ANALYTICAL AND NUMERICAL STUDY OF NATURAL CONVECTION HEAT TRANSFER IN A VERTICAL POROUS ANNULUS

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A study is made of quasi-steady state natural convection in a vertical annular porous layer when the inner wall is heated by a constant heat flux and the other walls are maintained adiabatic. On the basis of the Darcy-Oberbeck-Boussinesq equations, the problem is solved analytically in the limit of a long shallow annulus ($A \gg 1$), where $A$ is the aspect (height-to-gap width) ratio. The analytical solution for the flow and heat transfer, based on a parallel flow assumption, is derived in terms of the Darcy-Rayleigh number $R$ and radius ratio $K$. A numerical study of the same phenomenon, obtained by solving the complete system of governing equations, is conducted to assess the validity of the analytical results. It is demonstrated that the analytical solution predicts well the flow structure and the heat transfer for a wide range of $R$ and $K$. In the boundary layer regime it is shown that the Nusselt number, based on the gap width of porous annulus, is

$$Nu = R^2 \left[2K \ln K - (K^2 - 1)\right] / \left[2(K - 1)^2 (K + 1)^{3/2}\right].$$

KEYWORDS Natural convection  Heat transfer  Vertical porous annulus

INTRODUCTION

Over the past years considerable research efforts have been devoted to the study of heat transfer induced by buoyancy effects in a porous medium saturated with fluids. Interest in this natural convection phenomenon has been motivated by such diverse engineering problems as geothermal energy extraction, pollutant dispersion in aquifers, post accident heat removal from nuclear reactor rubble beds, and thermal insulation. The state-of-the-art has been summarized in a recent book by Nield and Bejan (1992).

The existing literature on this domain has focused considerable attention on natural convection in differentially heated vertical porous enclosures. For example, the case of a rectangular cavity with vertical walls at constant temperatures, the horizontal walls being insulated, has been studied theoretically by numerous investigators (Chan et al. (1970); Burns et al. (1976); Weber (1979); Shiralkar et al. (1983) and Poulikakos and Bejan (1983)). Several correlations for heat transfer coefficients have been proposed (Chan et al. (1970); Holst and Haziz (1972); Seki et al. (1978); Weber (1979) and Shiralkar et al. (1983)), these later covering a wide range of Rayleigh numbers and aspect ratios. A few studies (Bejan (1983); Prasad and Kulacki (1984a) and Vasseur et al. (1986)) have also been reported for the case where heating results from the imposition of a uniform heat flux as opposed to isothermal heating. The heat transfer rate based upon the mean temperature difference was higher as compared to the case with two
isothermal walls (Prasad and Kulacki (1984a)). The engineering importance of a constant heat flux boundary condition has been discussed by Cheng and Minkowycz (1977) in their study of free convection about a vertical plate embedded in a porous medium.

Natural convection in a porous medium bounded by vertical coaxial cylinders is not nearly as well studied as the rectangular counterpart. However annular cavities are also common and thus natural convection in such regions ought to be of interest to heat transfer engineers. What is known is through the efforts of Havstad and Burns (1982), Reda (1983), Prasad and Kulacki (1984b, 1985), Hickox and Gartling (1985), Prasad (1986) and Prasad et al. (1986). Havstad and Burns (1982) have considered a case where the vertical walls are held at constant temperature, the top and bottom of the cavity being insulated. The problem was investigated by using perturbation technique for low Rayleigh numbers regimes, an asymptotic solution valid for tall and narrow enclosures, and a finite difference numerical method. The same problem was considered by Hickox and Gartling (1985) using a finite element technique. An approximate analytical solution, valid for large aspect ratios and low Rayleigh numbers, was also developed. Although the works of Havstad and Burns (1982) and Hickox and Gartling (1985) have considered a wide range of radius ratios they are restricted to a relatively limited range of Rayleigh numbers. This range was widely extended by Prasad and Kulacki (1984b) who obtained numerical results for Rayleigh numbers up to $10^4$, aspect ratio $1 \leq A \leq 20$ and radius ratio $1 \leq K \leq 26$. The Rayleigh number criteria for flow regimes, namely conduction, asymptotic and boundary layer, were obtained and correlations for the heat transfer rates were also reported by these authors. The case of a short cylindrical annulus has also been considered by Prasad and Kulacki (1985) for $0.9 \leq A \leq 0.3$, $1 \leq K \leq 11$ and Rayleigh numbers smaller than $10^4$. The numerical solution was in good agreement with experimental values of the Nusselt number obtained for $K = 5.338$, $A = 1$ and Rayleigh numbers between 49 and 3582. More recently the problem of a vertical porous annulus has been studied numerically and experimentally by Prasad (1986) and Prasad et al. (1986) respectively for the case when its inner wall is heated by applying a constant heat flux while its outer wall is maintained isothermal. This configuration results in a higher rate of heat transfer compared to the isothermal heating. Higher the aspect ratio, larger is this difference whereas the effect of radius ratio is just reversed. A correlation for heat transfer coefficient has been obtained by Prasad (1986) for the range $1 \leq K \leq 10$, $1 \leq A \leq 50$ and Rayleigh numbers between $10^2$ and $10^4$. Results have also been presented for the asymptotic case when the heat transfer in an annulus is close to that for a vertical cylinder embedded in an infinite medium.

