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Modified Homotopy Perturbation Method for Solving Generalized Linear Complex Differential Equations

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Abstract

In this work, we have studied the linear complex differential equations which are used as mathematical models in many physically significant fields and applied science. The homotopy perturbation method (HPM) has been modified for solving generalized linear complex differential equations. Also, we have tested the modified HPM for the solving of different implementations which are show the efficiency and accuracy of the proposed method. The approximated solutions are agree well with analytical solutions for the tested problems.

Keywords: Homotopy perturbation method; Complex Differential Equations; Approximated solutions

1 Introduction

The most important mathematical models for physical phenomena is the differential equation. The motion of objects, fluid and heat flow, bending and cracking of materials, vibrations, chemical reactions and nuclear reactions are

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all modeled by systems of differential equations. Moreover, numerous mathematical models in science and engineering are expressed in terms of unknown quantities and their derivatives. Many applications of differential equations (DEs), particularly ODEs of different orders, can be found in the mathematical modeling of real life problems [7]. The homotopy perturbation method (HPM) is an efficient technique to nd the approximate solutions for ordinary and partial differential equations which describe different fields of science, physical phenomena, engineering, mechanics, and so on. HPM was proposed by Ji-Huan He in 1999 for solving linear and nonlinear differential equations and integral equations. Many researchers used HPM to approximate the solutions of differential equations and integral equations [10], [11], [5] & [9]. Complex differential equations and their solutions play a major role in science and engineering. When a mathematical model is formulated for a physical problem, it is often represented by complex differential equations that cannot be solved explicitly by analytic techniques. In this case, it is often necessary to move to approximation and numerical methods to find solutions of such equations. For instance, the vibrations of a one-mass system with two degrees of freedom are mostly described using the differential equation with a complex dependent variable which is usually linear complex differential equation. The solution of the differential equation clarifies the linear phenomena which occur in the system. The modern physics has some applications that using differential equations, ordinary or partial, in the complex domain. Recently, we have studied a wide class of complex differential equations(CDEs), when the input and output or mostly just the dependent variables are in the complex domain, which is used as mathematical models in many physically scientific fields and applied science. The approximated solutions of this class of differential equations have studied using modded homotopy perturbation method (HPM). This study will reveal the significance and consolidation of the most powerful fields in mathematics, complex analysis and differential equations. CDEs have been used in some applications in engineering and physics. For example, a time-harmonic form of Maxwell's equations and Schrodinger equation are examples of CPDEs and the six Painleve equations (PI-PVI), Riccati equation and Schwarzian equation, all are examples of CODEs.

The most important mathematical models for physical phenomena is the differential equation. Motion of objects, Fluid and heat flow, bending and cracking of materials, vibrations, chemical reactions and nuclear reactions are all modeled by systems of differential equations. Moreover, Numerous mathematical models in science and engineering are expressed in terms of unknown quantities and their derivatives. Many applications of differential equations (DEs), particularly ODEs of different orders, can be found in the mathematical modeling of real life problems [7]. The homotopy perturbation method (HPM) is an efficient technique to find the approximate solutions for ordinary and partial differential equations which describe different fields of science, physical phenomena, engineering, mechanics, and so on. HPM was proposed by Ji-Huan He in 1999 for solving linear and nonlinear differential equations and integral equations. Many researchers used HPM to approximate the solutions of differential equations and integral equations [10], [11], [5] & [6].

Complex tools are widely used in mathematical physics. The solution of mathematical model of physical problem is often made simpler through the use of complex analysis. Another particularly important application of complex numbers is in the quantum mechanics where they play a central role representing the state, or wave function, of a quantum system. The modern physics has some applications using differential equations, ordinary or partial, in complex domain. Recently, we have studied a wide class of complex differential equations(CDEs), when the input and output variables are in complex domain, which is used as mathematical models in many physically significant fields and applied science. The approximated solutions of this class of differential equations have studied using modefided homotopy perturbation method (HPM). This study will reveal the significance and consolidation of the most powerful fields in mathematics, complex analysis and differential equations. CDEs have been used in some applications in engineering and physics. For example, Time-Harmonic form of Maxwell's equations and Schrodinger equation are examples of CPDEs and the six Painleve equations (PI-PVI), Riccati equation and Schwarzian equation are a examples of CODEs. The objective of this paper is the studying of the approximated solutions of CODEs using HPM.

