INFINITE MATRICES, WAVELET COEFFICIENTS AND FRAMES

N. A. SHEIKH and M. MURSALEEN

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We study the action of $A$ on $f \in L^2(\mathbb{R})$ and on its wavelet coefficients, where $A = (a_{lmjk})_{lmjk}$ is a double infinite matrix. We find the frame condition for $A$-transform of $f \in L^2(\mathbb{R})$ whose wavelet series expansion is known.

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1. Introduction. The notation of frame goes back to Duffin and Schaeffer [7] in the early 1950s to deal with the problems in nonharmonic Fourier series. There has been renewed interest in the subject related to its role in wavelet theory. For a glance of the recent development and work on frames and related topics, see [3, 4, 5, 6, 9]. In this note, we will use the regular double infinite matrices (see [9, 10]) to obtain the frame conditions and wavelet coefficients.

2. Notations and known results. $\mathbb{N}$ is the set of positive integers, $\mathbb{Z}$ is the set of integers, $\mathbb{R}$ is the set of real numbers. The space $L^2(\mathbb{R})$ of measurable function $f$ is defined on the real line $\mathbb{R}$, that satisfies

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty. \tag{2.1}$$

The inner product of two square integrable functions $f, g \in L^2(\mathbb{R})$ is defined as

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)} \, dx, \quad \|f\|^2 = \langle f, f \rangle^{1/2}. \tag{2.2}$$

Every function $f \in L^2(\mathbb{R})$ can be written as

$$f(x) = \sum_{j,k \in \mathbb{Z}} C_{j,k} \psi_{j,k}(x). \tag{2.3}$$

This series representation of $f$ is called wavelet series. Analogous to the notation of Fourier coefficients, the wavelet coefficients $C_{j,k}$ are given by

$$C_{j,k} = \int_{-\infty}^{\infty} f(x) \overline{\psi_{j,k}(x)} \, dx = \langle f, \psi_{j,k} \rangle, \quad \psi_{j,k} = 2^{j/2} \varphi(2^j x - k). \tag{2.4}$$
Now, if we define an integral transform
\[
(W_{\psi}f)(b,a) = |a|^{-1/2} \int_{-\infty}^{\infty} f(x) \psi \left( \frac{x-b}{a} \right) dx, \quad f \in L^2(\mathbb{R}),
\] (2.5)
then the wavelet coefficients become
\[
C_{j,k} = (W_{\psi}f) \left( \frac{k}{2^j}, \frac{1}{2^j} \right),
\] (2.6)