The objective of the present investigation is to study the quasi-steady state natural convection in a vertical annular porous layer where the inner wall is heated by applying a constant heat flux while its other walls are insulated. The analysis proceeds as follows. First, the differential equations which describe the physical model considered here are formulated in a standard manner assuming the validity of the Darcy’s law and the Boussinesq approximation. Then by using a suitable finite difference numerical method, the governing equations are solved to obtain a detailed view of the velocity and temperature distributions within the cavity. This is followed by the derivation of a closed form analytical solution valid in the limit of a long shallow annulus. The
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Mathematical treatment, based on the parallel flow approach, follows that of Cormack et al. (1974). This type of flow is of interest in many practical applications such as the initial heat transfer through the insulation of heated vertical walls.

MATHEMATICAL FORMULATION

Consider a vertical annulus of height $H'$, inner radius $r'_i$ and outer radius $r'_o$, as shown in Figure 1. The annular region is filled with a rigid, fluid-saturated porous medium. All boundaries of the annular cavity are impermeable. The inner wall is heated by a constant heat flux $q'$ while all other boundaries are insulated. In the porous medium, the Darcy's law is assumed to hold and the viscous drag and inertial terms of the momentum equation are neglected because their magnitudes are of small order compared to the other terms for low Darcy and low particle Reynolds numbers. It is also assumed that the fluid properties are constant, except for the density variation in producing the buoyancy force. With these assumptions, the dimensional conservation equations for axisymmetric incompressible laminar flow are reduced to

$$\frac{\partial (r'u')}{\partial r'} + \frac{\partial (r'v')}{\partial z'} = 0$$

$$u' = -\frac{\kappa}{\mu} \frac{\partial p'}{\partial r'}$$

$$v' = -\frac{\kappa}{\mu} \left( \frac{\partial p'}{\partial z'} + \rho g \right)$$

![Figure 1: Schematic view of the physical configuration.](image-url)
\[ (\rho c)_p \frac{\partial T'}{\partial t'} + (\rho c)_f \left[ u' \frac{\partial T'}{\partial r'} + v' \frac{\partial T'}{\partial z'} \right] = k \left[ \frac{1}{r'} \frac{\partial}{\partial r'} \left( r' \frac{\partial T'}{\partial r'} \right) + \frac{\partial^2 T'}{\partial z'^2} \right] \quad (4) \]

\[ \rho = \rho_r \left[ 1 - \beta (T' - T'_r) \right] \quad (5) \]

The variables in the above equations are nondimensionalized as follows

\[ (r, z) = (r', z')/L' \]

\[ (u, v) = (u', v') L'/\alpha_f \]

\[ t = t' \alpha_f /L'^2 \]

\[ T = \frac{(T' - T'_r)}{\Delta T'} \]

\[ \Delta T' = q' L'/k \]

\[ p = p' k/(\mu \alpha_f) \]

where \( \alpha_f = k/(\rho c)_f \) and \( T'_r \) is the temperature at \( r' = r; \) and \( z' = 0 \).