2 Preliminary

2.1 High-Order Linear Complex Differential Equations

In this work, we will consider the following linear complex differential equations of order n:

$$\sum_{k=0}^{n} f_k(z)w^{(k)}(z) = f(z), \quad z, w(z) \in C, \tag{1}$$

with initial conditions

$$w^k(z_0) = \alpha_k, \tag{2}$$

for $k = 0, 1, 2, \ldots, n-1$.

Here $f_k(z)$ and $f(z)$ are functions in the complex domain and α_k are complex constants for all $k = 0, 1, 2, \dots, n-1$.

The implementation of proposed modified HPM for solving Linear CDEs (1) according the following algorithm.

3 Analysis of Homotopy Perturbation Method

To present a review of the homotopy perturbation method for solving the differential equations we will consider the following differential equations:

$$A(w) - f(z) = 0, \quad z \in \Omega, \tag{3}$$

with boundary conditions:

$$B(w, \frac{\partial w}{\partial z}) = 0, \quad z \in \partial\Omega, \tag{4}$$

where A is general differential operator, B is a boundary operator, $f(z)$ a known analytic function and $\partial\Omega$ is the boundary of the domain Ω. [8], [3], [2] & [1]

3.1 Homotopy Perturbation Method

In this section, we have present a brief description of the HPM, to illustrate the basic ideas of the homotopy perturbation method. The operator A in Equation (3) can be generally divided into two parts of L and N where L is linear part, while N is the nonlinear part in the DE, Therefore Equation (3) can be rewritten as follows [4] :

$$L(w) + N(w) - f(z) = 0. \tag{5}$$

By using homotopy technique, One can construct a homotopy

$$V(z, p) : \Omega \times [0, 1] \mapsto C$$

which satisfies:

$$H(v, p) = (1-p)[L(v) - L(w_0)] + p[L(v) + N(v) - f(z)] = 0, \tag{6}$$

or

$$H(v, p) = L(v) - L(w_0) + pL(w_0 + p[N(v) - f(z)]) = 0, \tag{7}$$

where $p \in [0, 1], \quad z \in \Omega \quad \& \quad p$ is called homotopy parameter and u_o is an initial approximation for the solution of equation (3) which satisfies the boundary conditions obviously, Using equation (6) or (7), we have the following equation:

$$H(v,0) = \quad L(v) - L(w_0) = 0 \tag{8}$$

and

$$H(v,1) = \quad L(v) + N(v) - f(z) = 0. \tag{9}$$

Assume that the solution of (6) or (7) can be expressed as a series in p as follows:

$$V = v_0 + pv_1 + p^2v_2 + p^3v_3 + \cdots = \sum_{i=0}^{\infty} p^i v_i \tag{10}$$

set $p \to 1$ results in the approximate solution of (3).

Consequently,

$$w(z) = \lim_{p\to 1} V = v_0 + v_2 + v_3 + \cdots = \sum_{i=0}^{\infty} v_i \tag{11}$$

It is worth to note that the major advantage of He's homotopy perturbation method is that the perturbation equation can be freely constructed in many ways and approximation can also be freely selected.

3.2 Modified Homotopy Perturbation Method

Firstly, we start with the initial approximation $w_0(z) = g(z)$.
Secondly, we can construct a homotopy for CDE (1) as follow:

$$H(w(z),p) = (1-p)(w^{(n)}(z) - w_0^{(n)}(z)) + p(\sum_{k=0}^{n} f_k(z)w^{(k)}(z) - f(z)) = 0. \tag{12}$$

Thirdly, using Taylor expansion about $z = 0$, by substituting the Taylor expansions for the coefficients functions. However,

$$H(w(t),p) = (1-p)(\sum_{i=0}^{\infty} p^i w_i^{(n)}(z) - w_0^{(n)}(z))$$

$$+p(\sum_{i=0}^{\infty}\sum_{k=0}^{n} p^i f_k(z)w_i^{(k)}(z) - f(z)) = 0. \tag{13}$$

