Optimizing production, shipment and inventory policies in a three-stage supply chain

Abstract
This paper considers an integrated production-inventory model for a three-stage supply chain involving multiple suppliers, multiple manufacturers and multiple retailers. The suppliers/manufacturers produce the raw materials/final goods at a finite rate and deliver the materials/goods in a number of batches to the manufacturers/retailers. Unlike existing models, we analyze the problem where the lead times from the manufacturers to the retailers are stochastic and shortage is allowed. We also explicitly include the transportation costs from the manufacturers to the retailers into the model. The numerical analysis shows that the coordination mechanism employed is more beneficial for the cases with less unpredictable lead times, lower shortage prices, and no transportation cost.

Keywords: Supply chain coordination; integrated production-inventory; stochastic lead time; joint economic lot sizing.
1. Introduction

The traditional inventory models assume that the policies are determined separately by a buyer or a vendor. This kind of decision making may not be suitable for today’s markets. Companies are realizing the necessity of coordination to have more efficient management of inventories across the supply chain (SC). Coordination mechanisms are either centralized or decentralized.

In the literature, a stream of research deals with centralized coordination in SCs referred to as joint economic lot sizing (JELS) or the integrated production-inventory problem. The present paper is related to this stream of research. The concept of joint optimization for buyer and vendor was first presented by Goyal (1976). He presented an integrated vendor-buyer model under the assumption of lot-for-lot policy for the shipments between SC members. Bannerjee (1986) considered the vendor delivering the whole batch to the buyer as a single shipment. Goyal (1988) then generalized that model by allowing a batch to be delivered as a number of equal-sized shipments. Goyal (1995) and Hill (1997) introduced models where the shipment size increases by a factor. Hill (1999) derived the structure of the optimal policy which includes shipments geometrically increasing in size followed by equal-sized shipments.

Many different directions have been added to the basic JELS model. Huang (2004) investigated the effect of quality on lot size. Hoque and Goyal (2006) developed an integrated inventory model with controllable lead time. Ertogral et al. (2007) added the transportation cost to the JELS models. Siajadi et al. (2006) introduced a methodology to obtain the lot size where multiple buyers demand one type of item from a single vendor. Recently, Sajadieh and Akbari (2009) added the pricing policy to the previous integrated vendor-buyer models. They considered the case in which the demand is price sensitive. Sajadieh et al. (2010) developed an integrated model where demand is dependent on the amount of items displayed on the shelf. Readers are referred to Glock (2012) and Ben-Daya et al. (2008) for reviews of the JELS models.

A recent trend encourages the integration of more than two SC stages. Munson and Rosenblatt (2001) developed a coordinating model in a three-stage SC with quantity discounts. Lee (2005) introduced an integrated inventory model where the raw material ordering from the supplier of the manufacturer is also taken into consideration. Pourakbar et al. (2007) then extended that model to a four-stage supply chain where backlogging is allowed. Khouja (2003) developed a three-stage SC model and considered three coordination
mechanisms between SC members, i.e., equal cycle time, integer multiplier, and integer power of two multipliers.

Cárdenas-Barrón (2007) extended Knouja’s model to an \( n \)-stage-multi-customer SC, considering the equal cycle time coordination mechanism. Recently, Ben-Daya and Al-Nassar (2008) generalized Knouja’s model by relaxing the assumption of starting shipment only after producing the whole lot. Jaber and Goyal (2008) extended this line of research by assuming a multiple suppliers, single vendor, and multiple buyers SC where the vendor could also be a supplier of some of the items. Chung and Wee (2007) algebraically developed a multi-stage inventory system with backorder considering JIT delivery. Recently, Yu et al. (2008) introduced a three-echelon supply chain for deteriorating items. The model developed considers a vertical integration of the producer, the distributor and the retailer and a horizontal integration of the producers.

As can be seen, coordination models in three-stage SCs are few. Moreover, almost all of them are developed under deterministic assumptions, which restrict their applicability. In this paper we intend to develop a stochastic JELS model for a three-stage SC including multiple suppliers, multiple manufacturers, and multiple retailers. A cost minimization model is derived for the problem where the lead times to retailers are stochastic. Moreover, the transportation cost is incorporated explicitly.

Following this section, the paper is organized as follows. Section 2 defines the problem and describes the notations and assumptions. Section 3 gives a discussion on the integrated production-inventory model. An exact algorithm is also developed in this section finding the optimal solution. Some numerical examples are presented in Section 4. Finally, the paper summaries and concludes in Section 5.

