Hankel matrix transformation of the Walsh–Fourier series

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ABSTRACT

The Hankel matrix has various applications. In this paper we prove that Hankel matrix is strongly regular and apply to obtain the necessary and sufficient conditions to sum the Walsh–Fourier series of a function of bounded variation.

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1. Introduction and preliminaries

A Hankel matrix \( H = (h_{n+k}) \) consists of constant on each diagonal orthogonal to the main diagonal and its \((n, k)\)th entry is a function of \(n + k\). The Hankel transform of the sequence \( x = (x_k) \) is defined as the sequence \( y = (y_n) \), where \( y_n = \sum_{k=0}^{n} h_{n+k}x_k \) provided that the series converges for each \( n = 0, 1, 2, \ldots \). An operator \( T \) which transforms \( x \) into \( y \) as described is called the operator induced by the Hankel matrix \( H \). That is, the Hankel matrix is of the form

\[
H = \begin{pmatrix}
  h_1 & h_2 & h_3 & \cdots & h_j & \cdots \\
  h_2 & h_3 & h_4 & \cdots & h_{j+1} & \cdots \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\
  h_j & h_{j+1} & h_{j+2} & \cdots & h_{j-1} & \cdots \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \ddots 
\end{pmatrix}
\]

A matrix \( A = (a_{nk})^{n,k=1}_{\infty} \) is said to be regular if it transforms the convergent sequences into convergent sequences leaving the limit invariant. The following are well-known necessary and sufficient conditions for a matrix \( A \) to be regular which are known as the Silverman–Toeplitz conditions:

(i) \( \sup_n \sum_k |a_{nk}| < \infty \),
(ii) \( \lim_n a_{nk} = 0 \) for each \( k \), and
(iii) \( \lim_n \sum_k a_{nk} = 1 \).

In particular, Hankel matrix \( H = (h_{n+k}) \) is a regular matrix [4].
Hankel matrices have a number of applications in various fields (c.f. [4,5,12,16]). Basic properties of the Hankel matrix can be found in [8,9]. Applications of Hankel operators to approximation theory, prediction theory, and linear system theory can be found in [14]. Corresponding Hankel transform has also many applications, e.g. analysis of central potential scattering [15], solenoidal magnetic field [13], and medical computed tomography [7], summing the trigonometric sequences, periodic and almost periodic sequences [4]. Walsh functions are very useful in Electronics and Electrical Engineering and our method of summing a Walsh–Fourier series through the Hankel matrices may provide an useful tool to deal with the sequences of zeros and ones which are used in many Engineering problems.

These matrices have a number of applications in various fields. Recently, Al-Homidan [4] obtained the sum of the conjugate Fourier series under certain conditions on the entries of Hankel matrix. In this paper, we prove that the matrix \( H \) is strongly regular. Further we apply Hankel matrix to find the sum of the Walsh–Fourier series.

2. Strongly regular matrices

First we recall the notion of almost convergence.

**Definition 2.1.** A linear functional \( L \) on \( l_\infty \) is said to be a **Banach limit** if it has the following properties: (i) \( L(x) \geq 0 \) if \( x \geq 0 \). (ii) \( L(e) = 1 \), and (iii) \( L(Sx) = L(x) \); where \( S \) is a shift operator defined by \( (Sx)_n = x_{n+1} \).

A bounded sequence \( x = (x_k) \) is said to be **almost convergent** to the value \( \ell \) [11] if all its Banach limits coincide, i.e. \( L(x) = \ell \) for all Banach limits \( L \).

Lorentz [11] established the following characterization:

**Theorem A.** A bounded sequence \( x = (x_k) \) is almost convergent to the number \( \ell \) if and only if \( t_m(x) \to \ell \) as \( m \to \infty \) uniformly in \( n \), where

\[
t_m = t_m(x) = \frac{1}{m+1} \sum_{j=0}^{m} x_{n+j}.
\]

The number \( \ell \) is called the generalized limit of \( x \), and we write \( \ell = f = \lim x \). We denote the set of all almost convergent sequences by \( f \), i.e.

\[
f := \{x \in l_\infty : \lim_{m} t_m(x) = \ell, \text{ uniformly in } n\}.
\]

**Remark 2.1.** (i) Let \( c \subseteq f \) and for \( x \in c, L(x) = \lim x \). That is, every convergent sequence is almost convergent to the same limit but not conversely. For example, the sequence \( x = (x_k) \) defined by

\[
x_k = \begin{cases} 1, & \text{if } k \text{ is odd}, \\ 0, & \text{if } k \text{ is even}
\end{cases}
\]

is not convergent but it is almost convergent to 1/2.