Fourthly, suppose that the solution of Equation (13) is in the form

$$w(z) \quad = \quad w_0(z) + pw_1(z) + p^2 w_2(z) + p^3 w_3(z) + \ldots \tag{14}$$

Fifthly, collecting terms of the same power of p gives, as show in the following equations:

$$p^0 : \quad w_0^{(n)}(z) \quad - w_0^{(n)}(z) = 0, \tag{15}$$

$$p^1 : \quad w_1^{(k)}(z) \quad + \sum_{k=0}^{n} f_k(z) w_0^{(k)}(z) - f(z) = 0, \tag{16}$$

$$p^2 : \quad w_2^{(k)}(z) \quad + \sum_{k=0}^{n} f_k(z) w_1^{(k)}(z) = 0, \tag{17}$$

$$p^3 : \quad w_3^{(k)}(z) \quad + \sum_{k=0}^{n} f_k(z) w_2^{(k)}(z) = 0, \tag{18}$$

$$p^4 : \quad w_4^{(k)}(z) \quad + \sum_{k=0}^{n} f_k(z) w_3^{(k)}(z) = 0, \tag{19}$$

$$p^5 : \quad w_5^{(k)}(z) \quad + \sum_{k=0}^{n} f_k(z) w_4^{(k)}(z) = 0, \tag{20}$$

$$\ldots$$

Hence, for $m = 2, 3, 4, \ldots$ we have,

$$p^m : \qquad w_m^{(k)}(z) + \sum_{k=0}^{n} f_k(z) w_{m-1}^{(k)}(z) = 0. \tag{21}$$

$$\ldots$$

Finally, using the Equations (15)-(21) with some simplifications, we get the following terms of the solution:

$$w_0(z) = g(z), \tag{22}$$

$$w_1(z) = -\underbrace{\int\int\dots\int}_{n-times}(\sum_{k=0}^{n} f_k(z)w_0^{(k)}(z) - f(z))\underbrace{dzdz\dots dz}_{n-times}, \tag{23}$$

$$w_2(z) = -\underbrace{\int\int\dots\int}_{n-times}(\sum_{k=0}^{n-1} f_k(z)w_1^{(k)}(z))\underbrace{dzdz\dots dz}_{n-times}, \tag{24}$$

$$w_3(z) = -\underbrace{\int\int\dots\int}_{n-times}(\sum_{k=0}^{n-1} f_k(z)w_2^{(k)}(z))\underbrace{dzdz\dots dz}_{n-times}, \tag{25}$$

$$w_4(z) = -\underbrace{\int\int\dots\int}_{n-times}(\sum_{k=0}^{n-1} f_k(z)w_3^{(k)}(z))\underbrace{dzdz\dots dz}_{n-times}, \tag{26}$$

and

$$w_5(z) = -\underbrace{\int\int\dots\int}_{n-times}(\sum_{k=0}^{n-1} f_k(z)w_4^{(k)}(z))\underbrace{dzdz\dots dz}_{n-times}, \tag{27}$$

$$\dots$$

Hence, the general term has the following form:

$$w_m(z) = -\underbrace{\int\int\dots\int}_{n-times}(\sum_{k=0}^{n-1} f_k(z)w_{m-1}^{(k)}(z))\underbrace{dzdz\dots dz}_{n-times}, \quad m = 2, 3, 4, \dots \tag{28}$$

$$\dots$$

Then the solution of Equation(1) is

$$w(z) = w_0(z) + w_1(z) + w_2(z) + w_3(z) + w_4(z) + w_5(z) + \dots \tag{29}$$

4 Implementations

In order to assess the accuracy of the solving generalized linear CDEs by HPM we will introduce some different examples in general and to compare the approximated solution with the exact solutions for these problems, we will consider the following.