2. Problem definition and notation

The model assumes a three-stage SC including \( n_s \) suppliers, \( n_m \) manufacturers, and \( n_r \) retailers. The problem is to coordinate production and inventory decisions so that the total chain cost is minimized. A centralized decision-making process is assumed where system-wide optimization is acceptable to all members involved. Inventory is continuously reviewed by each retailer. The retailer \( j \) orders a lot \( Q_{r,j} \) every time its inventory position reaches the reorder point \( r_j \). Each retailer is assigned to one manufacturer. However, each manufacturer may support more than one retailer. The same structure is employed considering supplier-manufacturer relation (see Figure 1). The shipments from one stage to another can be delivered during and after production phase. In other words, there is no need to complete a
production lot before starting shipments. We deal with integer-multiplier coordination mechanism between chain members (e.g. Khouja, 2003; and Ben-Daya and Al-Nassar, 2008). In this known mechanism, the suppliers'/manufacturers’ cycle time $T_s/T_m$ is an integer $m_s/m_m$ of the manufacturers'/retailers’ cycle time $T_m/T$; i.e., $T_m=m_m T$ and $T_s=m_s T$.

Other assumptions are as follows:

1. The demand rates are deterministic and constant. Moreover, the total demand at each stage is equal, that is $\sum_{j=1}^{n_s} D_{s,j} = \sum_{j=1}^{n_m} D_{m,j} = \sum_{j=1}^{n_r} D_{r,j} / f$ where $D_{i,j}$ is the demand rate of firm $j$ at stage $i$, $j \in \{1, \ldots, n\}$ and $i \in \{s, m, r\}$; and $f$ is the conversion factor of raw materials to finish products.

2. The finite production rates for the manufacturers and the suppliers are considered. Moreover, the production rate of each firm is greater than the demand rate assigned, i.e., $P_{s,j} > D_{s,j}$ and $P_{m,j} > D_{m,j}$, where $P_{i,j}$ is the production rate of firm $j$ at stage $i$, $j \in \{1, \ldots, n\}$ and $i \in \{s, m\}$.

3. The lead times to replenish the retailers’ orders for finished products consist of fixed and variable components. The shipment time from the manufacturer to the retailer is assumed to be the only variable component of lead time. This component is assumed to be stochastic and to follow an exponential distribution, i.e., $L_j \sim \text{exp}(\lambda_j)$, $j=1, \ldots, n_r$. The fixed component of lead time has no effect on the optimization because the demand is deterministic. Thus, without loss of generality; we eliminate it from our analysis. Moreover, the lead times to replenish raw materials for manufacturers are constant.

4. Shortages are allowed at the retailers’ stage and are assumed to be backordered. By considering the supply chain as a whole, the shortage for the SC is equal to the final customer’s unfulfilled demand. On the other hand, the shortage cost at the upper stages can be considered as an internal transaction between supply-chain members. Thus, it has no effect on total system costs.

5. Time horizon is infinite.

The notation adopted for cost parameters are as follows:

$h_i$ Inventory holding cost per unit per unit time at stage $i$, $i \in \{s, m, r\}$

$A_i$ Ordering/setup cost at stage $i$, $i \in \{s, m, r\}$
3. Model formulation

The optimal policy of the system is derived in this section. The expected ordering, inventory holding, and shortage costs of retailer per unit time for the single-manufacturer single-retailer case under stochastic lead time were obtained by Sajadieh et al. (2009) as follows:

\[
RC1_j(r_j, Q_{r,j}) = \frac{D_{r,j} A_j}{Q_{r,j}} + h_j (r_j + \frac{Q_{r,j}}{2} - \frac{D_{r,j}}{\lambda_j}) + (\pi + h_j) D_{r,j} \left[ e^{-r_j Q_{r,j}/\lambda_j} - e^{-(r_j + Q_{r,j})/\lambda_j} \right] \left( \lambda_j Q_{r,j} \right)
\]

Rather than assuming the transportation cost from the manufacturer to the retailer \( j \) to be a part of the ordering cost, its cost is taken as a function of the lot size \( Q_{r,j} \). The transportation cost format is with \( w \) breaking points, \( 0=b_{j,0}<b_{j,1}<b_{j,2}<\ldots<b_{j,w} \). The corresponding transportation costs per unit are \( c_{j,0}>c_{j,1}>\ldots>c_{j,w} \). Each unit of the product is charged a transportation cost of \( c_{j,k} \) if the shipment quantity \( Q_{r,j} \) falls in interval \([b_{j,k}, b_{j,k+1})\). Specifically, we have

\[
c_j = \begin{cases} 
c_{j,0} & b_{j,0} \leq Q_{r,j} < b_{j,1} \\
c_{j,1} & b_{j,1} \leq Q_{r,j} < b_{j,2} \\
\vdots \\
c_{j,w-1} & b_{j,w-1} \leq Q_{r,j} < b_{j,w} \\
c_{j,w} & b_{j,w} \leq Q_{r,j}
\end{cases}
\]

