(k(x), u) - \varepsilon f_u(k(x), u)f_k(k(x), u)k(x) \right) \\
= \varepsilon \frac{\partial}{\partial x} \left( (\lambda^2 - f_u(k(x), u)^2) \frac{\partial u}{\partial x} \right). \quad (4)
$$

Hence, a necessary condition for the coefficient of the dissipation in (4) to be positive is

$$|f_u(k(x), u)| \leq \lambda \quad \forall u. \quad (5)$$

This condition, known by sub-characteristic condition [7], ensures that the characteristic speeds of the relaxation system (3) are at least as large as the characteristic speeds of the original Eq. (1). For conservation laws with continuous flux function (i.e. $k_L = k_R$), strong convergence of solution of (3) to the unique entropy solution of (1) has been shown in many references, see [13,10] among others. The case with discontinuous flux function has been addressed for instance in [8]. The main advantage of numerically solving the relaxation system (3) over the original conservation law (1) lies in the special structure of the linear characteristic fields and localized lower order terms. In fact, the linear hyperbolic nature of (3) makes it possible to approximate its solution easily by under-resolved stable numerical discretization that uses neither Riemann solvers spatially nor nonlinear system of algebraic equations solvers temporally.

It is clear from the expansion (4) that the characteristic speeds $\lambda$ are related to the numerical diffusion. In general, $\lambda$ can be chosen large enough such that the sub-characteristic condition (5) is satisfied. However, the numerical diffusion increases with $\lambda$ and for accuracy reasons $\lambda$ should be chosen as small as possible. For the flux function (2), a simple choice may be

$$\lambda = \max\{k_L, k_R\} g'(u). \quad (6)$$

It is easy to verify that $\lambda$ in (6) satisfies the sub-characteristic condition (5). For more discussions on the selection of characteristic speeds in relaxation methods we refer to [15,1].

Let us define the Riemann invariants

$$\varphi = \frac{1}{2} \left( u + \frac{u}{\lambda} \right), \quad \psi = \frac{1}{2} \left( u - \frac{u}{\lambda} \right). \quad (7)$$
The equations (3) can be reformulated in a \(2 \times 2\) diagonalizable system as
\[
\begin{align*}
\frac{\partial \phi}{\partial t} + \lambda \frac{\partial \phi}{\partial x} &= -\frac{1}{\varepsilon} (\phi - \mathcal{M}_1), \\
\frac{\partial \psi}{\partial t} - \lambda \frac{\partial \psi}{\partial x} &= -\frac{1}{\varepsilon} (\psi - \mathcal{M}_2),
\end{align*}
\]
where the Maxwellians \(\mathcal{M}_1\) and \(\mathcal{M}_2\) are given by
\[
\mathcal{M}_1 = \frac{1}{2} \left( u + \frac{f(k(x), u)}{\lambda} \right), \quad \mathcal{M}_2 = \frac{1}{2} \left( u - \frac{f(k(x), u)}{\lambda} \right).
\]
Note that solution of (8) has the property that it propagates at finite speeds (+\(\lambda\) or \(-\lambda\)) along linear characteristic curves. The positive speed is a velocity of forward propagation and the negative one is a velocity of backward propagation. Similar interpretation is commonly used in kinetic theory of rarefied gases and its discrete velocity models. The links between relaxation and kinetic methods have been discussed in [14,9,2].

3. Numerical relaxation methods

In many kinetic methods for discrete velocity models the spatial and time discretizations are treated by method of lines. Here, the relaxation system (3) is numerically solved using a splitting operator where the advection stage and relaxation stage are discretized separately. To discretize the system (3) we assume, for simplicity, an equally spaced grid with grid space \(\Delta x\) and a uniform time step \(\Delta t\). We use the notation \(\omega^n_i\) to denote the value of a function \(\omega\) at position \(x = x_i\) and time \(t = t_n\). In this section, we first recall the second-order MUSCL relaxation scheme. Then a characteristics-based relaxation scheme is formulated.

