Intuitionistic fuzzy 2-metric space and its completion

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ABSTRACT

Recently, Mursaleen and Lohani [Mursaleen M, Lohani Danish. Intuitionistic fuzzy 2-normed space and some related concepts. Chaos, Solitons & Fractals (2008), doi:10.1016/j.chaos.2008.11.006] have introduced the concept of intuitionistic fuzzy 2-normed space. In this paper, we introduce the concept of intuitionistic fuzzy 2-metric space and study its completion.

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1. Introduction and preliminaries

The theory of fuzzy sets [29] has become an area of active research for the last forty years. It has a wide range of applications in the field of science and engineering, e.g. population dynamics [5], chaos control [17], computer programming [20], nonlinear dynamical systems [22], medicine [4] etc. Fuzzy topology is one of the most important and useful tool studied by various authors, e.g. [14–16,19,21,23,28]. The most fascinating application of the fuzzy topology in quantum physics arises in $e^{\psi}$-theory due to El Naschie [7–13] who presented the relation of fuzzy Kähler interpolation of $e^{\psi}$ to the recent work on cosmo-topology and the Poincaré dodecahedral conjecture and gave various applications and results of $e^{\psi}$-theory from nano technology to brain research.

Recently, the concepts of intuitionistic fuzzy sets, intuitionistic fuzzy metric spaces, intuitionistic fuzzy topological spaces and intuitionistic fuzzy normed spaces have been introduced and studied in [2,3,6,25,26], respectively. Quite recently, the concept of intuitionistic fuzzy 2-normed spaces has been introduced by Mursaleen and Lohani Danish [24]. Certainly, there are some situations where the ordinary metric does not work and the concept of intuitionistic fuzzy metric seems to be more suitable in such cases.

Our aim for this paper is to introduce the concept of intuitionistic fuzzy 2-metric space and study its completion which may be a useful functional tool in the development of fuzzy topology. For completion of fuzzy metric spaces and intuitionistic fuzzy metric spaces, we refer to [15,21,1], respectively.

We recall some notations and basic definitions used in this paper.

Definition 1.1 [27]. A binary operation $*: [0, 1] \times [0, 1] \rightarrow [0, 1]$ is said to be a continuous t-norm if it satisfies the following conditions:

(a) $*$ is associative and commutative,
(b) $*$ is continuous,
(c) $a * 1 = a$ for all $a \in [0, 1]$,
(d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0, 1]$. 
A binary operation $\diamond : [0, 1] \times [0, 1] \to [0, 1]$ is said to be a continuous $t$-conorm if it satisfies the conditions (a), (b), (d) as above and $a \diamond 0 = a$ for all $a \in [0, 1]$.

**Definition 1.2** [24]. The five-tuple $(X, \mu, \nu, \ast, \circ)$ is said to be an intuitionistic fuzzy 2-normed space (for short, IF-2-NS) if $X$ is any vector space, $\ast$ is a continuous $t$-norm, $\circ$ is a continuous $t$-conorm, and $\mu$ and $\nu$ are fuzzy sets on $X \times X \times (0, \infty)$ satisfying the following conditions. For every $x, y, z \in X$, and $s, t > 0$

(a) $\mu(x, y; t) + \nu(x, y; t) \leq 1$,
(b) $\mu(x, y; t) > 0$,
(c) $\mu(x, y; t) = 1$ if and only if $x$ and $y$ are linearly dependent,
(d) $\mu(2x, y; t) = \mu(x, y; \frac{t}{2})$ for each $x \neq 0$,
(e) $\mu(x, z; t) \ast \mu(y, z; s) \leq \mu(x + y, z; t + s)$,
(f) $\mu(x, y; \cdot) : (0, \infty) \to [0, 1]$ is continuous,
(g) $\lim_{s \to 0} \mu(x, y; t) = 1$ and $\lim_{s \to 0} \nu(x, y; t) = 0$,
(h) $\mu(x, y; t) = \mu(y, x; t)$

In this case $(\mu, \nu)$ is called an intuitionistic fuzzy 2-norm on $X$, and we denote it by $(\mu, \nu)\circ$.

**Definition 1.3** [18]. Let $X$ be a nonempty set. A real valued function $d$ on $X \times X \times X$ is said to be a 2-metric on $X$ if

(a) given distinct elements $x, y, z$ of $X$, there exists an element $s$ of $X$ such that $d(x, y, s) = 0$;
(b) $d(x, z) = 0$ when at least two of $x, y, z$ are equal;
(c) $d(x, y) = d(y, x)$ for all $x, y, z$ in $X$;
(d) $d(x, y, z) \leq d(x, y, w) + d(w, z, x)$ for all $x, y, z, w$ in $X$.