(ii) A periodic sequence \( (x_n) \) for which numbers \( N \) and \( p \) (the period) exist such that \( x_{n+p} = x_n \) holds for \( n \geq N \) is almost convergent to the value \( L(x_n) = \frac{1}{p} (x_N + x_{N+1} + \cdots + x_{N+p-1}) \). For example the periodic sequence \( (1,0,0,1,0,0,1,\ldots) \) is almost convergent to 1/3.

(iii) We say that a sequence \( (x_n) \) is **almost periodic** if for every \( \epsilon > 0 \), there are two natural numbers \( N \) and \( r \), such that in every interval \( (k,k+r), \ k > 0 \), at least one "\( \epsilon \)-period" \( p \) exists. More precisely \( |x_{n+p} - x_n| < \epsilon \), for \( n \geq N \) must hold for this \( p \). Thus it is easy to see that every almost periodic sequence is almost convergent. But there are almost convergent sequences which are not almost periodic. For example, the sequence \( x = (x_k) \) defined by

\[
x_k = \begin{cases} 1, & \text{if } k = n^2, \\ 0, & \text{if } k \neq n^2; \ n \in \mathbb{N}
\end{cases}
\]

is almost convergent to 0) but not almost periodic.

Lorentz [11] defined and characterized the strongly regular matrix as follows:

**Definition 2.2.** A matrix \( A = (a_{nk}) \) is said to be **strongly regular** if it sums all almost convergent sequences and \( \lim Ax = f = \lim x \) for all \( x \in f \).

**Theorem B.** \( A \) is strongly regular if and only if

(i) \( A \) is regular, and

(ii) \( \lim_{n \to \infty} \sum_{k=0}^{n} |a_{nk} - a_{n,k+1}| = 0. \)
Of course, every strongly regular matrix is regular but not conversely.

King [10] used the notion of almost convergence to define and characterize the almost regular matrices. A matrix $A$ is said to be almost regular if it transforms the convergent sequences into almost convergent sequences and $f - \lim Ax = \lim x$. Of course, every regular matrix is almost regular but not conversely.

We observe that the Hankel matrix is strongly regular. First we note the following lemma which can be obtained directly by Lemma 2 of Das [3].

**Lemma 2.1.** Let $H = (h_{n,k})$ be a Hankel matrix (which is of course regular) such that $\lim_n |h_{n+k}| = 0$. Then there exists a bounded sequence $x$ such that $\|x\| \leq 1$ and

$$\lim \sup_n \sum_k h_{n,k}x_k = \lim \sup_n \sum_k |h_{n,k}|$$

(2.1)

Now we are ready to prove the following:

**Theorem 2.2.** The Hankel matrix $H = (h_{n,k})$ is strongly regular.

**Proof.** Since in [4] it was proved that the matrix $H = (h_{n,k})$ is regular, we have to prove the condition (ii) of Theorem B. To prove (ii), let us construct two increasing sequences of natural numbers $(m_k)$ and $(p_k)$ and define the sequence $x = (x_k)$ by

$$x_{2p_k+1+2l} = (-1)^l \text{sgn}(h_{2p_k+2l+m_k} - h_{2p_k+2l+1-m_k}),$$

$$x_{2p_k+2l+1} = -x_{2p_k+2l}, \quad l = 1, \ldots, p_k - p_{k-1} \text{ and } k \geq 1;$$

otherwise $x_k = 0$.

Then $x$ is bounded and almost convergent to 0; also $\|x\| \leq 1$. Since $H$ is regular and regularity implies almost regularity, we have $f - \lim Hx = \lim x$. Hence by Lemma 2.1, we have

$$0 = \lim \sup_n \sum_k (h_{n,k} - h_{n+k+1})x_k = \lim \sup_n \sum_k |h_{n,k} - h_{n+k+1}|$$

i.e. (ii) holds for $H = (h_{n,k})$. Hence $H$ is strongly regular, i.e. it sums every almost convergent sequence with $\lim Hx = f - \lim x$. If $f - \lim x = 0$, then we cannot say that $\lim x = 0$.

This completes the proof of the theorem. □

**Remark 2.2.** Since $H$ is strongly regular and every almost periodic sequence is almost convergent, we have the following result which is stronger than Theorem 4.5 of [4].

**Theorem 2.3.** Every Hankel matrix $H$ sums almost periodic sequences.

3. Main result

Let us define a sequence of functions $h_0(x), h_1(x), \ldots, h_0(x)$ which satisfy the following conditions:

$$h_0(x) = \begin{cases} 1, & \text{for } 0 \leq x \leq \frac{1}{2}; \\ -1, & \text{for } \frac{1}{2} \leq x < 1; \end{cases}$$

$h_0(x+1) = h_0(x)$ and $h_n(x) = h_0(2^n x)$. $n = 1, 2, \ldots$. The functions $h_n(x)$ are called the Rademacher’s functions.