4.1 Problem 1

Consider the following CDE:

$$w'(z) - w(z) = 0, \quad w > 0, z \in C \tag{30}$$

subject to the initial condition

$$w(0) = 1,$$

with the exact solution

$$w(z) = e^z.$$

Comparing Equation(30) with Equation(1), we have $n = 1, \quad f_0(z) = -1, \quad f_1(z) = 1$,and $\quad f(z) = 0$
The initial approximation has the form $w_0(z) = 1$ substituting Equation (30) into Equations(22)-(28), we have

$$w_1(z) = z \tag{31}$$

$$w_2(z) = \frac{z^2}{2!} \tag{32}$$

$$w_3(z) = \frac{z^3}{3!} \tag{33}$$

$$w_4(z) = \frac{z^4}{4!} \tag{34}$$

Then, the general solution of Equation (30) is written as follow:

$$\begin{aligned} w(z) &= w_0(z) + w_1(z) + w_2(z) + w_3(z) + w_4(z) + \ldots \\ &= e^z \end{aligned} \tag{35}$$

4.2 Problem 2

Consider the following CDE:

$$w''(z) + w(z) = 0, \quad -1 < w > 1, z \in C \tag{36}$$

subject to the initial condition

$$w(0) = 0, \quad w'(0) = 1$$

with the exact solution

$$w(z) = \sin(z).$$

Comparing Equation(36) with Equation(1), we have $n = 2,\quad f_0(z) = 1,\quad f_1(z) = 0$
$, f_2(z) = 1\quad , and\quad f(z) = 0$
The initial approximation has the form $w_0(z) = z$ substituting Equation (36) into Equations(22)-(28), we have

$$w_1(z) = -\frac{z^3}{3!} \quad (37)$$
$$w_2(z) = \frac{z^5}{5!} \quad (38)$$
$$w_3(z) = -\frac{z^7}{7!} \quad (39)$$
$$w_4(z) = \frac{z^9}{9!} \quad (40)$$
$$\dots$$
$$w_m(z) = (-1)^m \frac{z^{2m+1}}{(2m+1)!} \quad (41)$$

Then, the general solution of equation (36) is written as follow:

$$\begin{aligned} w(z) &= w_0(z) + w_1(z) + w_2(z) + w_3(z) + w_4(z) + \dots \\ &= \sin(z) \end{aligned} \quad (42)$$

4.3 Problem 3

Consider the following Lienard CDE:

$$w''(z) - w(z) + z = 0, \quad z \in C \quad (43)$$

subject to the initial condition

$$w(0) = 2\alpha, \quad w'(0) = \beta$$

Where $\alpha = a_1 + ib_1,\quad \beta = a_2 + ib_2$, with the exact solution

$$w(z) = c_1 e^z + c_2 e^{-z} + z.$$

Comparing Equation(43) with Equation(1), we have $n = 2,\quad f_0(z) = -1,\quad f_1(z) = 0$,
$f_2(z) = 1\quad , and\quad f(z) = z$
The initial approximation has the form $w_0(z) = 2\alpha + \beta z$ substituting Equation

(43) into Equations(22)-(28), we have

$$w_1(z) = \frac{2\alpha z^2}{2!} + (\beta - 1)\frac{z^3}{3!}, \quad (44)$$

$$w_2(z) = \frac{2\alpha z^4}{4!} + (\beta - 1)\frac{z^5}{5!}, \quad (45)$$

$$w_3(z) = \frac{2\alpha z^6}{6!} + (\beta - 1)\frac{z^7}{7!}, \quad (46)$$

$$w_4(z) = \frac{2\alpha z^8}{8!} + (\beta - 1)\frac{z^9}{9!}, \quad (47)$$

$$\dots$$

$$w_m(z) = \frac{2\alpha z^{2m}}{2m!} + (\beta - 1)\frac{z^{2m+1}}{(2m+1)!}, \quad (48)$$

Suppose $\beta = 1$ then, the general solution of Equation (43) is written as follow:

$$\begin{aligned} w(z) &= w_0(z) + w_1(z) + w_2(z) + w_3(z) + w_4(z) + \dots \\ &= \alpha(e^z + e^{-z}) + z \end{aligned} \quad (49)$$

4.4 Problem 4

Consider the following CDE:

$$w'''(z) + iw(z) = 0, \quad z \in C \quad (50)$$

subject to the initial conditions

$$w(0) = 1, \quad w'(0) = i, \quad w''(0) = -1$$

with the exact solution

$$w(z) = ae^{ix}.$$

Comparing Equation(50) with Equation(1), we have $n = 3$, $f_0(z) = i$, $f_1(z) = 0$,
$f_2(z) = 0$, $f_3(z) = 1$, and $f(z) = 0$
From the initial conditions evince $z = ix$
Then, the initial approximation has the form $w_0(z) = 1 + ix - \frac{x^2}{2!}$ substituting