3.1. Eulerian relaxation method

In an Eulerian framework, relaxation schemes combine upwinding reconstructions for the spatial derivatives and implicit–explicit (IMEX) Runge–Kutta schemes for time integrations. Here, we briefly formulate a second-order relaxation scheme and for more details on its implementation and other higher order relaxations methods refer to [7,1,15].

Integrating (3) with respect to \(x\) and keeping the time \(t\) continuous we obtain the following discrete system:
\[
\begin{align*}
\frac{d u_i}{dt} + \frac{v_{i+\frac{1}{2}} - v_{i-\frac{1}{2}}}{\Delta x} &= 0, \\
\frac{d v_i}{dt} + \lambda^2 \frac{u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}}{\Delta x} &= -\frac{1}{\varepsilon} (v_i - f(k_i, u_i)).
\end{align*}
\]
Time integration of (9) is carried out using a second-order IMEX scheme as follows. Given \( \{u_i^n, v_i^n\}, \{u_i^{n+1}, v_i^{n+1}\} \) are updated by

\[
\begin{align*}
  u_i^* &= u_i^n, \\
  v_i^* &= v_i^n + \frac{\Delta t}{\epsilon} (v_i^* - f(k_i, u_i^*)) , \\
  u_i^{(1)} &= u_i^* - \Delta t \frac{v_i^{n+1} - v_i^n}{\Delta x}, \\
  v_i^{(1)} &= v_i^* - \Delta t \lambda^2 \frac{u_i^{n+1} - u_i^n}{\Delta x}, \\
  u_i^{(2)} &= u_i^{**} - \Delta t \frac{v_i^{*+} - v_i^{*-}}{\Delta x}, \\
  v_i^{(2)} &= v_i^{**} - \Delta t \lambda^2 \frac{u_i^{*+} - u_i^{*-}}{\Delta x},
\end{align*}
\]

(10)

\[
\begin{align*}
  u_i^{n+1} &= \frac{1}{2} (u_i^n + u_i^{(2)}), \\
  v_i^{n+1} &= \frac{1}{2} (v_i^n + v_i^{(2)}).
\end{align*}
\]

It is noteworthy that using the procedure (10) neither linear algebraic equations nor nonlinear source terms can arise. However, since the advective part in (9) is treated explicitly, this relaxation scheme is conditionally stable such as the time step \( \Delta t \) has to satisfy the CFL condition

\[
\text{CFL} = \lambda^2 \frac{\Delta t}{\Delta x} \leq 1.
\]

(11)

The spatial discretization (9) is completed when a numerical construction of the fluxes is chosen. Since the system (3) has linear characteristics, upwind schemes can easily be implemented. For example, the van Leer’s MUSCL scheme applied to \( \phi \) and \( \psi \) in (8) gives

\[
\begin{align*}
  \phi_{i+\frac{1}{2}} &= \phi_i + \frac{1}{2} \Delta x \sigma_i^+, \\
  \psi_{i+\frac{1}{2}} &= \psi_{i+1} + \frac{1}{2} \Delta x \sigma_{i+1}^-, 
\end{align*}
\]

(12)

where \( \sigma_i^+ \) and \( \sigma_i^- \) are the slope of \( \phi \) and \( \psi \) on the \( i \)th cell, respectively. For \( \omega^+ = \phi \) and \( \omega^- = \psi \), the slopes \( \sigma^\pm \) are defined by

\[
\sigma_i^\pm = \frac{1}{\Delta x} (\omega_i^\pm - \omega_{i+1}^\pm) \Phi(\theta_i^\pm), \quad \theta_i^\pm = \frac{\omega_i^\pm - \omega_{i+1}^\pm}{\omega_{i+1}^\pm - \omega_i^\pm},
\]

with \( \Phi(\theta) \) defines the van Leer’s slope limiter function [18],

\[
\Phi(\theta) = \frac{|\theta| + \theta}{1 + |\theta|}.
\]