The transportation conditions can be satisfied by the following constraints, where the order quantity for retailer \( j \) is to be in only one appropriate interval \( k \) of transportation pricing schedule.

\[-M (1 - z_{j,k}) + b_{j,k} \leq Q_{r,j} < b_{j,k+1} + M (1 - z_{j,k}) \quad j = 1, 2, \ldots, n_j \quad k = 0, 1, \ldots, w \]

\[\sum_{k=0}^{w} z_{j,k} = 1 \quad j = 1, 2, \ldots, n_j \]

where \( M \) and \( b_{j,w+1} \) are sufficiently large numbers. \( z_{j,k}=1 \) if the order quantity for retailer \( j \) falls in the shipment interval \( k \), otherwise \( z_{j,k}=0 \). Therefore, the transportation cost for retailer \( j \) per unit time is obtained as follows:

\[
RC2_j = D_{r,j} \sum_{k=0}^{w} c_{j,k} z_{j,k}
\]

As the cycle time for the retailers is the same, the order quantity \( Q_{r,j} \) can be replaced by \( D_{r,j} T \). Total cost of retailer \( j \) can then be summarized as:
As the retailers employ the inventory position to order the products, the manufacturers face no uncertainty as to the orders received. The manufacturer \( m \) produces \( T D_{m,j} \) product units at each setup and transfers them in \( m \) equal-sized shipments to the retailers assigned. The average inventory of finished products for manufacturer \( j \) is calculated by subtracting the average products sent to the retailers from the total system inventory (see Ben-Daya and Al-Nassar, 2008).

\[
AIM_1(j) = \frac{TD_{m,j}}{P_{m,j}} + \frac{(P_{m,j} - D_{m,j})_m TD_{m,j}}{2P_{m,j}} - \frac{TD_{m,j}}{2} \left( \frac{m_s - 1}{(1 - D_{m,j})_j + D_{m,j}} \right)
\]

Moreover, the raw materials are received by the manufacturers as an EOQ model. The average inventory of raw materials for manufacturer \( j \) can then be obtained as:

\[
AIM_2(j) = \frac{m_s J D_{m,j}^2}{2P_{m,j}}
\]

Hence, the total cost for manufacturer \( j \) is as follows:

\[
TCM_1(m_j) = \frac{A_j}{m_j T} + h_s \frac{m_s J D_{m,j}^2}{2P_{m,j}} + h_m \frac{TD_{m,j}}{2} \left( \frac{m_s - 1}{(1 - D_{m,j})_j + D_{m,j}} \right)
\]

The total cost for supplier \( j \) is obtained using the same approach

\[
TCM_2(s_j) = \frac{A_s}{m_s m_j T} + h_s \frac{m_s J D_{s,j}^2}{2P_{s,j}} + h_m \frac{TD_{s,j}}{2} \left( \frac{m_s - 1}{(1 - D_{s,j})_j + D_{s,j}} \right)
\]

The total SC cost can then be summarized as

\[
TCM_r_j = \sum_{j=1}^{n} TCM_r \sum_{i=1}^{n} TCM_m + \sum_{j=1}^{n} TCS_s
\]

\[
= \sum_{j=1}^{n} \left[ \frac{A_{r,j}}{T} \sum_{i=1}^{n} C_{j,i} \frac{K_{i,j,k}}{A_{r,j}} + h_{r,j} + D_{r,j} \frac{T}{2} \left( \frac{1}{\lambda_{r,j}} \right) + (\pi + h_{r,j}) D_{r,j} \frac{T}{\lambda_{r,j}^2} \left[ e^{-\lambda_{r,j} D_{r,j}} - e^{-\lambda_{r,j} (D_{r,j} + T)} \right] \right]
\]

\[
+ \sum_{j=1}^{n} \left[ \frac{A_{m,j}}{m_j m_T} + h_{m,j} \frac{m_s J D_{m,j}^2}{2P_{m,j}} + h_m \frac{TD_{m,j}}{2} \left( \frac{m_s - 1}{(1 - D_{m,j})_j + D_{m,j}} \right) \right]
\]

\[
+ \sum_{j=1}^{n} \left[ \frac{A_{s,j}}{m_s m_j T} + h_s \frac{m_s J D_{s,j}^2}{2P_{s,j}} + h_m \frac{TD_{s,j}}{2} \left( \frac{m_s - 1}{(1 - D_{s,j})_j + D_{s,j}} \right) \right]
\]
It can easily be shown that \( TSC \) is strictly convex in \( r_j \), \( j=1,\ldots, n \) for given values of all other variables. Taking the first partial derivative of Equation (4) with respect to \( r_j \), we have

\[
\frac{\partial TSC}{\partial r_j} = h_j - \frac{\pi + h_j}{\lambda_j T} \left[ e^{-\left(\pi + h_j\right) T} - e^{-\left(r_j/p_{s,j} + T\lambda_j\right)} \right] = 0
\]