Eliminating the pressure from Eqs. (2) and (3) in the usual way, assuming the Boussinesq approximation and substituting the non-dimensional variables defined by Eq. (6), Eqs. (2) to (5) can be reduced to the following coupled system for the stream function \( \Psi \) and temperature \( T \).

\[ \frac{\partial^2 \Psi}{\partial r'^2} + \frac{\partial^2 \Psi}{\partial z'^2} + \frac{1}{r'} \frac{\partial \Psi}{\partial r'} = -R_r \frac{\partial T'}{\partial r'} \quad (7) \]

\[ \frac{\partial T'}{\partial t'} + u' \frac{\partial T'}{\partial r'} + v' \frac{\partial T'}{\partial z'} = -1 \frac{\partial}{\partial r'} \left( r' \frac{\partial T'}{\partial r'} \right) + \frac{\partial^2 T'}{\partial z'^2} \quad (8) \]

\[ u = \frac{1}{r'} \frac{\partial \Psi}{\partial z'} \quad v = -\frac{1}{r'} \frac{\partial \Psi}{\partial r'} \quad (9) \]

where \( R = g \beta q' k L'^2 /\alpha \nu k \) is a Rayleigh number based on the constant heat flux \( q' \). The heat capacity ratio \( \sigma = (\rho c)_p/(\rho c)_f \) results from the fact that the thermal inertia of the medium depends on the inertias of the solid and fluid.

The boundary conditions on \( \Psi \) and \( T \) are

\[ r = \frac{1}{K - 1} : \quad \Psi = 0, \quad \frac{\partial T'}{\partial r'} = -1 \]

\[ r = \frac{K}{K - 1} : \quad \Psi = 0, \quad \frac{\partial T'}{\partial r'} = 0 \quad (10) \]

\[ z = \pm \frac{H}{2} : \quad \Psi = 0, \quad \frac{\partial T'}{\partial z'} = 0 \]
Due to the thermal boundary conditions considered here, the phenomenon is clearly a transient one. The heat transfer collected by the inner wall cannot leave through the other insulated walls and consequently the temperature field within the vertical annulus must rise with time. However the temperature gradients themselves are expected to become time independent after an initial transient. Since the flow field depends on temperature gradients, it is therefore expected to reach a steady state. The temperature may be redefined as \( T^* = T - St \) so that its asymptotic time dependence is treated in a separate term. Dropping the asterisk from now on, Eq. (8) becomes

\[
\frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = - \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{\partial^2 T}{\partial z^2} + S
\]

where \( S = -2/(K + 1) \) acts as fictitious dimensionless source term.

Eqs. (7) and (11), together with the boundary conditions (10) completely determine the problem in terms of three dimensionless parameters, namely the Rayleigh number \( R \), the radius ratio \( K \) and the annulus height \( H \).

The Nusselt number for the present problem is defined as follows

\[
Nu = \frac{\Delta T_C}{\Delta T}
\]

with \( \Delta T = T(1/(K - 1),0) - T(K/(K - 1),0) \) being the temperature difference between the vertical walls at mid height and \( \Delta T_C \) the corresponding temperature difference in the pure conduction regime.

**NUMERICAL METHOD**

To obtain the numerical solution to Eqs. (7), (9) to (11) a finite difference scheme was used. The transient form of the energy Eq. (11) was solved using the time-marching, finite difference technique. The first and second derivatives were approximated by central differences and the time derivative by a first order forward derivative. The finite difference form of the energy equation was written in conservative form for the nonlinear convective terms in order to preserve the conservative property. The Poisson Eq. (7) was solved by the Successive-Over-Relaxation method. The iterative procedure was carried out until a steady-state was obtained. The convergence criteria for all field variables are

\[ |\Phi_{\text{new}} - \Phi_{\text{old}}|_{\text{ext}} \leq 10^{-4} \]

where \( \Phi \) stands for any of the field variables and the subscript ext denotes the extremum value over the grid points.

As a result of a grid independence study, grid size of \((61) \times (81)\) was chosen for the majority of the calculations. For high radius ratio \( K \), a grid size of \((81) \times (101)\) was employed. The time step was selected so that no numerical oscillations occurred. During the iteration, the field always evolved smoothly from arbitrary initial data to final steady states.
The accuracy of the numerical model was verified by comparing results from the present investigation with those obtained by Prasad and Kulacki (1984b) for an annulus when the vertical walls are held at constant temperature. Maximum differences between the two investigations were within 2%. Also, in the pure conduction regime \((R = 0)\), the numerical model predicted the analytical solution

\[
T_c = \frac{S}{4(K - 1)^2} \left[ (r^2(K - 1)^2 - 1) - 2K^2 \ln r(K - 1) \right]
\]

with an accuracy of less than 0.5% deviation.

