Hahn-Banach Theorem in Vector Spaces

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Abstract. In this paper we introduce a new extension to Hahn-Banach Theorem and consider its relation with the linear operators. At the end we give some applications of this theorem.

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1. Introduction

Huang and Zhang [2] introduced the notion of cone metric spaces and some fixed point theorems for contractive mappings were proved in these spaces. The results in [2] were generalized by Sh.Rezapour and R. Hamlbarani in [6]. Suppose that $\preceq$ is a partial order on a set $S$ and $A \subseteq S$. The greatest lower bound of $A$ is unique, if it exists. It is denoted by $\inf(A)$. Similarly, the least upper bound of $A$ is unique, if it exists, and is denoted by $\sup(A)$.

Let $E$ be a linear space and $P$ a subset of $E$. $P$ is called a cone if

(i) $P$ is closed, non-empty and $P \neq \{0\}$.
(ii) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers $a, b$.
(iii) $P \cap -P = \{0\}$.

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For a given cone \( P \subseteq E \), we can define a partial ordering \( \leq \) with respect to \( P \) by \( x \leq y \) if and only if \( y - x \in P \). Note that \( x < y \) will stand for \( x \leq y \) and \( x \neq y \), while \( x \ll y \) will stand for \( y - x \in \text{int}P \), where \( \text{int}P \) denotes the interior of \( P \).

\( P \) is called the normal cone of \( E \), if there is a number \( M > 0 \) such that for all \( x, y \in P \), \( 0 \leq x \leq y \) implies \( \|x\| \leq M\|y\| \).

The least positive number satisfying the above inequality is called the normal constant of \( P \).

2. Main Results

Hahn-Banach Theorem is one of the important theorems in analysis and many authors have investigated on this theorem and its applications ([2-6]).

In the sequel we assume that \((E, \|\cdot\|)\) is a Banach algebra that is ordered by a normal cone \( P \) with constant normal \( M=1 \), \( \text{int}P \neq \emptyset \) and \( \leq \) is partial ordering with respect to \( P \). We recall that a Banach algebra is a pair \((E, \|\cdot\|)\), where \( E \) is an algebra and \( \|\cdot\| \) is a complete norm such that \( \|xy\| \leq \|x\| \|y\| \).

Definition 2.1. Let \( X \) be a vector space and \( p \) be a map from vector space \( X \) into \( E \). We call that \( p \) is a sublinear map if \( p(tx)=tp(x) \) and \( p(x+y) \leq p(x) + p(y) \) whenever \( t > 0 \) and \( x, y \in X \).

Theorem 2.2. [Hahn- Banach Theorem] Let \( Y \) be a subspace of a vector space \( X \) and \( p : X \to E \) a sublinear map. If the linear map \( T_0 : Y \to E \) satisfies \( T_0(y) \leq p(y) \) for every \( y \in Y \), then there is a linear map \( T : X \to E \) such that \( T|_Y = T_0 \) and \( T(x) \leq p(x) \) whenever \( x \in X \).

Proof. Let \( x_1 \in X \setminus Y \) and \( Y_1 = Y \oplus \{x_1\} \). Note that each member of \( Y_1 \) can be expressed in the form \( y + tx_1 \), where \( y \in Y \) and \( t \) is a scalar, in exactly one way. For \( y_1, y_2 \in Y \),

\[
T_0(y_1) + T_0(y_2) = T_0(y_1 + y_2) \\
\leq p(y_1 - x_1 + y_2 + x_1) \\
\leq p(y_1 - x_1) + p(y_2 + x_1).
\]
Then
\[ \sup \{ T_0(y) - p(y - x_1) : y \in Y \} \leq \inf \{ p(y + x_1) - T_0(y) : y \in Y \} \]
and so for some \( t_1 \in E \)
\[ \sup \{ T_0(y) - p(y - x_1) : y \in Y \} \leq t_1 \leq \inf \{ p(y + x_1) - T_0(y) : y \in Y \}. \]

For any \( y \in Y \) and scalar \( t \), define \( T_1(y + tx_1) = T_0(y) + t.t_1 \). It is easy to check that \( T_1 \) is a linear map whose restriction to \( Y \) is \( T_0 \). Therefore
\[ T_1(y + tx_1) = t(T_0(t^{-1}y) + t_1) \leq tp(t^{-1}y + x_1) = p(y + tx_1) \]
and
\[ T_1(y - tx_1) = t(T_0(t^{-1}y) - t_1) \leq tp(t^{-1}y - x_1) = p(y - tx_1). \]

So \( T_1(x) \leq p(x) \) whenever \( x \in Y_1 \).