Thus, we get

\[
r_j = \frac{D_{r,j}}{h_j} \ln \left[ \left( \pi + h_j \right)(1 - e^{-\lambda_j T}) / (h_j \lambda_j T) \right]
\]

As seen in (5), the optimal reorder points are independent of each other. We are then allowed to substitute (5) into (4) for all the reorder points. After simplification we have

\[
TSC(m_s, m_m, T, z_{1,0}, \ldots, z_{n,w}) = \sum_{j=1}^{n} \left( D_j \sum_{k=0}^{w} c_{j,k} z_{j,k} + h_j D_j \left[ \frac{T}{2} + \frac{1}{\lambda_j} \ln \left[ \left( \pi + h_j \right)(1 - e^{-\lambda_j T}) / (h_j \lambda_j T) \right] \right] \right) + \frac{T}{2} \sum_{j=1}^{n} D_{s,j} \left[ h_j m_s D_{s,j} / P_{s,j} + h_m \left( m_m - 1 \right) (1 - D_{s,j} / P_{s,j}) + D_{s,j} / P_{s,j} \right]
\]

\[
+ \frac{m_s A_s + n_s A_j / m_s + n_s A_j / (m_s m_m)}{T} + \frac{h_m T}{2} \sum_{j=1}^{n} D_{s,j} \left( m_s - 1 \right) (1 - D_{s,j} / P_{s,j}) + D_{s,j} / P_{s,j}
\]

**Proposition 1.** The cycle time at the retailers’ stage cannot be more than \( 2\pi/(\pi+h_j)\min(1/\lambda_j) \), i.e.

\[
T_{\text{max}} = \frac{2\pi}{(\pi+h_j)\min(1/\lambda_j)}
\]

**Proof.** All the reorder points are assumed to be positive, i.e. \( r_j \geq 0, \quad j=1,\ldots,n \). Therefore, based on equation (5), we have

\[
(\pi + h_j)(1 - \exp(-\lambda_j T)) \geq h_j \lambda_j T \quad \forall \ j=1,\ldots,n
\]

The Maclaurin series for \( \exp(-\lambda_j T) \) is as \( \sum_{i=0}^{\infty} (-\lambda_j T)^i / i! \). For simplicity reasons, however, we just consider its first three terms. The inequality will then be simplified as \( \lambda_j T \leq 2\pi/(\pi+h_j) \).

This inequality should be satisfied for all \( j=1,\ldots,n_r \). Hence, we can conclude that 
\[
T^{\text{max}} = 2\pi/(\pi + h_r) \min(1/\lambda_j).
\]

**Proposition 2.** For given values of \( m_m \) and \( m_s \), the optimal cycle time at the retailers’ stage cannot be less than \( T_0 \), where \( T_0 \) is the optimal cycle time assuming \( z_{j,k} = 0 \ \forall \ j,k \).

**Proof.** Equalizing all \( z_{j,k} \) to zero means that there is no transportation cost between the manufacturers and the retailers. As \( \text{TSC} \) is convex in \( T \) for given values of \( m_m \) and \( m_s \) (see Appendix), changing (increasing or decreasing) the retailers’ cycle time from \( T_0 \) will increase the total system cost, assuming that the transportation cost is excluded. Moreover, as \( c_{j,0} > c_{j,1} > \cdots > c_{j,w} \), decreasing the cycle time cannot decrease the transportation cost. Thus, \( \text{TSC}(T) > \text{TSC}(T_0) \), where \( T < T_0 \).

**Proposition 3.** If for the given values of \( m_m \) and \( m_s \), the optimal cycle time at the retailers’ stage is equal to \( T^{\text{max}} \), then \( T^{\text{max}} \) is also the optimal cycle time for the cases with \( m_m \leq m_m^0 \) and \( m_s \leq m_s^0 \).

**Proof.** The total system cost can be rewritten as 
\[
a_0 + a_1 G_1(T) + a_2 G_2(m_m T) + a_3 G_3(m_m m_s T),
\]
where \( a_0, a_1, a_2, \) and \( a_3 \) are the parameters independent of \( T, m_m, \) and \( m_s \), and \( G_i(x) \) is a function of \( x \). Suppose that the cost is minimized for given values of \( m_m \) and \( m_s \). Therefore, if we decrease \( m_m \) to \( m_m - 1 \) and keep \( m_s \) unchanged, then the cycle time that minimizes \( \phi_1 = a_0 + a_1 G_1(T) \) remains unchanged.