APPROXIMATE ANALYTICAL SOLUTION

In this section an approximate solution, valid for quasi-steady state, is presented for the case of a long shallow annulus \((A = H/(K - 1) \gg 1)\). A preliminary examination of the numerical results (Figure 2) clearly shows that, in the core region, i.e., except in regions close to the upper and lower boundaries, the flow can be considered parallel. Similar results have been observed in the past for the case of a shallow rectangular cavity as discussed in detail by Cormack et al. (1974), Walker and Homsy (1978), and Vasseur et al. (1987, 1989). For this situation the flow and temperature fields in the central part of the cavity must be respectively of the following form

\[
\begin{align*}
u & = 0 \quad v = v(r) \\ T & = Cz + O(r)
\end{align*}
\]

where \(C\) is the unknown but constant temperature gradient in the \(z\) direction.

Substituting Eqs. (14) and (15) into Eqs. (7) and (11) and rearranging one obtains respectively

\[
\begin{align*}
\frac{d^2 v}{dr^2} + \frac{1}{r} \frac{dv}{dr} - \alpha^2 v & = SR \\
\frac{d^2 \theta}{dr^2} + \frac{1}{r} \frac{d\theta}{dr} & = vC + S
\end{align*}
\]

where \(\alpha^2 = RC\).

Solutions of Eqs. (16) and (17) satisfying the boundary conditions given by Eq. (10) are

\[
\begin{align*}
v & = DI_o(\alpha r) + EK_o(\alpha r) - F \\
T & = Cz + \frac{v}{R} - \frac{1}{R} \left[ DI_o(\alpha r) + EK_o(\alpha r) - F \right]
\end{align*}
\]
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FIGURE 2 Isotherms and streamlines for

a) $R = 10$, $K = 2$, $\Psi_{\text{min}} = -0.4929$, $T_{\text{max}} = 0.1606$, $T_{\text{min}} = -0.5289$

b) $R = 10^2$, $K = 2$, $\Psi_{\text{min}} = -2.4224$, $T_{\text{max}} = 0.4747$, $T_{\text{min}} = -0.5636$

c) $R = 4 \times 10^2$, $K = 2$, $\Psi_{\text{min}} = -3.9345$, $T_{\text{max}} = 0.4194$, $T_{\text{min}} = -0.4420$

d) $R = 4 \times 10^2$, $K = 1.2$, $\Psi_{\text{min}} = -17.1980$, $T_{\text{max}} = 0.4670$, $T_{\text{min}} = -0.4940$

e) $R = 4 \times 10^2$, $K = 1.5$, $\Psi_{\text{min}} = -7.2010$, $T_{\text{max}} = 0.4506$, $T_{\text{min}} = -0.4720$

f) $R = 4 \times 10^2$, $K = 2.5$, $\Psi_{\text{min}} = -2.8824$, $T_{\text{max}} = 0.3871$, $T_{\text{min}} = -0.4063$
where $F = RS/\alpha^2$, $I_j$ and $K_j$ are modified Bessel functions of $j^{th}$ order, and the constants $D$ and $E$ are given by

$$D = \frac{RK_1(m)}{\alpha} \frac{1}{[I_1(m)K_1(n) - I_1(n)K_1(m)]}$$

(20)

$$E = \frac{RI_1(m)}{\alpha} \frac{1}{[I_1(m)K_1(n) - I_1(n)K_1(m)]}$$

(21)

where $n = \alpha/(K - 1)$, $m = \alpha K/(K - 1)$.

The thermal boundary conditions in the $z$ direction cannot be satisfied exactly with the parallel flow approximation. However, the value of the vertical temperature gradient $C$ may be obtained simply by considering the arbitrary control volume of Figure 1. Integration of the energy Eq. (11), together with the first and last boundary conditions of Eq. (10), yields the following energy flux integral at any $z$ section

$$C = S \int_{1/(K - 1)}^{K/(K - 1)} rv \partial dr$$

(22)

Substituting Eqs. (18) and (19) into (22) and integrating yields

$$2\alpha^2(K^2 - 1) + \frac{4R^2(K - 1)}{\alpha^2(K + 1)} = \{K^2[(DI_o(m) + EK_o(m))^2 - (DI_1(m) + EK_1(m))^2] - [(DI_o(n) + EK_o(n))^2 - (DI_1(n) + EK_1(n))^2]\}$$