Equation (50) into Equations(22)-(28), we have

$$
\begin{aligned}
w_1(z) &= -i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!}, && (51)\\
w_2(z) &= -\frac{x^6}{6!} - i\frac{x^7}{7!} + \frac{x^8}{8!}, && (52)\\
w_3(z) &= i\frac{x^9}{9!} - \frac{x^{10}}{10!} - i\frac{x^{11}}{11!}, && (53)\\
w_4(z) &= \frac{x^{12}}{12!} + i\frac{x^{13}}{13!} - \frac{x^{14}}{14!}, && (54)\\
&\ldots
\end{aligned}
$$

Then, the general solution of Equation (50) is written as follow:

$$
\begin{aligned}
w(z) &= w_0(z) + w_1(z) + w_2(z) + w_3(z) + w_4(z) + \ldots\\
&= e^{ix} && (55)
\end{aligned}
$$

4.5 Problem 5

Consider the following CDE:

$$w'''(z) + 3w''(z) - 4w(z) = 0, \quad z \in C \tag{56}$$

subject to the initial condition

$$w(0) = a, \quad w'(0) = b, \quad w''(0) = c$$

Where $a, \quad b, \quad c \in C$, with the exact solution

$$w(z) = c_1 e^z + c_2 e^{-2z} + c_3 z e^{-2z}.$$

Comparing Equation(56) with Equation(1), we have $n = 2, \quad f_0(z) = -1, \quad f_1(z) = 0$,
$f_2(z) = 1$, *and* $f(z) = z$
The initial approximation has the form $w_0(z) = a + bz + c\frac{z^2}{2!}$ substituting

Equation (56) into Equations(22)-(28), we have

$$\begin{aligned} w_1(z) &= 4a\frac{z^3}{3!} + 4b\frac{z^4}{4!} + 4c\frac{z^5}{5!}, & (57) \\ w_2(z) &= 16a\frac{z^6}{6!} + 16b\frac{z^7}{7!} + 16c\frac{z^8}{8!}, & (58) \\ w_3(z) &= 64a\frac{z^9}{9!} + 64b\frac{z^{10}}{10!} + 64c\frac{z^{11}}{11!}, & (59) \\ w_4(z) &= 256a\frac{z^{12}}{12!} + 256b\frac{z^{13}}{13!} + 256c\frac{z^{14}}{14!}, & (60) \\ &\dots \\ w_m(z) &= 4^m(a\frac{z^{3m}}{3m!} + b\frac{z^{3m+1}}{(3m+1)!} + c\frac{z^{3m+2}}{(3m+2)!}), & (61) \end{aligned}$$

Suppose $\beta = 1$ then, the general solution of Equation (56) is written as follow:

$$\begin{aligned} w(z) &= w_0(z) + w_1(z) + w_2(z) + w_3(z) + w_4(z) + \dots \\ &= a + bz + c\frac{z^2}{2!} + 4a\frac{z^3}{3!} + 4b\frac{z^4}{4!} + 4c\frac{z^5}{5!} + 16a\frac{z^6}{6!} + 16b\frac{z^7}{7!} + 16c\frac{z^8}{8!} \\ w_3(z) &= 64a\frac{z^9}{9!} + 64b\frac{z^{10}}{10!} + 64c\frac{z^{11}}{11!} + 256a\frac{z^{12}}{12!} + 256b\frac{z^{13}}{13!} + 256c\frac{z^{14}}{14!} \\ &+ \dots + 4^m(a\frac{z^{3m}}{3m!} + b\frac{z^{3m+1}}{(3m+1)!} + c\frac{z^{3m+2}}{(3m+2)!}) + \dots \quad (62) \end{aligned}$$

5 Discussion and Conclusion

In this paper, the approximated solution of a class of CDEs has been studied. HPM has been modified for solving generalized linear CDEs. Also, we have tested the HPM on the solving of different implementations which are show the efficiency and accuracy of the proposed method. The approximated solutions are agree well with analytical solutions for the tested problems Moreover, the approximated solutions using the proposed method proved to be more accurate.

Conflict of Interests. The authors declare that there is no conflict of interests regarding the publication of this paper.

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