The Walsh functions are defined by $\phi_0(x) = 1$,

$$\phi_n(x) = h_n(x)h_{n_1}(x)\ldots h_{n_t}(x), \quad 0 \leq x \leq 1,$$

for $n = 2^n + 2^n + \cdots 2^n$; where the integers $n_i$ are uniquely determined by $n_{i+1} < n_i$.

Let us recall some basic properties of Walsh functions (see [6,1]).

For each fixed $x \in [0, 1)$ and for all $t \in [0, 1)$.

(i) $\phi_n(x+t) = \phi_n(x)\phi_n(t)$,

(ii) $\int_0^1 f(x+t)dt = \int_0^1 f(t)dt$, and

(iii) $\int_0^1 f(t)\phi_n(x+t)dt = \int_0^1 f(x+t)\phi_n(t)dt$;

where $+$ denotes the operation in the dyadic group, the set of all sequences $s = (s_n)$, $s_n = 0, 1$ for $n = 1, 2, \ldots$, is addition modulo 2 in each coordinate.
Let for \( x \in [0, 1) \),
\[
J_k(x) = \int_0^x \phi_k(t) dt, \quad k = 0, 1, 2, \ldots
\]
It is easy to see that \( J_k(x) = 0 \) for \( x = 0, 1 \).

Let \( f \) be \( L \)-integrable and periodic with period 1, and let the Walsh–Fourier series of \( f \) be
\[
\sum_{n=1}^{\infty} c_n \phi_n(x),
\]
where
\[
c_n = \int_0^1 f(x) \phi_n(x) dx
\]
are called the Walsh–Fourier coefficients of \( f \).

In [4], author obtained the sum of the conjugate Fourier series under certain conditions on the entries of Hankel matrix. Our object here is to find necessary and sufficient conditions for Hankel matrix to sum the Walsh–Fourier series.

We shall need the following lemma which is known as the Banach Weak Convergence Theorem [2].

**Lemma A.** \( \lim_{n \to \infty} \int_0^1 g_n h_k \, dx = 0 \) for all \( h_k \in BV[0, 1] \), if and only if \( \| g_n \| < \infty \) for all \( n \) and \( \lim_{n \to \infty} g_n = 0 \); where \( BV[0, 1] \) denotes the set of all functions of bounded variations on \([0, 1]\).

**Theorem 3.1.** Let \( H = (h_{n+k}) \) be a Hankel matrix. Let \( z_k(x) = c_k \phi_k(x) \) for an \( L \)-integrable function \( f \in BV[0, 1] \). Then for every \( x \in [0, 1) \)
\[
\lim_{n \to \infty} \sum_{k=1}^{\infty} h_{n+k} z_k(x) = 0
\]
if and only if
\[
\lim_{n \to \infty} \sum_{k=1}^{\infty} h_{n+k} J_k(x) = 0,
\]
where \( x \) is a point at which \( f(x) \) is of bounded variation.

**Proof.** We have
\[
z_k(x) = c_k \phi_k(x) = \int_0^1 f(t) \phi_k(t) \phi_k(x) dt = \int_0^1 f(t) \phi_k(x+t) dt = \int_0^1 (x+t) \phi_k(t) dt,
\]
where \( x+t \) belongs to the set \( \Omega \) of dyadic rationals in \([0, 1)\), in particular each element of \( \Omega \) has the form \( p/2^n \) for some non-negative integers \( p \) and \( n \), \( 0 \leq p < 2^n \). Now, on integration by parts, we obtain
\[
z_k(x) = \left[ f(x+t) J_k(t) \right]_0^1 - \int_0^1 J_k(t) df(x+t) = -\int_0^1 J_k(t) df(x+t), \quad \text{since } J_k(x) = 0 \text{ for } x = 0, 1.
\]

Hence
\[
\sum_{k=1}^{\infty} h_{n+k} z_k(x) = -\int_0^1 D_n(t) dq_n(t), \quad (3.1)
\]
where
\[
D_n(t) = \sum_{k=1}^{\infty} h_{n+k} J_k(t) \quad (3.2)
\]
and \( q_n(t) = f(x+t) \). Write, for any \( t \in \mathbb{R} \), \( g_n = (D_n(t)) \).

By the regularity of \( H \), it follows that \( \| g_n \| < \infty \) for all \( n \) and \( g_n \to 0 \). Hence by Lemma A, \( \int_0^1 D_n(t) \, dq_n(t) \to 0 \).

Now, taking the limit as \( n \to \infty \) in (3.1) and (3.2), we get the desired result.

This completes the proof of the theorem. \( \square \)

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References