A sequence \( \{x_n\} \) in a Hilbert space \( H \) is a frame if there exist constants \( c_1 \) and \( c_2 \), \( 0 < c_1 \leq c_2 < \infty \), such that
\[
c_1 \|f\|^2 \leq \sum_{n \in \mathbb{Z}} |\langle f, x_n \rangle|^2 \leq c_2 \|f\|^2,
\] (2.7)
for all \( f \in H \). The supremum of all such numbers \( c_1 \) and infimum of all such numbers \( c_2 \) are called the frame bounds of the frame. The frame is called tight frame when \( c_1 = c_2 \) and is called normalized tight frame when \( c_1 = c_2 = 1 \). Any orthonormal basis in a Hilbert space \( H \) is a normalized tight frame. The connection between frames and numerically stable reconstruction from discretized wavelet was pointed out by Grossmann et al. [8]. In 1985, they defined that a wavelet function \( \psi \in L^2(\mathbb{R}) \), constitutes a frame with frame bounds \( c_1 \) and \( c_2 \), if any \( f \in L^2(\mathbb{R}) \) such that
\[
c_1 \|f\|^2 \leq \sum_{j,k \in \mathbb{Z}} |\langle f, \psi_{j,k} \rangle|^2 \leq c_2 \|f\|^2.
\] (2.8)
Again, it is said to be tight if \( c_1 = c_2 \) and is said to be exact if it ceases to be frame by removing any of its elements. There are many examples proposed by Daubechies et al. [6]. For further details, one can refer to [1, 5, 6]. Chui and Shi [2] proved that \( \{\psi_{j,k}\} \) is a frame for \( L^2(\mathbb{R}) \) with bounds \( c_1 \) and \( c_2 \), if for some \( a > 1 \) and \( b > 0 \), the Fourier transform \( \hat{\psi} \) satisfies
\[
c_1 \leq \frac{1}{b} \sum_{j \in \mathbb{Z}} |\hat{\psi}(a^j w)|^2 \leq c_2 \text{ a.e.},
\] (2.9)
for some constants \( c_1 \) and \( c_2 \). By integrating each term in
\[
\frac{c_1}{|w|} \leq \frac{1}{b} \sum_{j \in \mathbb{Z}} \frac{|\hat{\psi}(a^j w)|^2}{|w|} \leq \frac{c_2}{|w|}
\] (2.10)
over \( 1 \leq |w| \leq a \), we have
\[
2c_1 \log a \leq \frac{1}{b} \sum_{j \in \mathbb{Z}} \int_{1 \leq |w| \leq a} \frac{|\hat{\psi}(a^j w)|^2}{|w|} dw \leq 2c_2 \log a,
\] (2.11)
which immediately yields
\[
c_1 \leq \frac{1}{2b \log a} \int_{-\infty}^{\infty} \frac{|\hat{\psi}(a^j w)|^2}{|w|} dw \leq c_2.
\] (2.12)
The above condition known as compactibility condition was also observed by Daubechies [4] by using techniques from trace class operators. The above constants were given by frame bounds, see [2].

Let $A = (a_{mn})$ be a double infinite matrix of real numbers. Then, $A$-transform of a double sequence $x = (x_{jk})$ is

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{mnjk}x_{jk},$$

which is called $A$-means or $A$-transform of the sequence $x = (x_{ij})$. This definition is due to Móricz and Rhoades [9].

A double matrix $A = (a_{mn})$ is said to be regular (see [10]) if the following conditions hold:

(i) $\lim_{m,n \to \infty} \sum_{j,k=0}^{\infty} a_{mnjk} = 1$,

(ii) $\lim_{m,n \to \infty} \sum_{j=0}^{\infty} |a_{mnjk}| = 0, \ (k = 0, 1, 2, \ldots)$,

(iii) $\lim_{m,n \to \infty} \sum_{k=0}^{\infty} |a_{mnjk}| = 0, \ (j = 0, 1, 2, \ldots)$,

(iv) $\|A\| = \sup_{m,n \geq 0} \sum_{j,k=0}^{\infty} |a_{mn}| < \infty$.

Either of conditions (ii) and (iii) implies that

$$\lim_{m,n \to \infty} a_{mnjk} = 0.$$  \hfill (2.14)

In this note, we establish the frame condition by using $A$-transform of nonnegative regular matrix, also we find action of the matrix $A$ on wavelet coefficients.

3. Main results. In this section, we prove the following theorems.

**Theorem 3.1.** Let $A = (a_{ij})$ be a double nonnegative regular matrix. If

$$f(x) = \sum_{j,k \in \mathbb{Z}} C_{j,k}\psi_{j,k}(x)$$  \hfill (3.1)

is a wavelet expansion of $f \in L^2(\mathbb{R})$ with wavelet coefficients

$$C_{j,k} = \int_{-\infty}^{\infty} f(x)\psi_{j,k}(x)dx = \langle f, \psi_{j,k} \rangle,$$  \hfill (3.2)

then the frame condition for $A$-transform of $f \in L^2(\mathbb{R})$ is

$$c_1 \|f\|^2 \leq \sum_{i,l \in \mathbb{Z}} |\langle Af, \psi_{i,l} \rangle|^2 \leq c_2 \|f\|^2,$$  \hfill (3.3)

where $Af$ is the $A$-transform of $f$ and $0 < c_1 \leq c_2 < \infty$.