(23)

The $z$-temperature gradient $C$ in the velocity and temperature field solutions, Eqs. (18) and (19), can be obtained for any combination of the controlling parameters $R$ and $K$ by numerically solving the above transcendental equation. Typical numerical values of $C$ are presented in Figure 3 as a function of Rayleigh number $R$ and radius ratio $K$. The curve for $K \to 1$ corresponds to the limiting case of a rectangular cavity. For a fixed value of $R$ it is observed that $C$ decreases with increasing $K$. This is due to the fact that the strength of the flow circulation is reduced as $K$ is made larger (see Figures 2(d), (e), (c) and (i)), resulting in a lower vertical thermal stratification. For a fixed value of $K$ Figure 3 indicates that, as expected, $C \to 0$ as the value of $R \to 0$, i.e., a case of the pure conduction regime. As $R$ is made larger, $C$ increases first with $R$, passes through a maximum and then begins to decrease asymptotically toward $C \to 0$ as $R \to \infty$. That is also expected since for very large $R$, i.e., in the boundary layer regime, the temperature field is characterized by a sharp temperature drop through the very thin thermal boundary layer near the inner cylinder. Outside this thermal layer, the fluid is almost isothermal with a very low thermal stratification in the vertical direction.

Since the temperature in the core of the annulus varies linearly with $z$, the difference between the temperature at the inner and outer walls remains constant along the vertical direction. Thus, from Eq. (19) it is readily found that $\Delta T$ defined earlier
The following (12) is given by

$$
\Delta T = \frac{1}{R} \left\{ D[I_0(n) - I_0(m)] + E[K_0(n) - K_0(m)] \right\}
$$

(24)

The corresponding temperature difference, in the pure conduction regime is

$$
\Delta T_p = \frac{2K^2 \ln K - (K^2 - 1)}{2(K - 1)(K^2 - 1)}
$$

(25)

The Nusselt number, Eq. (12), is thus given by

$$
Nu = \frac{R[2K^2 \ln K - (K^2 - 1)]}{2(K - 1)(K^2 - 1)\{D[I_0(n) - I_0(m)] + E[K_0(n) - K_0(m)]\}}
$$

(26)

The boundary layer regime may be deduced from the above results. For this situation \( R \to \infty \) and an asymptotic analysis of the modified Bessel functions then leads to the following formulae

$$
v = \frac{R}{\alpha} \left[ \frac{e^{\frac{(K-1)^{-1}}{r}}}{\sqrt{r(K-1)}} - \frac{2}{\alpha^2(K+1)} \right]
$$

(27)

$$
T = Cz + \left[ \frac{e^{\frac{(K-1)^{-1}}{r}}}{\alpha \sqrt{r(K-1)}} - \frac{1}{\alpha} \right]
$$

(28)

$$
Nu = \frac{\alpha[2K^2 \ln K - (K^2 - 1)]}{2(K - 1)(K^2 - 1)}
$$

(29)
where

\[ C = R^{-1/5} (K + 1)^{-2/5}, \quad \alpha = \sqrt{(RC)} = R^{2/5} (K + 1)^{-1/5} \]  

(30)

The asymptotic behaviour of \( C \) for very large \( R \), Eq. (30), is also indicated in Figure 3 as dotted lines for comparison.

The velocities and temperatures at the inner and outer cylinders are

\[ v_i = \frac{R}{\alpha} \left[ 1 - \frac{2}{\alpha (K + 1)} \right] \]  

(31)

\[ v_o = -\frac{2R}{\alpha^2 (K + 1)} \]  

(32)

\[ T_i = Cz \]  

(33)

\[ T_o = Cz - \frac{1}{\alpha} \]  

(34)

An other limit of particular interest is the case of the rectangular cavity for which \( \alpha \gg 1 \) and \( K \to 1 \). For this particular situation it may be shown that the present analytical solution yields

\[ v = \frac{R}{\alpha} \left[ \frac{\cosh \alpha (x - 1)}{\sinh \alpha} - \frac{1}{\alpha} \right] \]  

(35)

\[ \theta = \frac{1}{\alpha \sinh \alpha} \left[ \cosh \alpha (x - 1) - \cosh \alpha \right] \]  

(36)

\[ \text{Nu} = \frac{\alpha \sinh \alpha}{2 (\cosh \alpha - 1)} \]  

(37)

\[ \alpha^2 + \frac{R^2}{\alpha^4} = \frac{R^2}{2 \alpha^3 \sinh^2 \alpha} \left[ \sinh \alpha \cosh \alpha + \alpha \right] \]  

(38)

where coordinate \( x \) is taken on the left wall of the rectangular cavity. Eqs. (35)–(38) correspond to those obtained earlier by Vasseur et al. (1986) for a rectangular cavity.