Once \( \phi_{i+\frac{1}{2}} \) and \( \psi_{i+\frac{1}{2}} \) are reconstructed using (12), the numerical fluxes \( u_{i+\frac{1}{2}} \) and \( v_{i+\frac{1}{2}} \) in the relaxation system are obtained from (7) as

\[
\begin{align*}
  u_{i+\frac{1}{2}} &= \phi_{i+\frac{1}{2}} + \psi_{i+\frac{1}{2}}, \quad \text{and} \quad
  v_{i+\frac{1}{2}} = \lambda \left( \phi_{i+\frac{1}{2}} - \psi_{i+\frac{1}{2}} \right),
\end{align*}
\]

respectively. Note that if we set \( \Phi = 0 \), the spatial discretization (12) reduces to the first-order upwind scheme. Higher order discretizations can be reconstructed following similar manner, we refer to [16,15,1] for a third-order relaxation scheme.
3.2. Lagrangian relaxation method

Lagrangian or characteristic methods treat the advection by using a Lagrangian tracking algorithm along the characteristic curves while keeping the convenience of a fixed computational mesh. The numerical information from the previous time level is projected from the background Eulerian mesh into the Lagrangian mesh and the required quantities in the present time level are calculated by interpolation. The significant advantage of these methods is that, due to the Lagrangian treatment of advection, the CFL restriction is eliminated and the time truncation errors are reduced in the Eulerian methods.

The starting point is to determine the characteristics curves associated with the transport part of the system (8). Let denote by

\[ \chi^+_i = x_i - \lambda \Delta t \quad \text{and} \quad \chi^-_i = x_i + \lambda \Delta t, \]

the two characteristic curves associated with the kinetic variables \( \varphi \) and \( \psi \), respectively. The characteristic lines \( \chi^+_i \) are also known by departure points of a particle passing through the gridpoint \( x_i \) at time \( t_{n+1} \). Once the the departure points are calculated and located, the solution of the advection stage in (8) is simply

\[ \varphi^{n+1}_i = \tilde{\varphi}^n_i := \varphi(t_n, \chi^+_i), \quad \psi^{n+1}_i = \tilde{\psi}^n_i := \psi(t_n, \chi^-_i). \]

In general \( \chi^+_i \) in (13) will not coincide with the spatial position of a gridpoint. A search locate algorithm is then required to find the host cell where such point is located. In our case, this step can be simple as index checking or ad hoc searching. Therefore, the solutions in (14) must be obtained by interpolation from known values at gridpoints of the cell where \( \chi^+_i \) belong. For the examples presented in this paper, we used the cubic spline interpolation. Other interpolation procedures can also be applicable.

Assuming that suitable approximations are performed to \( \tilde{\varphi}^n_i \) and \( \tilde{\psi}^n_i \), then the Lagrangian solutions \( \hat{u}^n_i \) and \( \hat{v}^n_i \) in the relaxation system are obtained from (7) as

\[ \hat{u}^n_i = \hat{\varphi}^n_i + \hat{\psi}^n_i \quad \text{and} \quad \hat{v}^n_i = \lambda(\hat{\varphi}^n_i - \hat{\psi}^n_i), \]

respectively. The time integration of the relaxation stage in (9) is carried out using the same IMEX scheme as in (10). Hence, given \( \{u^n_i, v^n_i\} \), we compute \( \{u^{n+1}_i, v^{n+1}_i\} \) as follows:

\[ u^n_i = \hat{u}^n_i, \quad v^n_i = \hat{v}^n_i + \tfrac{\Delta t}{\varepsilon}(v^*_i - f(k_i, u^n_i)), \]
\[ u^{n*}_i = u^n_i, \quad v^{n*}_i = v^n_i - \tfrac{\Delta t}{\varepsilon}(v^{n*}_i - f(k_i, u^{n*}_i)) - \tfrac{2\Delta t}{\varepsilon}(v^n_i - f(k_i, u^n_i)), \]
\[ u^{n+1}_i = \frac{1}{2}(\hat{u}^n_i + u^{n*}_i), \quad v^{n+1}_i = \frac{1}{2}(\hat{v}^n_i + v^{n*}_i). \]
We should mention that, since the characteristic curves (13) are calculated exactly, the time stepping (15) is unconditionally stable such that the selection of time steps is based only on accuracy reasons.