Assuming \( u = m_m T \), we can rewrite \( \phi_2 = a_2 G_2(m_m T) + a_3 G_3(m_m m_s T) \) as \( a_2 G_2(u) + a_3 G_3(u) \). Consider that \( u^* \) minimizes this cost function. So if we decrease \( m_m \) to \( m_m - 1 \) and keep \( m_s \) unchanged, the cycle time that minimizes \( \phi_2 \) increases to \( m_m T/(m_m - 1) \). As the total cost is the sum of \( \phi_1 \) and \( \phi_2 \), the cycle time that minimizes \( \phi_1 + \phi_2 \) increases when moving to \( m_m - 1 \). The above proof can be extended for the case in which \( m_s \) decreases to \( m_s - 1 \), and \( m_m \) remains unchanged.

Moreover, based on Proposition 1, the cycle time at the retailers’ stage cannot be more than \( T^{\text{max}} \). Thus, decreasing \( m_m \) and \( m_s \) form \( m_m^0 \) and \( m_s^0 \) do not increase the cycle time. Therefore, for any number of shipments less than or equal to \( m_m^0 \) and \( m_s^0 \), the optimal cycle time will be the maximum possible amount, i.e. \( T^{\text{max}} \).

Based on Proposition 3, the optimal value of \( T \) is inversely related to \( m_m \) and \( m_s \). That is intuitively true, because as the number of shipments increases, the production batch at the
manufacturers and the suppliers increases. Therefore, the cycle time and consequently the
order quantity should decrease to prevent huge amounts of production batches at the
manufacturers and the suppliers. In other words, as the number of shipments increases, the
production batch is split into a larger number of smaller shipment quantities.

**Proposition 4.** Integer multipliers for the cycle time at the manufacturers and at the suppliers
stages are finite. In other words, there are upper bounds for \(m_m\) and \(m_s\).

**Proof.** Taking the second partial derivative of \(TSC\) with respect to \(m_m\) and \(m_s\), we have

\[
\frac{\partial^2 TSC}{\partial m_m^2} = \frac{2(n_m A_m + n_s A_s / m_s)}{m_m T} > 0
\]

\[
\frac{\partial^2 TSC}{\partial m_s^2} = \frac{2n_s A_s}{m_s m_s T} > 0
\]

Hence, \(TSC\) is convex in \(m_m\) and \(m_s\) respectively for given values of other variables.

Equalizing to zero the first partial derivative of \(TSC\) with respect to \(m_m\) and \(m_s\), we obtain

\[
m_m^2 = \frac{2(n_m A_m + n_s A_s / m_s)}{[\theta_1 + h_1 \sum_{j=1}^{n_s} D_{s,j} (D_{s,j} / P_{s,j}) + (m_s - 1)(1 - D_{s,j} / P_{s,j}))]}T^2
\]

\[
m_s^2 = \frac{2n_s A_s}{\theta_2 m_m^2 T^2}
\]

Where \(\theta_1 = h_1 \sum_{j=1}^{n_s} D_{s,j} (D_{s,j} / P_{s,j}) + h_m \sum_{j=1}^{n_m} D_{m,j} (1 - D_{m,j} / P_{m,j})\) and

\[
\theta_2 = h_1 \sum_{j=1}^{n_s} D_{s,j} (1 - D_{s,j} / P_{s,j}).
\]

As can be seen, \(\theta_1\) and \(\theta_2\) are positive variables and independent of \(m_m\) and \(m_s\). Thus,
there is a inverse relation between \(m_s\) and the optimal value of \(m_m\), as well as between \(m_m\) and
the optimal value of \(m_s\). The upper bound for the optimal value of \(m_m\) and \(m_s\) is then obtained
at \(m_s = 1\) and \(m_m = 1\), respectively. Moreover, the optimal values of \(m_m\) and \(m_s\) are inversely
related to \(T\). The retailers’ cycle time is equal to \(T = Q_r / D_{r,j}\). Although the order quantities are
theoretically considered to be greater than zero, the order quantity in practice is not less than
one. In other words, a smallest unit of product can always be defined that the contracts
between retailer and manufacturer are based on. The smallest value for \(T\) can then be
obtained as \(T_{\min} = \min\{1 / D_{r,j}\}\).