The second step of the proof is to show that the first step can be repeated until a linear map is obtained. It is dominated by \( p \) and its restriction to \( Y \) is \( T_0 \). Let \( \mathcal{U} \) be the collection of all linear maps \( G \) such that the domain of \( G \) is a subspace of \( X \) that includes \( Y \), the restriction of \( G \) to \( Y \) is \( T_0 \), and \( G \) dominated by \( p \). Define a preorder \( \preceq \) on \( \mathcal{U} \) by declaring that \( G_1 \preceq G_2 \) whenever \( G_1 \) is the restriction of \( G_2 \) to a subspace of the domain of \( G_2 \).

It is easy to see that each nonempty chain \( \mathcal{C} \) in \( \mathcal{U} \) has an upper bound in \( \mathcal{U} \). Consider the linear map whose domain is the union of the domains of the members of \( \mathcal{C} \) and which agrees at each point \( z \) of \( Z \) with every member of \( \mathcal{C} \) that is defined at \( z \). By Zorn’s lemma, the preorder set \( \mathcal{U} \) has a maximal element \( T \). The domain of \( T \) is all of \( X \). On the other hand with by applying the first step there is a \( T_1 \) in \( \mathcal{U} \) such that \( T \preceq T_1 \), but \( T_1 \not\preceq T \). This \( T \) satisfies all that is required. \( \square \)

**Proposition 2.3.** Let \( Y \) be a closed subspace of a linear normed space \( X \) and \( T_0 : Y \to E \) be an injective bounded linear map. Then there exists a bounded linear map \( T : X \to E \) such that \( \|T\| = \|T_0\| \) and \( T|_Y = T_0 \).

**Proof.** For every nonzero element \( x \in X \) define \( p(x) = \|T_0\| \|x\| \frac{1}{\|T_0(x)\|} \) and \( p(0) = 0 \). Since for every nonzero element \( x \in X \), we have
\[ \|T_0(x)\| \leq \|T_0\| \|x\| T_0(x). \]
and so \( T_0(x) \leq p(x) \). Now by Theorem 2.2., there exists a linear map 
\( T : X \rightarrow E \) such that \( T|_Y = T_0 \) and \( T(x) \leq p(x) \) whenever \( x \in X \). 
Since \( P \) is a normal cone with constant normal 1, \( \|T(x)\| \leq \|T_0\| \|x\| \) 
and \( \|T(x)\| \leq \|T_0\| \). Therefore \( \|T\| = \|T_0\| \). □

**Theorem 2.4.** Let \( X \) be a linear normed space and \( 0 \neq x \in X \). Then 
for every \( e \in S_E \) there is a linear map \( T_e : X \rightarrow E \) such that 
\( \|T_e\| = 1 \), \( T_e(x) = \|x\| e \), where \( S_E = \{x \in E : \|x\| = 1\} \).

**Proof.** Define \( G_e : \langle x \rangle \rightarrow E \) by \( G_e(\alpha x) = \alpha \|x\| e \) for every scalar \( \alpha \). 
Clearly \( G_e \) is injective, linear and \( G_e(x) = \|x\| e \). Also for \( \alpha \neq 0 \),
\[
\|G_e(\alpha x)\| = |\alpha| \|x\| = \|\alpha x\|.
\]
Since \( E \) is ordered by a normal cone \( P \) with constant normal \( M = 1 \), 
then \( \|G_e\| \leq 1 \). Also since,
\[
\|G_e\| \|x\| \geq \|G_e(x)\| = \|x\|,
\]
so \( \|G_e\| \geq 1 \). Hence \( \|G_e\| = 1 \). Let \( T_e \) be then Hahn-Banach extension 
of \( G_e \) from proposition 2.3, so the proof is complete. □

In the following we introduce immediate consequence of the above theorem.

**Corollary 2.5.** Let \( X \) be a linear normed space and \( x \neq y \in X \). Then 
there is a linear map \( T : X \rightarrow E \) such that \( Tx \neq Ty \).

**Corollary 2.6.** Let \( X \) be a linear normed space and \( x \in X \). Then
\[
\|x\| = \sup_{T \in \mathcal{B}} \|Tx\|,
\]
where \( \mathcal{B} = \{T : X \rightarrow E : T \) is a linear map and \( \|T\| = 1\} \).