4. Applications

In this section, we present some numerical results obtained with relaxation scheme discussed in the above section. In all our computations, the characteristic speeds $\lambda$ are calculated at each time step as

$$\dot{\lambda} = \max_x \{ f_u(k(x), u) \} + \tau,$$

where $\tau$ is a safety parameter set to 0.1 for all examples to avoid that $\lambda$ vanishes. We will use $\varepsilon = 10^{-10}$, unless otherwise stated. Here, the CFL number is fixed and the time step is adjusted according to

$$\Delta t = \text{CFL} \frac{\Delta x}{\lambda^2}.$$

In the sequel, upwind, MUSCL and characteristic stand for relaxation schemes using respectively, upwind, MUSCL and characteristics methods in their reconstructions.

4.1. Accuracy test problem

Our first example is a conservation law with exact steady state solution which can be used to quantify the results obtained by our relaxation methods. This example can also serve to test the ability of the above relaxation schemes to converge to the correct entropy solution. The problem statement is given by Eqs. (1) and (2) where

$$g(u) = u(1 - u), \quad k(x) = \begin{cases} 2, & \text{if } 0 \leq x \leq 2.5, \\ \frac{25 - 2x}{10}, & \text{if } 2.5 < x < 7.5, \\ 1, & \text{if } 7.5 \leq x \leq 10, \end{cases}$$

and an initial condition given by

$$u_0(x) = \begin{cases} 0.9, & \text{if } 0 \leq x \leq 2.5, \\ \frac{1 + \sqrt{0.28}}{2}, & \text{if } 2.5 < x \leq 10. \end{cases}$$

This problem has a steady exact solution defined as

$$u_\infty(x) = \begin{cases} 0.9, & \text{if } 0 \leq x \leq 2.5, \\ \frac{1}{2} + \frac{\sqrt{k(x)^2 - 0.72k(x)}}{2k(x)}, & \text{if } 2.5 < x < 7.5, \\ \frac{1 + \sqrt{0.28}}{2}, & \text{if } 7.5 \leq x \leq 10. \end{cases}$$
We have computed the approximate solution at $t = 10$. At this time the approximated solutions are almost stationary, and therefore error norms can be calculated. We consider the $L^\infty$- and $L^1$-error norms defined as

$$\max_{1 \leq i \leq M} |u_i - u_\infty(x_i)| \quad \text{and} \quad \sum_{i=1}^M |u_i - u_\infty(x_i)| \Delta x,$$

respectively. In (16), $u_i$ and $u_\infty(x_i)$ are respectively, the computed and exact steady state solutions at gridpoint $x_i$, whereas $M$ stands for the number of gridpoints used in the spatial discretization. The obtained results are listed in Table 4.1 for two different CFL numbers. For CFL = 0.5, the results provided by the MUSCL scheme are more accurate than the results provided by upwind and characteristic schemes. The error norms (when the mesh is refined) for all the schemes are very close each to other. Increasing the CFL number from 0.5 to 3, the upwind and MUSCL schemes go unstable (‘–’ in Table 4.1 corresponds to runs where the upwind and MUSCL schemes become unstable). As expected, the characteristic scheme is not subject to CFL condition, it still producing accurate results with CFL number six times larger than that used in upwind or MUSCL schemes.

Our next concern is to test the asymptotic-preserving property of the relaxation schemes. To this end, results obtained by the MUSCL scheme using $\varepsilon = 1$, $10^{-1}$ and $10^{-3}$ are depicted with the exact solution in Fig. 4.1. We have used $M = 200$ in these computations. The upwind and characteristic schemes give similar results and are not included here for sake of brevity. At the relaxing regime $\varepsilon = 1$ and $\varepsilon = 10^{-1}$, the approximate solutions are far from the correct asymptotic solution. Decreasing the relaxation rate to $\varepsilon = 10^{-3}$, the approximate solution fits well with the exact steady state solution. This agree