**Theorem 3.2.** If $C_{j,k}$ are the wavelet coefficients of $f \in L^2(\mathbb{R})$, that is, $C_{j,k} = \langle f, \psi_{j,k} \rangle$, then the $d_{l,m}$ are the wavelet coefficients of $Af$, where $\{d_{l,m}\}$ is defined as the $A$-transform of $\{C_{j,k}\}$ by

$$d_{l,m} = \sum_{j,k=\infty}^{\infty} a_{lmjk}C_{jk}.$$  \hfill (3.4)
Theorem 3.3. Let $A = (a_{lm,jk})$ be a double nonnegative matrix whose elements are $\langle \psi_{j,k}, \psi_{l,m} \rangle$. Then, $\{ \psi_{j,k} \}$ constitutes a frame of $L^2(\mathbb{R})$ if and only if $\{ \psi_{l,m} \}$ constitutes a frame of $L^2(\mathbb{R})$, where $C_{j,k} = \langle f, \psi_{j,k} \rangle$ and $d_{l,m} = \langle f, \psi_{l,m} \rangle$.

Proof of Theorem 3.1. We can write

$$f(x) = \sum_{j,k \in \mathbb{Z}} \langle f, \psi_{j,k} \rangle \psi_{j,k}.$$ (3.5)

If we take $A$-transform of $f$, we get

$$Af(x) = \sum_{i,l \in \mathbb{Z}} \langle Af, \psi_{i,l} \rangle \psi_{i,l},$$ (3.6)

and therefore

$$\sum_{i,l \in \mathbb{Z}} |\langle Af, \psi_{i,l} \rangle|^2 \leq \sum_{i,l \in \mathbb{Z}} \int_{-\infty}^{\infty} |Af(x)|^2 |\psi_{i,l}(x)|^2 dx \leq \|A\|^2 \|f\|_2^2 \sum_{i,l \in \mathbb{Z}} \|\psi_{i,l}\|_2^2.$$ (3.7)

Since $A$ is regular matrix and $\|\psi_{i,l}\|_2 = 1$, therefore

$$\sum_{i,l \in \mathbb{Z}} |\langle Af, \psi_{i,l} \rangle|^2 \leq c_2 \|f\|_2^2,$$ (3.8)

where $c_2$ is positive constant.

Now, for any arbitrarily $f \in L^2(\mathbb{R})$, define

$$\tilde{f} = \left[ \sum_{i,l \in \mathbb{Z}} |\langle Af, \psi_{i,l} \rangle|^2 \right]^{-1/2} f.$$ (3.9)

Clearly,

$$\langle A\tilde{f}, \psi_{i,l} \rangle = \left[ \sum_{i,l \in \mathbb{Z}} |\langle Af, \psi_{i,l} \rangle|^2 \right]^{-1/2} \langle Af, \psi_{i,l} \rangle,$$ (3.10)

then

$$\sum_{i,l \in \mathbb{Z}} |\langle Af, \psi_{i,l} \rangle|^2 \leq 1.$$ (3.11)
Hence, if there exists \( \alpha \) a positive constant, then
\[
\|A\hat{f}\|^2 \leq \alpha,
\]
\[
\left[ \sum_{i,l \in \mathbb{Z}} |\langle A f, \psi_{i,l} \rangle|^2 \right]^{-1} \|Af\|^2 \leq \alpha.
\]  

(3.12)

Since \( A \) is regular, we have
\[
\left[ \sum_{i,l \in \mathbb{Z}} |\langle A f, \psi_{i,l} \rangle|^2 \right]^{-1} \|f\|^2 \leq \alpha_1 \left( = \frac{\alpha}{\|A\|^2} \right),
\]

(3.13)

where \( \alpha_1 \) is another positive constant. Therefore,
\[
c_1 \|f\|^2 \leq \sum_{i,l \in \mathbb{Z}} |\langle A f, \psi_{i,l} \rangle|^2,
\]

(3.14)

where \( c_1 = \alpha > 0 \).