In the boundary layer regime, \( R \to \infty \), and the above formula reduce to

\[ v = \frac{R}{\alpha} \left[ \frac{1}{e^{2x}} - \frac{1}{\alpha} \right] \]  

(39)

\[ \theta = \frac{1}{\alpha \left[ e^{2x} - 1 \right]} \]  

(40)
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\[ C = \frac{R^{-1/5}}{2^{2/5}} \]  
\[ Nu = \frac{R^{2/5}}{2^{6/5}} \]

RESULTS AND DISCUSSION

As it is seen from the governing equations and the boundary conditions, the flow and the temperature fields for the present problem are functions of three parameters, namely \( R \), \( A \) and \( K \). In the preceding section a closed form solution was obtained by assuming that a parallel flow can describe well the basic flow features. The parallel flow approximation originates from the study of infinitely thin fluid and porous layers. For the case of a vertical rectangular porous layer, heated from the sides by a uniform heat flux, it was demonstrated by Vasseur et al. (1986) that the parallel flow approximation gives a reasonable prediction for the flow and heat transfer provided that the aspect ratio of the layer is made greater than 2. For the problem considered in this study a test was first made to determine the smallest \( A \) which could most economically provide results reasonably close to the large aspect ratio approximation. The results are indicated in Table I, which shows the Nusselt number \( Nu \) and maximum stream function \( \Psi_{\text{ext}} \) obtained numerically for the case \( R = 400 \), \( K = 1.5 \) and various \( A \). The analytical values predicted here (\( A \rightarrow \infty \)) are also given for reference. It can be observed that the results for \( A = 4 \) are sufficiently close to the analytical values. For this reason all the numerical results presented here were obtained for this particular aspect ratio. Thus in the limit of a tall shallow annulus the problem is governed by only two parameters \( R \) and \( K \). The effects of those parameters on the flow and temperature fields will now be discussed.

Figures 2(a) to (f) depict the streamlines (left) and isotherms (right) generated by the numerical scheme for some typical values of \( R \) and \( K \). The regions shown are the right-hand halves of vertical section, i.e. the axis of symmetry is to the left (see Figure 1). Figure 2(a) with \( R = 10 \), and \( K = 2 \) represents the results obtained for a pseudo-conduction regime. The isotherms are almost vertical and dense near the inner wall. The flow consists of an asymmetric single cell filling the entire cavity and rotating.
slowly in the clockwise direction (counter-clockwise in the left half of the cavity). At higher Rayleigh numbers the evolution of the flow structure can be observed from Figures 2(b) and (c). The development of a boundary layer regime near the inner wall, with an increasing $R$, is clearly illustrated by the steepness of the velocity and temperature profiles near that boundary. The temperature field is also seen to be strongly affected by the increase in buoyancy forces. The formation of a constant vertical thermal stratification in the core of the cavity is also observed. Effects of increasing curvature can be seen by comparing Figures 2(d), (e), (c) and (f) corresponding to a radius ratio of $K = 1.2, 1.5, 2$ and $2.5$ respectively.

From these last figures, it is seen that $\Psi_{\text{ext}}$ decreases with $K$. $\Psi_{\text{ext}}$ is a direct measure of the convective flow circulating within the cavity. While interpreting the streamline patterns, it must be recalled that in an annular geometry, the velocity is inversely proportional to the radial position in addition to be inversely proportional to the streamline spacing.

Figures 4(a) and 4(b) show respectively the velocity and temperature distributions across the annulus at elevation $z = 0$ at various dimensionless times $t$ for $R = 10^3$.
and $K = 1.5$. The numerical computation was started from the pure conduction temperature field Eq. (13), the saturating fluid being initially at rest. It is seen that the solution evolves toward a quasi steady state for which the dimensionless temperature involved in Eq. (11) becomes time independent. Such a steady state is always reached if enough time (dimensionless time of the order of unity) is allowed to the system.