<table>
<thead>
<tr>
<th>$M$</th>
<th>CFL = 0.5</th>
<th>CFL = 3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Upwind</td>
<td>MUSCL</td>
</tr>
<tr>
<td>$L^\infty$-error</td>
<td>100</td>
<td>$1.265E-2$</td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>$6.960E-3$</td>
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<td></td>
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<td>$1.899E-3$</td>
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<tr>
<td></td>
<td>1600</td>
<td>$9.659E-4$</td>
</tr>
<tr>
<td>$L^1$-error</td>
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<td>$2.079E-2$</td>
</tr>
<tr>
<td></td>
<td>200</td>
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</tr>
<tr>
<td></td>
<td>1600</td>
<td>$1.243E-3$</td>
</tr>
</tbody>
</table>
well with the formal convergence analysis of relaxation system to the original conservation law as $\varepsilon \to 0$.

4.2. Traffic flow problem

Traffic flow models remain challenging for numerical solution, even though great progress has been made in the development of modern shock capturing methods for equations of conservation laws in the last decades. Taking into account the nature of vehicular roads, these models offer a realistic one-dimensional conservation law with discontinuous coefficient entries. The well-known Lighthill–Whitham and Richards model [12] for traffic flows can be written in conservation form as (1) where $u(t, x) = a(x)\rho(t, x)$ with $a(x)$ and $\rho(t, x)$ are the lane number and the density per-lane, respectively. The flux function is

\[ f(k(x), u) = k(x)u(1 - u), \quad k(x) = \frac{v(x)}{v_{\text{max}}}, \]

![Fig. 4.1. Solution of the accuracy test problem at different relaxation rates $\varepsilon$.](image-url)
where $v(x)$ is the free flow velocity at the point $x$ and $v_{\text{max}} = \max_x \in \mathbb{R} v(x)$ is the maximum speed. Two situations are selected namely a bottleneck traffic and a traffic jam situation.

We consider a road with length of 10 km with an initial density $\rho(x) = 0.2 \text{ veh/km}$. The coefficients $a(x)$ and $v(x)$ are discontinuous functions given by

$$a(x) = \begin{cases} 4, & \text{if } x < 3 \text{ km}, \\ 2, & \text{if } x \geq 3 \text{ km}, \end{cases} \quad v(x) = \begin{cases} 1, & \text{if } x < 3 \text{ km}, \\ 0.6, & \text{if } x \geq 3 \text{ km}. \end{cases}$$

The computational road is divided into 100 gridpoints, the CFL is set to 0.75, and the duration of this simulation is 900 s. To display the results we use dimensionless variables obtained by scaling the space $x$ and time $t$ by $x/L$ and $tv_{\text{max}}/L$, respectively. The evolution of dimensionless density is shown in Fig. 4.2. Our relaxation scheme captures the correct jam structure and advect the moving fronts without deteriorating the location of backwards and forwards waves. A comparison between the computed results by the three relaxation schemes is displayed in Fig. 4.3 for $t = 1.3$. It is clear that extensive numerical diffusion is introduced by the upwind scheme, while the characteristic results are less diffusive than those obtained by the upwind scheme.

The second test case consists of a jam situation and its recovery. Here the road length is 2 km, $a(x) = 1$ and

![Fig. 4.2. Density evolution for the bottleneck situation in traffic flow problem.](image-url)
Initially the density $\rho(x) = 0.5$ veh/km and the duration of this simulation is 90 s.

This problem is more complicated than the previous one since the flux function depends on space and time as well. The evolution of the density in time–space phase domain is plotted in Fig. 4.4. As can be seen, good behavior is recovered by the MUSCL scheme without any significant loss of accuracy. In Fig. 4.5, we plot the density approximated by the three schemes at time $t = 1.5$. The upwind solution is very diffusive and a deterioration of the accuracy is clearly seen in the characteristic solution. The accuracy of the characteristic scheme degrades may be due to the dependence of flux function on the time variable.

Note that the exact solutions to these examples are not available, but the computed solutions seem to converge to the physically relevant solutions in
Fig. 4.4. Density evolution for the jam situation and its recovery in traffic flow problem.