Combining (3.8) and (3.14), we have
\[
c_1 \|f\|^2 \leq \sum_{i,l \in \mathbb{Z}} |\langle A f, \psi_{i,l} \rangle|^2 \leq c_2 \|f\|^2.
\]

(3.15)

This completes the proof. 

\[\square\]

**Proof of Theorem 3.2.** We can write
\[
\langle Af, \psi_{j,k}\rangle = \int_{-\infty}^{\infty} Af(x) \psi_{l,m}(x)dx
\]
\[
= \int_{-\infty}^{\infty} \sum_{j,k=-\infty}^{\infty} a_{l,m} \psi_{j,k}(x) \psi_{l,m}(x)dx.
\]

(3.16)

Now,
\[
\sum_{l,m=-\infty}^{\infty} \langle Af, \psi_{l,m}\rangle \psi_{l,m} = \sum_{l,m=-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{j,k=-\infty}^{\infty} a_{l,m} \psi_{j,k}(x) \psi_{l,m}(x)dx
\]
\[
= \sum_{l,m=-\infty}^{\infty} d_{l,m} \psi_{l,m} \int_{-\infty}^{\infty} \|\psi_{l,m}(x)\|^2 dx
\]
\[
= \sum_{l,m=-\infty}^{\infty} d_{l,m} \psi_{l,m}.
\]

(3.17)

Therefore,
\[
\sum_{l,m=-\infty}^{\infty} d_{l,m} \psi_{l,m} = \sum_{l,m=-\infty}^{\infty} \langle Af, \psi_{l,m}\rangle \psi_{l,m}.
\]

(3.18)

This implies that \( d_{l,m} \) are wavelet coefficients of \( Af \).
Thus,

\[ d_{l,m} = \langle f, \psi_{l,m} \rangle. \]  \hspace{1cm} (3.19)

This completes the proof. \hfill \Box

**Proof of Theorem 3.3.** We observe that

\[
\begin{align*}
    a_{lmjk} C_{j,k} &= \langle \psi_{j,k}, \psi_{l,m} \rangle \langle f, \psi_{j,k} \rangle \\
    &= \int_{-\infty}^{\infty} \psi_{j,k}(x) \overline{\psi_{l,m}(x)} dx \int_{-\infty}^{\infty} f(x) \psi_{j,k}(x) dx \\
    &= \int_{-\infty}^{\infty} f(x) \overline{\psi_{l,m}(x)} dx \int_{-\infty}^{\infty} \psi_{j,k}(x) \psi_{j,k}(x) dx \\
    &= \int_{-\infty}^{\infty} f(x) \overline{\psi_{l,m}(x)} dx \\
    &= \langle f, \psi_{l,m} \rangle,
\end{align*}
\]

that is, \( a_{lmjk} C_{j,k} = d_{l,m} \).

Now,

\[
\begin{align*}
    \sum_{l,m} \left| d_{l,m} \right|^2 &= \sum_{l,m} \left| a_{lmjk} C_{j,k} \right|^2 = \sum_{l,m} \left| \langle f, \psi_{l,m} \rangle \right|^2 \\
    &= \frac{1}{(2\pi)^2} \sum_{l,m} \left| \langle \hat{f}, \hat{\psi}_{l,m} \rangle \right|^2, \hspace{1cm} (3.20)
\end{align*}
\]

by Parseval’s formula for trigonometric Fourier series.

Now

\[
\begin{align*}
    \frac{1}{(2\pi)^2} \sum_{l,m} \left| \int_0^{2\pi} \sum_{p=-\infty}^{\infty} \hat{f}(w+2\pi p) \overline{\hat{\psi}(w+2\pi p)} e^{ilmw} dw \right|^2 &= p \\
    &= \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{p=-\infty}^{\infty} \hat{f}(w+2\pi p) \overline{\hat{\psi}(w+2\pi p)} dw \right|^2, \hspace{1cm} (3.21)
\end{align*}
\]

by Parseval’s formula for trigonometric Fourier series.