Therefore, the upper bounds for the optimal values of \(m_m\) and \(m_s\) are as follows
\[ UB_m = \max\left\{ 1, \frac{\sqrt{2(n_mA_m + n_RA_A)}}{(\theta_1 + h_1 \sum_{j=1}^{n_m} D_{r,j}/P_{n,j})^{1/2} T^{\min}} \right\} \] (7)

\[ UB_s = \max\left\{ 1, \frac{\sqrt{2n_A}}{\theta_2^{1/2} T^{\min}} \right\} \] (8)

3.1. Solution Algorithm

The algorithm suggested is based on the above discussions. It runs below the upper bounds of \( m_s \) and \( m_m \). Following the algorithm’s steps, the globally optimal values of all decision variables are obtained. Propositions 1, 2, 3, and 4 are used in Steps 2, 4 and 5; Steps 2, 3 and 5; Step 8; and Steps 1, 9 and 10, respectively.

Step 0. Set \( TSC^{\text{opt}} \) to an arbitrarily large number.

Step 1. Calculate \( T^{\max} \), \( UB_m \), \( UB_s \) using Equations (6), (7) and (8) respectively. Set \( m_m = UB_m \), \( m_s = UB_s \), \( LB_m = 1 \).

Step 2. Find the optimal value of \( T \) that minimizes \( TSC \), assuming \( z_{j,k} = 0 \ \forall j, k \). Set that value to \( T_0 \). If \( T_0 > T^{\max} \), then \( T_0 = T^{\max} \).

Step 3. If \( T_0 \geq b_{j,w}/D_{r,j} \ \forall j \), then go to Step 7.

Step 4. Find corresponding \( k \) and \( l \) for each retailer where \( T_0 D_{r,j} \in [b_{j,k}, b_{j,k+1}] \) and \( T^{\max}_0 D_{r,j} \in [b_{j,l}, b_{j,l+1}] \).

Step 5. Calculate \( TSC(m_s, m_m, T_0) \), \( TSC(m_s, m_m, b_{j,k+1}/D_{r,j}) \), \ldots, \( TSC(m_s, m_m, b_{j,l}/D_{r,j}) \) for all retailers. The minimum \( TSC \) is the optimal solution for given \( m_m \) and \( m_s \).

Step 6. Compute the values of \( r_j \) using Equation (5).

Step 7. If \( TSC < TSC^{\text{opt}} \), then set \( TSC^{\text{opt}} = TSC \), \( T^{\text{opt}} = T \), \( m_m^{\text{opt}} = m_m \), \( m_s^{\text{opt}} = m_s \), \( r_j^{\text{opt}} = r_j \ \forall j \), and \( z_{j,k}^{\text{opt}} = z_{j,k} \ \forall j, k \).

Step 8. If \( T = T^{\max} \), then \( LB_m = m_m \).

Step 9. Decline \( m_m \) by 1. If \( m_m \geq LB_m \), then go to Step 2.

Step 10. Decline \( m_s \) by 1. If \( m_s \geq 1 \), then \( m_s = UB_m \), and go to Step 2.

3.2. Non-coordinated supply chain

As opposed to the integrated model where there is a central decision maker, we also obtain the optimal policies if each facility is managed separately. In other words, in non-coordinated cases the supply chain members do not share their information. We assume that first the retailers make the ordering and shipment decisions; then the manufacturers find their optimal
production policies, and finally the suppliers find theirs.

If there is no coordination, each retailer first finds \( r_j, T_j, \) and \( z_{j,k} \) where \( TCR_j \) is minimized. The optimal values of decision variables can be easily found employing Steps 3-8 modified by the algorithm in Section 3.1. We assume that in non-coordinated cases, each manufacturer manages each of its retailers separately. The same scheme is employed considering suppliers-manufacturers relation. Therefore, the manufacturer \( i \) should find an integer multiple \( m_{m,j} \) of cycle time of retailer \( j \) where \( TCM_{i,j} \) is minimized (as each retailer is assigned to just one manufacturer, there is no need to add index \( i \) to \( m_{m,j} \)).

\[
TCM_{i,j}(m_{m,j}) = \frac{A_{m,i}}{m_{m,j}T_j} + h_s \frac{m_{m,j}T_j fD_{r,j}^2}{2P_{m,i}} + h_m \frac{T_j D_{r,j}}{2} \left( (m_{m,j} - 1)(1 - \frac{D_{r,j}}{P_{m,i}}) + \frac{D_{r,j}}{P_{m,i}} \right)
\]

The optimal value of \( m_{m,j} \) is obtained using the following conditions

\[
m_{m,j}^*(m_{m,j}^* - 1) \leq \frac{2A_{m,i}/T_j^2}{h_s fD_{r,j}^2/P_{m,i} + h_m D_{r,j}(1 - D_{r,j}/P_{m,i})} \leq m_{m,j}^*(m_{m,j}^* + 1)
\]  (9)

We assume that the manufacturer \( i \) orders raw materials based on the average of its production batch for different retailers, i.e.