Fig. 4.5. Comparison of the results for the jam situation and its recovery in traffic flow problem.
both test cases. The MUSCL scheme capture the shock accurately, do not diffuse the fronts or exhibit oscillations near the steep gradients.

4.3. Two-phase flow problem

The Buckley–Leverett equation has serve as one of the simplest model of two-phase flow in a porous medium [11]. Here the governing equations are (1) where the flux function is not convex and is given by

$$f(k(x), u) = \frac{u^2}{u^2 + k(x)(1-u)^2}, \quad k(x) = \begin{cases} 50, & \text{if } 0 \leq x < 0.5, \\ 5, & \text{if } 0.5 \leq x \leq 1. \end{cases} \tag{17}$$

The initial condition is

$$u_0(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq 1 - \frac{1}{\sqrt{2}}, \\ 1, & \text{if } 1 - \frac{1}{\sqrt{2}} \leq x \leq 1. \end{cases}$$

The spatial domain is discretized into 200 gridpoints and a CFL = 0.75 is used. Fig. 4.6 shows the evolution of MUSCL solution in time–space phase domain. The contact discontinuity is perfectly approximated and the monotonicity of the rarefaction wave is maintained. In Fig. 4.7, comparison of the solutions computed by the three relaxation schemes is shown at time $t = 0.2$. For this test case, MUSCL yields the best resolution, upwind is very diffusive, and characteristic gives comparable results which are less diffusive than upwind.

Next we introduce gravitational effects in the Buckley–Leverett model. Thus, the flux function in (17) changes to

$$f(k(x), u) = \frac{u^2}{u^2 + k(x)(1-u)^2}, \quad k(x) = \begin{cases} 50, & \text{if } 0 \leq x < 0.5, \\ 5, & \text{if } 0.5 \leq x \leq 1. \end{cases} \tag{17}$$

![Fig. 4.6. Solution evolution for the Buckley–Leverett model in two-phase flow problem.](image-url)
This problem can be seen as a model for the governing equation of the saturation of a fluid in a gravity field flowing in a one-dimensional medium with discontinuous permeability $k(x)$. We use the same coefficient $k(x)$ and initial data as the previous test. Again, 200 gridpoints are used in our computations and $\text{CFL} = 0.75$. The results are displayed in Figs. 4.8 and 4.9. As in the previous test, the MUSCL scheme resolves this example more accurately than the other two schemes. In Fig. 4.9, an undershoot is detected on the contact discontinuity in the characteristic solution.

In summary, MUSCL scheme seems to produce satisfactory results and to exhibit a typical second-order behavior in all these examples. The upwind results are much more smeared out and exhibited typical first-order behavior. On the other hand, the characteristic scheme can be an alternative when problems under consideration require large CFL numbers.
Fig. 4.8. Solution evolution for the Buckley–Leverett model with gravitation in two-phase flow problem.

Fig. 4.9. Results for the Buckley–Leverett model with gravitation in two-phase flow problem.
5. Conclusions

We have presented a class of uniformly stable methods for numerical solution of conservation laws with discontinuous flux function. Relaxation approximation is used to transform the nonlinear equation to a semilinear system with linear characteristics. To approximate its solution, we have implemented relaxation scheme in Eulerian and Lagrangian frameworks. The methods are easy to implement and do not require Riemann solvers or special front tracking techniques. In addition, the Eulerian relaxation scheme is uniformly stable under the standard CFL condition, while the Lagrangian relaxation scheme is unconditionally stable.

Numerical results are shown for several test examples on traffic flows and two-phase flow in porous media. The obtained results demonstrate good shock resolution with high accuracy in smooth regions and without any nonphysical oscillations near the shock areas. The convergence to the exact steady state solution has been also tested. Finally combining relaxation approximation with characteristic method show considerable promise in developing explicit but unconditionally stable solvers for conservation laws with discontinuous flux function. Furthermore, borrowing the idea of flux-corrected techniques, it feasible to improve the accuracy of the characteristic reconstruction in Section 3.2. This issue and extension of these methods to systems of conservation laws will be examined in a forthcoming work.

References