Now

\[
\begin{align*}
    \left| \sum_{p=-\infty}^{\infty} \hat{f}(w+2\pi p) \overline{\hat{\psi}(w+2\pi p)} \right|^2 &= \left( \sum_{p=-\infty}^{\infty} \hat{f}(w+2\pi p) \overline{\hat{\psi}(w+2\pi p)} \right) \times \left( \sum_{q=-\infty}^{\infty} \hat{f}(w+2\pi q) \overline{\hat{\psi}(w+2\pi q)} \right). \hspace{1cm} (3.22)
\end{align*}
\]

Let \( f(w) = \sum_{q=-\infty}^{\infty} \hat{f}(w+2\pi q) \overline{\hat{\psi}(w+2\pi q)}. \)
Therefore,

\[ p = \frac{1}{2\pi} \int_{0}^{2\pi} \left| \sum_{p=-\infty}^{\infty} \hat{f}(w + 2\pi p) \hat{\psi}(w + 2\pi p) d\omega \right|^2 \]

\[ = \frac{1}{2\pi} \left( \int_{-\infty}^{\infty} \hat{f}(w) \hat{\psi}(w) F(w) d\omega \right) \]

\[ = \frac{1}{2\pi} \left( \sum_{q=-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(w) \hat{\psi}(w) \hat{f}(w + 2\pi q) \hat{\psi}(w + 2\pi q) d\omega \right) \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(w) \hat{\psi}(w) \hat{f}(w) \hat{\psi}(w) d\omega \]

\[ = \| f \|_2^2, \quad (3.23) \]

that is,

\[ \sum_{l,m} |d_{lm}|^2 = \| f \|_2^2, \quad f \in L^2(\mathbb{R}). \quad (3.24) \]

Therefore, for a regular matrix \( A = (a_{lmjk}) \), we have

\[ c_1 \| f \|_2^2 \leq \sum_{l,m} |d_{lm}|^2 \leq c_2 \| f \|_2^2 \quad (3.25) \]

if and only if

\[ c_1' \| f \|_2^2 \leq \sum_{j,k} |c_{jk}|^2 \leq c_2' \| f \|_2^2, \quad (3.26) \]

where, \( 0 \leq c_1', c_2' < \infty \). This completes the proof. \( \square \)

REFERENCES


N. A. Sheikh: Department of Mathematics, National Institute of Technology, Srinagar, Kashmir 190006, Jammu and Kashmir, India

*E-mail address*: neyaznit@yahoo.co.in

M. Mursaleen: Department of Mathematics, Aligarh Muslim University, Aligarh 202002, Uttar Pradesh, India

*Current address*: Department of Mathematics, Faculty of Science, P.O. Box 80203, King Abdul Aziz University, Jeddah, Kingdom of Saudi Arabia

*E-mail address*: mursaleen@postmark.net
Special Issue on

Human-Centric Applications of Distributed Camera Networks

Call for Papers

In a camera network, access to multiple sources of visual data often allows for making more comprehensive interpretation of events and activities. Vision-based sensing fits well within the notion of pervasive sensing and computing environments, enabling novel user-centric applications. In such applications, the actions of the users and their interactions with the environment are detected and interpreted by the network of cameras, and proper services or responses are offered based on the context.

Gesture recognition problems have been extensively studied in human computer interactions (HCIs), where often a set of predefined gestures is used for delivering instructions to machines. However, passive gestures predominate in behavior descriptions in many applications. Some traditional application examples include surveillance and security applications, while novel application classes arise in emergency detection in elderly care and assisted living, video conferencing, creating human models for gaming and virtual environments, and biomechanics applications analyzing human movements. Through pervasive visual sensing and collaborative processing, distributed camera networks offer the potential of a generalized HCI environment, in which the network reacts to various intentional gestures of the users stated, for example, via hand movements or gazing at a region of interest, as well as to unintentional posture changes caused by events such as accidental falls in assisted living applications.