Figures 5(a) and 6(a) display the axial velocity distribution at mid height for various values of $R$ and $K$ respectively. The corresponding temperature profiles are presented in Figures 5(b) and 6(b). The analytical results are continuous lines; numerical results shown as circles are seen to agree well. In all these graphs it was found convenient to renormalize the radial coordinate $r$ according to $r^* = r - (K - 1)^{-1}$. With this normalization $r^*$ varies between 0 and 1 for all values of the radius ratio $K$.

The velocity profiles of Figures 5(a) and 6(a) indicate that the magnitude of the axial velocity is considerably higher near the inner cylinder than the outer one. This point

![Graphs showing velocity and temperature distributions](image-url)

**FIGURE 5** a) Velocity profiles and b) temperature profiles for $K = 2$ and various Rayleigh numbers.
magnitude of the velocity near the outer wall becomes smaller with increasing $K$ in agreement with the results of Figure 2.

Figure 7 presents the Nusselt number $N_u$, as a function of $R$ for various values of $K$. The analytical (Eq. (26)) and numerical results are seen to agree well. At very low Rayleigh numbers (conduction regime), the Nusselt number approaches unity for all radius ratios $K$. This follows from the fact that $N_u$ has been normalized with respect to the pure conduction value, $1/\Delta T_c = [2(K - 1)(K^2 - 1)]/[(K^2 - 1) - 2K^2 \ln K]$. An increase in $R$ is naturally always associated with the decrease of temperature difference $\Delta T$. As indicated by Figure 7, $N_u$ depends on $K$. At a given $R$, higher the value of $K$, lower is $N_u$. For very large Rayleigh numbers, i.e. in the boundary layer regime, the asymptotic behaviour of the Nusselt number, as predicted by Eqs. (29) and (30), is represented in Figure 7 as dotted lines. As indicated earlier, the start of the boundary layer regime depends upon $K$. Higher the radius ratio, larger is the Rayleigh number required to reach the boundary layer regime. Thus, when $K = 1$, (i.e. for a rectangular

![Figure 7](image1.png)  
**FIGURE 7** Effects of Rayleigh number on the heat transfer rate for various aspect ratio $K$.

![Figure 8](image2.png)  
**FIGURE 8** Curvature effects on Nusselt number for various Rayleigh numbers.
cavity), the boundary layer regime is reached at \( R \approx 5 \times 10^2 \) while for \( K = 10 \) it is at \( R \approx 2 \times 10^5 \).

To confirm the curvature effects, the Nusselt number is plotted against \( K \) in Figure 8 for various Rayleigh numbers. For any fixed \( R \), the Nusselt number is seen to decrease as the radius ratio is made larger. This result follows from the fact that with increasing \( K \) the ratio of outer and inner surface areas is enhanced and the fluid volume associated with the strong temperature and velocity gradients near the inner boundary reduces. Elsewhere, i.e. away from the inner boundary, the fluid motion is weaker; thus the conduction plays a larger role in the heat transfer. A similar behavior has been reported by Havstad and Burns (1982) and Prasad and Kulacki (1984b) in their studies of the convective heat transfer within a vertical porous annulus whose vertical walls were at constant temperatures.

CONCLUSIONS

The problem of natural convection heat transfer in a vertical annular porous layer, where the inner wall is heated by a constant heat flux and the other walls are maintained adiabatic, was investigated analytically and numerically. At a sufficiently large time after heating, a quasi-steady state is reached, for which local temperature gradients, velocities and other parameters become very nearly independent of time. For quasi-steady state regime, the governing equations are solved analytically in the limit of a long shallow annulus. The solution gives the flow and temperature fields and the Nusselt number as a function of the Rayleigh number and radius ratio. The main conclusions of the present analysis are:

1° The parallel flow model derived in this study has been found to be quite accurate in predicting the flow structure and heat transfer for a wide range of the governing parameters. Numerically it has been demonstrated that the validity of the analytical solution depends essentially on the aspect ratio \( A \) of the cavity. In general the present theory is valid provided that \( A \) is made greater than approximately 4. However, for very large Rayleigh numbers, i.e. in the boundary layer regime, the flow was observed to remain parallel even for an aspect ratio of 2.