**Proof.** By Theorem 2.4., there is a linear map \( T : X \rightarrow E \) such that 
\( \|T\| = 1 \), \( \|T(x)\| = \|x\| \). Then \( \|x\| = \|T(x)\| \leq \sup_{T \in \mathcal{B}} \|Tx\| \). On the 
other hand since \( \|T(x)\| \leq \|T\| \|x\| \), and so \( \sup_{T \in \mathcal{B}} \|Tx\| \leq \|x\| \). □
We recall that a point \( g_0 \in Y \) is said to be a best approximation for \( x \in X \) if and only if \( \|x - g_0\| = \|x + Y\| = d(x,Y) \). The set of all best approximations of \( x \in X \) in \( Y \) is shown by \( P_Y(x) \). In the other words,

\[
P_Y(x) = \{ g_0 \in Y : \|x - g_0\| = d(x,Y) \},
\]

If \( P_Y(x) \) is non-empty for every \( x \in X \), then \( Y \) is called a Proximinal set. The set \( Y \) is Chebyshev if \( P_Y(x) \) is a singleton set for every \( x \in X \) (see [2-6]).

Now we want to present some applications of new extension Hahn-Banach theorem in approximation theory.

**Proposition 2.7.** Let \( Y \) be a closed subspace of a linear normed space \( X \), and \( x \in X \setminus Y \). Then for every \( e \in S_E \) there is a linear map \( T_e : Y \oplus \langle x \rangle \to E \) such that \( \|T_e\| = 1 \), \( T_e x = d(x,Y)e, T_e|Y = 0 \).

**Proof.** Define \( T_e : Y \oplus \langle x \rangle \to E \) by \( T_e(y + \alpha x) = \alpha d(x,Y)e \) for every \( y \in Y \) and scalar \( \alpha \). It is clear that \( T_e \) is linear, \( T_e x = d(x,Y)e \) and \( T_e|Y = 0 \). For any \( y \in Y \) and scalar \( \alpha \neq 0 \),

\[
\|T_e(y + \alpha x)\| = |\alpha|d(x,Y) \leq \|y + \alpha x\|,
\]

so \( \|T_e\| \leq 1 \). Also since,

\[
\|T_e\| \|x - y\| \geq \|T_e(x - y)\| = d(x,Y) \quad y \in Y,
\]

so \( \|T_e\| \geq 1 \). Hence \( \|T_e\| = 1 \). \( \square \)

**Theorem 2.8.** Let \( Y \) be a closed subspace of a cone norm space \( X \). Suppose that \( x \in X \setminus Y \) and \( g_0 \in Y \). Then \( g_0 \in P_Y(x) \) iff for every \( e \in S_E \) there is a linear map \( T_e : Y \oplus \langle x \rangle \to E \) such that

\[
\|T_e\| = 1, \quad T_e(x - g_0) = \|x - g_0\|e, T_e|Y = 0.
\]

**Proof.** Assume \( g_0 \in P_Y(x) \). Since \( x \in X \setminus Y \), \( \|x - g_0\| = d(x,Y) \) and so by Proposition 2.7., there is a linear map \( T_e : Y \oplus \langle x \rangle \to E \) such that

\[
\|T_e\| = 1, \quad T_e(x - g_0) = \|x - g_0\|e, T_e|Y = 0.
\]
Conversely suppose there is a linear map \( T : Y \oplus \langle x \rangle \to E \) such that \( \|T_e\| = 1 \), \( T_e(x - g_0) = \|x - g_0\|e, T_e|_Y = 0 \). Then
\[
\|x - g_0\| = \|T_e(x - g_0)\| = \|T_e(x - g)\| \leq \|T_e\| \|x - g\| = \|x - g\|
\]
and so \( g_0 \in P_Y(x) \). □

**Corollary 2.9.** Suppose \( X \) is a normed linear spaces and \( x, y \in X \). Then \( x \perp y \) iff for every \( e \in S_E \) there is a linear map \( T_e : \langle y \rangle \oplus \langle x \rangle \to E \) such that \( \|T_e\| = 1 \), \( T_e(x) = \|x\|e, T_e(y) = 0 \).

It is clear that \( \ell_\infty \) is a Banach algebra and \( P = \{ \{x_n\} \in \ell_\infty : x_n \geq 0, \ for \ all \ n \} \) is a normal cone with constant normal \( M = 1 \). Also in [1] proved that for every linear map \( T_0 : Y \to \ell_\infty \) there is a linear map \( T : X \to \ell_\infty \) such that \( \|T\| = \|T_0\| \) and \( T|_Y = T_0 \). Consequently we have following result.

**Corollary 2.10.** Let \( Y \) be a closed subspace of a linear normed space \( X \), and \( x \in X \setminus Y \). Then \( M \subseteq P_Y(x) \) iff for every \( e \in S_{\ell_\infty} \), there is a linear map \( T : X \to \ell_\infty \) such that for every \( g \in M \)
\[
\|T_e\| = 1, \ T_e x = \|x - g\|e, T_e|_Y = 0.
\]

**References**


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