Application development based on visual information obtained via multiple cameras requires new methodologies to efficiently fuse the data in the network. In a multi camera network, the option to employ local processing of acquired video at the source camera facilitates operation of scalable vision networks by avoiding transfer of raw images. Embedded processing utilizes the increasingly available computing power at the source to extract features from the images, which are exchanged with other cameras. Additional motivation for distributed processing stems from an effort to preserve privacy of the network users while offering services in applications such as assisted living. In a distributed processing framework, data fusion can occur across the three dimensions of 3D space (multiple views), time, and feature levels.

The goal of this special issue is to provide a coverage of the various approaches to human-centric application development in a multi camera setting. In particular, approaches based on the distributed processing of acquired video sequences, model-based approaches for human behavior monitoring, vision-based information fusion and collaborative decision making, and interfaces between the vision network and high-level reasoning modules that provide interpretative deductions will fit well within the scope of the special issue. The special issue also aims to provide insight into algorithm and system development topics pertaining to real-world application design for smart environments.

Original papers, previously unpublished and not currently under review by another journal, are solicited to cover one or more of the following topics:

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Hamid Aghajan, Stanford University, CA, USA; hamid@wsnl.stanford.edu
Richard Kleihorst, NXP Semiconductors Research, Eindhoven, The Netherlands; richard.kleihorst@nxp.com
Bernhard Rinner, University of Klagenfurt, Carinthia, Austria; bernhard.rinner@uni-klu.ac.at
Wayne Wolf, Georgia Institute of Technology, GA, USA; wolf@ece.gatech.edu
Special Issue on

Climate Change and Infectious Disease

Call for Papers

Virtually every atmospheric scientist agrees that climate change—most of it anthropogenic—is occurring rapidly. This includes, but is not limited to, global warming. Other variables include changes in rainfall, weather-related natural hazards, and humidity. The Intergovernmental Panel on Climate Change (IPCC) issued a major report earlier this year establishing, without a doubt, that global warming is occurring, and that it is due to human activities.

Beginning about two decades ago, scientists began studying (and speculating) how global warming might affect the distribution of infectious disease, with almost total emphasis on vector-borne diseases. Much of the speculation was based upon the prediction that if mean temperatures increase over time with greater distance from the equator, there would be a northward and southward movement of vectors, and therefore the prevalence of vector-borne diseases would increase in temperate zones. The reality has been more elusive, and predictive epidemiology has not yet allowed us to come to conclusive predictions that have been tested concerning the relationship between climate change and infectious disease. The impact of climate change on infectious disease is not limited to vector-borne disease, or to infections directly impacting human health. Climate change may affect patterns of disease among plants and animals, impacting the human food supply, or indirectly affecting human disease patterns as the host range for disease reservoirs change.

In this special issue, Interdisciplinary Perspectives on Infectious Diseases is soliciting cross-cutting, interdisciplinary articles that take new and broad perspectives ranging from what we might learn from previous climate changes on disease spread to integrating evolutionary and ecologic theory with epidemiologic evidence in order to identify key areas for study in order to predict the impact of ongoing climate change on the spread of infectious diseases. We especially encourage papers addressing broad questions like the following. How do the dynamics of the drivers of climate change affect downstream patterns of disease in human, other animals, and plants? Is climate change an evolutionary pressure for pathogens? Can climate change and infectious disease be integrated in a systems framework? What are the relationships between climate change at the macro level and microbes at the micro level?

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<table>
<thead>
<tr>
<th>Event</th>
<th>Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>Manuscript Due</td>
<td>January 1, 2008</td>
</tr>
<tr>
<td>First Round of Reviews</td>
<td>April 1, 2008</td>
</tr>
<tr>
<td>Publication Date</td>
<td>July 1, 2008</td>
</tr>
</tbody>
</table>

Guest Editors

Bettina Fries, Albert-Einstein College of Medicine, Yeshira University, NY 10461, USA; fries@aecom.yu.edu

Jonathan D. Mayer, Division of Allergy and Infectious Diseases, University of Washington, WA 98195, USA; jmayer@u.washington.edu