\[
T_{m,i} = \frac{\sum_{j=1}^{n} I_{i,j} m_{m,j} T_{m,j} D_{r,j}}{\sum_{j=1}^{n} I_{i,j} D_{r,j}}
\]

where \( I_{i,j} \) is 1 if retailer \( j \) is assigned to manufacturer \( i \), 0 otherwise. The supplier \( k \) then finds an integer multiple \( m_{s,i} \) of cycle time of manufacturer \( i \) where \( TCS_{k,i} \) is minimized.

\[
TCS_{k,i}(m_{s,i}) = \frac{A_i}{m_{s,i}T_{m,i}} + h_s \frac{fT_{m,i}D_{m,i}}{2} \left( (m_{s,i} - 1)(1 - \frac{fD_{m,i}}{P_{s,k}}) + \frac{fD_{m,i}}{P_{s,k}} \right)
\]

The optimal value of \( m_{s,i} \) for given \( T_{m,i} \) is obtained using the following conditions

\[
m_{s,i}^*(m_{s,i}^* - 1) \leq \frac{2A_i}{h_s fT_{m,i}^2D_{m,i}(1 - fD_{m,i}/P_{s,k})} \leq m_{s,i}^*(m_{s,i}^* + 1)
\]  (10)

The expected total system cost of the non-coordinated model \( TSC_{NC} \) is equal to the sum of the SC members expected costs.

4. Numerical study

Consider a supply chain including seven retailers, three manufacturers, and one supplier. Table 1 shows the relevant data which is the same as used by Khouja (2003) for common parameters. The other values (shortage cost and lead time parameter) will be analyzed for a wide range. Assume that manufacturer 1, 2, and 3 supply retailers 1-3; 4-5; and 6-7, respectively.
In order to concentrate on shortage cost and lead-time variability, first assume that there is no transportation cost, i.e. \( w=0 \), and \( c_{j,p}=0 \). Table 2 summarizes the results of this example. The expected total costs for coordinated and non-integrated supply chains are \( TSC=103,682.1 \), and \( TSC_{NC}=116,419.2 \), respectively.

The effects of lead-time variability as well as shortage costs are analyzed, using the models in Section 3. For this purpose, two sets of parameter specifications are considered; ten levels for lead-times parameter: \( \lambda \in [0.05, 0.1, 1, 5, 10, 15, 20, 25, 30, 35] \), and three levels for shortage costs: \( \pi \in [20, 40, 60] \). The saving percentage obtained by coordination is defined as \( PS=(TSC_{NC}-TSC)/TSC_{NC} \times 100 \). The coordination benefit should be shared by all members in equitable fashion. A fair method is then necessary for allocating profit in order to persuade the retailers to cooperate with the manufacturers and suppliers. One way is to allocate the joint total cost to each member as follows (see Wu and Ouyang, 2003; and Ouyang et al., 2004)

\[
TCR_j^{C} = TCR_j/TSC_{NC}, \quad TCM_j^{C} = TCM_j/TSC_{NC}, \quad \text{and} \quad TCS_j^{C} = TCS_j/TSC_{NC}
\]

where \( TCR_j^{C} \), \( TCM_j^{C} \), and \( TCS_j^{C} \) are the cost of retailer, manufacturer, and supplier \( j \) under coordination, respectively.

Figure 2 illustrates that the saving percentage in total cost obtained by coordination increases by lead-time parameter. Intuitively, one might expect that the coordination is more beneficial for environments with high variable lead time. However, the numerical results show that \( PS \) decreases as lead-time standard deviation \( (1/\lambda) \) increases. In fact, coordination is not generally recommended when the retailers’ lead times are too unpredictable. The reason is that under the non-coordinated supply chain the retailers are more flexible and can select their policy freely, while they have to observe the same ordering cycle time under coordination. The importance of lead time can be better understood when we see that increasing \( \lambda \) from 0.05 to 35 raises \( PS \) from 0% to 15.4% on average.

From Figure 2 we also see that the shortage cost decreases the benefits of coordination. This is due to the fact that the coordination is advantageous for upstream stages, i.e. manufacturers and suppliers, while it increases the costs at the retailers’ stage. As shortage
cost increases, the cost saving at downstream stage becomes more important. Therefore the coordination which boosts the costs at the retailers’ stage becomes less attractive. Increasing the shortage cost from 20 to 40 and from 40 to 60, decreases PS by 1.39% and 0.65% on average, respectively. Moreover, the decrease in PS when moving from one shortage cost to another remains almost unchanged for a wide range of \( \lambda \).