The pair $(X, d)$ is then called a 2-metric space.

As an example of a 2-metric space, take $X = \mathbb{R}^3$ being equipped with the 2-metric $d(x, y, z)$: the area of the triangle spanned by $x$, $y$, and $z$, which may be given explicitly by the formula,

$$d(x, y, z) = |x_1(y_2z_3 - z_2y_3) - x_2(y_1z_3 - z_1y_3) + x_3(y_1z_2 - z_1y_2)|,$$

where

$$x = (x_1, x_2, x_3), \quad y = (y_1, y_2, y_3), \quad z = (z_1, z_2, z_3).$$

2. Intuitionistic fuzzy 2-metric space

In this section, we introduce the concept of intuitionistic fuzzy 2-metric space and define some more related concepts.

**Definition 2.1.** The 5-tuple $(X, M, N, \ast, \circ)$ is said to be an intuitionistic fuzzy 2-metric space (for short, IF-2-MS) if $X$ is any nonempty set, $\ast$ is a continuous $t$-norm, $\circ$ is a continuous $t$-conorm, and $M, N$ fuzzy sets on $X \times X \times (0, \infty)$, satisfying the following conditions. For each $x, y, z, w \in X$ and $s, t > 0$,

(a) $M(x, y, z; t) + N(x, y, z; t) \leq 1$,
(b) given distinct elements $x, y$ of $X$, there exists an element $z$ of $X$ such that $M(x, y, z; t) > 0$,
(c) $M(x, y, z; t) = 1$ if at least two of $x, y, z$ are equal,
(d) $M(x, z, y; t) = M(x, z, y; t)$ for all $x, y, z$ in $X$,
(e) $M(x, y, w; t) \ast M(x, w, z; s) \ast M(w, y, z; r) \leq M(x, y, z; t + s + r)$ for all $x, y, z, w \in X$,
(f) $M(x, y, z; \cdot) : (0, \infty) \to [0, 1]$ is continuous,
(g) $M(x, y, z; t) \leq 1$,
(h) $N(x, y, z; t) = 0$ if at least two of $x, y, z$ are equal,
(i) $N(x, y, z; t) = N(x, y, z; t)$ for all $x, y, z$ in $X$,
(j) $N(x, y, w; t) \circ N(x, w, z; s) \circ N(w, y, z; r) \geq N(x, y, z; t + s + r)$,
(k) $N(x, y, z; \cdot) : (0, \infty) \to [0, 1]$ is continuous,
In this case $(M, N)$ is called an intuitionistic fuzzy 2-metric space. Let $(M, N)$ be an intuitionistic fuzzy 2-metric space on X and denote it by $(M, N)$. The functions $M(x, y, z; t)$ and $N(x, y, z; t)$ denote the degree of nearness and the degree of non nearness between $x, y$ and $z$ with respect to $t$, respectively.

**Remark 2.1.** In an intuitionistic fuzzy 2-metric space $(X, M, N, *, \circ)$, $M(x, y, z; :)$ is non-decreasing and $N(x, y, z; :)$ is non-increasing for all $x, y, z \in X$.

Let $(X, d)$ be a metric space. Denote $a * b = ab$ and $\alpha b = \min\{1, a + b\}$ for all $a, b \in [0, 1]$ and let $M_d$ and $N_d$ be fuzzy sets on $X^3 \times (0, \infty)$ defined by:

$$M_d(x, y, z; t) = \frac{h t^0}{h t^0 + md(x, y, z)}, \quad N_d(x, y, z; t) = \frac{d(x, y, z)}{kt^0 + md(x, y, z)}$$

for all $h, k, m, n \in \mathbb{R}^+$. Then $(X, M_d, N_d, *, \circ)$ is an intuitionistic fuzzy 2-metric space.

**Definition 2.2.** Let $(X, M, N, *, \circ)$ be an intuitionistic fuzzy 2-metric space, and let $r \in (0, 1)$, $t > 0$ and $x \in X$. The set $B(x, r, t) = \{y \in X : M(x, y, z; t) > 1 - r, N(x, y, z; t) < r, \forall z \in X\}$ is called the open ball with center $x$ and radius $r$ with respect to $t$.

**Definition 2.3.** Let $(X, M, N, *, \circ)$ be an intuitionistic fuzzy 2-metric space, then a set $U \subseteq X$ is said to be an open set if each of its points is the center of some open ball contained in $U$. The open set in an intuitionistic fuzzy 2-metric space $(X, M, N, *, \circ)$ will be denoted by $U$.