2° At low Rayleigh numbers the dependence of the Nusselt number of \( K \) cannot be separated from its dependence on \( R \). In the boundary layer regime the flow is concentrated along the inner wall to form a boundary layer regime with a constant thickness of the order \( \alpha^{-1} = (K + 1)^{1/5} R^{-2/5} \). The effects of curvature on the heat transfer in this flow regime are expressed by Eq. (29).

3° The main features of the approximate theoretical solution have been tested by a numerical solution of the full governing equations in the range \( 0 \leq R \leq 5 \times 10^4 \) and \( 1 \leq K \leq 3 \). A good agreement was found between the parallel flow approximation and the numerical simulation.

ACKNOWLEDGMENTS

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# NOMENCLATURE

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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</thead>
<tbody>
<tr>
<td>$A$</td>
<td>annulus aspect ratio, $H/(K - 1)$</td>
</tr>
<tr>
<td>$C$</td>
<td>dimensionless temperature gradient in $z$ direction</td>
</tr>
<tr>
<td>$D, E$</td>
<td>constants, Eqs. (20)-(21)</td>
</tr>
<tr>
<td>$F$</td>
<td>constant, $RS/\alpha^2$</td>
</tr>
<tr>
<td>$g$</td>
<td>gravitational acceleration</td>
</tr>
<tr>
<td>$H'$</td>
<td>height of the annulus</td>
</tr>
<tr>
<td>$H$</td>
<td>dimensionless height of the annulus, $H'/L'$</td>
</tr>
<tr>
<td>$I_n, K_n$</td>
<td>modified Bessel functions</td>
</tr>
<tr>
<td>$K$</td>
<td>radius ratio, $r_o'/r_i'$</td>
</tr>
<tr>
<td>$k$</td>
<td>thermal conductivity of the saturated porous medium</td>
</tr>
<tr>
<td>$L'$</td>
<td>gap width of porous annulus, $(r_o'/r_i')$</td>
</tr>
<tr>
<td>$Nu$</td>
<td>Nusselt number, Eq. (12)</td>
</tr>
<tr>
<td>$p'$</td>
<td>fluid pressure</td>
</tr>
<tr>
<td>$q'$</td>
<td>constant heat flux</td>
</tr>
<tr>
<td>$R$</td>
<td>Rayleigh number, $(g\beta q'L'^2)/(k\nu\alpha)$</td>
</tr>
<tr>
<td>$S$</td>
<td>constant, $2/(K + 1)$</td>
</tr>
<tr>
<td>$t'$</td>
<td>time</td>
</tr>
<tr>
<td>$T'$</td>
<td>temperature</td>
</tr>
<tr>
<td>$u'$</td>
<td>velocity component in $r'$-direction</td>
</tr>
<tr>
<td>$v'$</td>
<td>velocity component in $z'$-direction</td>
</tr>
</tbody>
</table>

## Greek Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>constant used in the analytical solution, $\sqrt{RC}$</td>
</tr>
<tr>
<td>$\alpha_f$</td>
<td>thermal diffusivity of fluid</td>
</tr>
<tr>
<td>$\beta$</td>
<td>thermal expansion coefficient</td>
</tr>
<tr>
<td>$\Delta T$</td>
<td>dimensionless temperature difference between vertical boundaries</td>
</tr>
<tr>
<td>$\theta$</td>
<td>$r$ dependent temperature term</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>permeability of the porous medium</td>
</tr>
<tr>
<td>$\mu$</td>
<td>dynamic viscosity of fluid</td>
</tr>
<tr>
<td>$\rho$</td>
<td>density of fluid</td>
</tr>
<tr>
<td>$(\rho c)_f$</td>
<td>heat capacity of fluid</td>
</tr>
<tr>
<td>$(\rho c)_p$</td>
<td>heat capacity of saturated porous medium</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>heat capacity ratio, $(\rho c)_p/(\rho c)_f$</td>
</tr>
<tr>
<td>$\Psi$</td>
<td>dimensionless stream function, $\Psi'/\alpha_f L'$</td>
</tr>
</tbody>
</table>

## Subscripts

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$c$</td>
<td>pure conduction</td>
</tr>
<tr>
<td>$f$</td>
<td>fluid</td>
</tr>
<tr>
<td>$i$</td>
<td>inner cylinder</td>
</tr>
<tr>
<td>$o$</td>
<td>outer cylinder</td>
</tr>
<tr>
<td>$p$</td>
<td>saturated porous medium</td>
</tr>
</tbody>
</table>
REFERENCES