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Figure 2
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Now assume that the following transportation cost structure is employed:

<table>
<thead>
<tr>
<th>Range</th>
<th>Unit transportation cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0 \leq Q_{r,j} &lt; 1000 )</td>
<td>0.4</td>
</tr>
<tr>
<td>( 1000 \leq Q_{r,j} &lt; 4000 )</td>
<td>0.25</td>
</tr>
<tr>
<td>( 4000 \leq Q_{r,j} )</td>
<td>0.17</td>
</tr>
</tbody>
</table>

Figure 3 shows the savings percentage when the transportation cost is included. Comparing Figures 2 and 3 shows that the coordination is even less attractive when there is a transportation cost between manufacturers and retailers. The reason is that in the non-coordinated model, each retailer can employ higher discount levels, whereas in the coordinated case, any increase in one of the retailer’s order quantity will also increase the order quantity of all the other retailers. The results show that PS is almost half when the transportation cost is included.

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Figure 3
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5. Conclusions

Centralized coordination models in three-stage SCs are not only few, but also limited to deterministic conditions. The present paper is a contribution along this line of research. It introduces an integrated production-inventory model to minimize the total cost of a three-stage supply chain. Unlike the models found in the literature, the presented JELS model considers a general multiple-supplier, multiple-manufacturer, multiple-retailer SC where the retailers’ lead times are stochastic, shortage is allowed, and the transportation costs are included. We derived the expected cost functions and proposed an analytic solution algorithm to find the globally optimal policy.

In this paper we demonstrated that although coordination between supply chain members
is beneficial, there are also some disadvantages especially when considering stochastic lead time, shortage, and transportation costs. In rare situations these disadvantages may even outweigh the benefits of coordination. This is due to the fact that forcing all retailers to observe the same ordering cycle time reduces the flexibility of the supply chain at its downstream stage. In other words, in the coordinated supply chains, the retailers cannot easily change their policy. The reason is that any change in the cycle time of one of retailers affects the costs of all retailers, whereas in the non-coordinated case each retailer is free to optimize its own cost.

Considering the transportation cost can also reduce the benefit of coordination. That is the result of inflexibility as well. For example, if a retailer in a coordinated supply chain wants to increase its order quantity to use discount opportunity in transportation costs, this increase will enlarge the order quantity of all the other retailers. Although the coordination acts as a constraint for the retailers, its benefit is obtained at the upstream stages, i.e., manufacturers and suppliers. If the savings in the costs of manufacturers and suppliers outweigh the extra inventory, shortage and transportation costs at the retailers, then coordination is proposed. Otherwise it may be better to stop coordinating. However, based on numerical research in this paper, coordination benefit offsets the detriment for a very wide range of parameters.

Further research can be extended to consider other stochastic parameters such as demand and/or production rate. The models presented in this article are limited in the sense that each retailer can obtain materials from only one manufacturer. More complex structures can be studied where the orders are split between the manufacturers. Another extension is to consider other distributions for lead times. Considering other coordination mechanisms such as the powers of two multipliers mechanism is also proposed for future research.

Appendix

The second partial derivative of $TSC$ with respect to $T$ is as follows:

\[
\frac{\partial^2 TSC}{\partial T^2} = \sum_{j=1}^{n} h_j D_{s,j} (1-e^{-\lambda_j T})^2 - h_j D_{s,j} \lambda_j^2 T^2 e^{-\lambda_j T} \frac{m_j m_j}{m_j m_j} + 2(n_j A_j + n_j A_j + n_j A_j) \frac{1}{m_j m_j} T^3
\]

The above expression can be rewritten as

\[
\frac{\partial^2 TSC}{\partial T^2} = \sum_{j=1}^{n} \beta_1 (1-e^{-\lambda_j T})^2 - \lambda_j^2 T^2 e^{-\lambda_j T} + \beta_2
\]

where $\beta_1 = h_j D_{s,j} (1-\lambda_j T^2 - \lambda_j T^2)$, and $\beta_2 = 2(n_j m_j A_j + n_j A_j + n_j A_j) / m_j m_j T^3$.
It can be seen that $\beta_1$ and $\beta_2$ are positive. We then need to show the positivity of $\gamma = \left(1 - e^{-\lambda_j T}\right)^2 - \lambda_j^2 T^2 e^{-\lambda_j T}$ to conclude the convexity. Assuming $x = \exp(-\lambda_j T)$, we update $\gamma$ as $\gamma = \left(1-x\right)^2 - 2x \ln x$. As $\lambda_j T > 0$, $x$ can vary between 0 and 1. Moreover, $\ln x$ is negative for $0 \leq x \leq 1$. Therefore, $\gamma$ is positive, and consequently $TSC$ is convex in $T$. However, no closed form solution exists for optimal value of $T$. Thus, we use one dimensional search algorithms (e.g. Fibonacci, and Newton), which are employed by optimization software, to find optimal value of $T$. As $TSC$ is convex in $T$, the solution found is the global optimum.

**References**