**Definition 2.4.** Let $(X, M, N, *, \circ)$ be an intuitionistic fuzzy 2-metric space. A sequence $(x_n)$ in $X$ is said to be Cauchy if for each $\epsilon > 0$ and each $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, z; t) > 1 - \epsilon$ and $N(x_n, x_m, z; t) < \epsilon$ for all $n, m > n_0$ and for all $z \in X$.

**Definition 2.5.** Let $(X, M, N, *, \circ)$ be an intuitionistic fuzzy 2-metric space. A sequence $x = (x_n)$ is said to be convergent to $L \in X$, with respect to the intuitionistic fuzzy 2-metric $(M, N)$, if, for every $\epsilon > 0$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $M(x_n, L; t) > 1 - \epsilon$ and $N(x_n, L; t) < \epsilon$ for all $k \geq k_0$ and for all $z \in X$. In this case we write $(M, N) \rightarrow \lim x = L$ or $x_k \rightarrow_L (k \rightarrow \infty)$.

**Definition 2.6.** Let $(X, M, N, *, \circ)$ be an intuitionistic fuzzy 2-metric space. Define $\tau_{(M, N)} = \{A \subseteq X : \text{for each } x \in A, \text{there exists } t > 0 \text{ and } r \in (0, 1) \text{ such that } B(x, r, r) \subseteq A\}$. Then $\tau_{(M, N)}$ is a topology on $(X, M, N, *, \circ)$.

**Definition 2.7.** An intuitionistic fuzzy 2-metric space $(X, M, N, *, \circ)$ is said to be complete if every Cauchy sequence in it is convergent with respect to $\tau_{(M, N)}$.

**Definition 2.8.** Let $X$ be any nonempty set and $(Y, M, N, *, \circ)$ be an intuitionistic fuzzy 2-metric space. Then a sequence $(f_n)$ of functions from $X$ to $Y$ is said to converge uniformly to a function $f$ from $X$ to $Y$ if given $t > 0$ and $r \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that $M(f_n(x), f(x), z; t) > 1 - r$ and $N(f_n(x), f(x), z; t) < r$ for all $n \geq n_0$ and for all $x \in X$ and $z \in Y$.

**Definition 2.9.** Let $(X, M, N, *, \circ)$ and $(Y, M', N', *, \circ')$ be two intuitionistic fuzzy 2-metric spaces. A mapping $f : X \rightarrow Y$ is called an isometry if for each $x, y, z \in X$ and $t > 0$, $M(x, y, z; t) = M'(f(x), f(y), f(z); t)$ and $N(x, y, z; t) = N'(f(x), f(y), f(z); t)$.

**Definition 2.10.** Let $(X, M, N, *, \circ)$ and $(Y, M', N', *, \circ')$ be two intuitionistic fuzzy 2-metric spaces. Then $X$ and $Y$ are called isometric if there is an isometry between $X$ and $Y$.

**Definition 2.11.** Let $(X, M, N, *, \circ)$ be an intuitionistic fuzzy 2-metric space. Then the completion of $(X, M, N, *, \circ)$ is a complete intuitionistic fuzzy 2-metric space $(\overline{X}, M', N', *, \circ')$ such that $(X, M, N, *, \circ)$ is isometric to a dense subspace of $\overline{X}$.

**Definition 2.12.** Let $(X, M, N, *, \circ)$ and $(Y, M', N', *, \circ')$ be two intuitionistic fuzzy 2-metric spaces. A mapping $f : X \rightarrow Y$ is uniformly continuous if for each $\epsilon \in (0, 1)$ and each $t > 0$, there exist $\delta \in (0, 1)$ and $s > 0$ such that $M'(f(x), f(y), f(z); t) > 1 - \epsilon$ and $N'(f(x), f(y), f(z); t) < \epsilon$ whenever $M(x, y, z; s) > 1 - \delta$ and $N(x, y, z; s) < \delta$ for all $z \in X$.

**Definition 2.13.** Let $(X, M, N, *, \circ)$ and $(Y, M', N', *, \circ')$ be two intuitionistic fuzzy 2-metric spaces. Then, $X$ and $Y$ are called uniformly isomorphic if there is a bijection $f : X \rightarrow Y$ such that both $f$ and $f^{-1}$ are uniformly continuous in the sense of **Definition 2.12**. In this case, we say that $f$ is a uniform isomorphism between $(X, M, N, *, \circ)$ and $(Y, M', N', *, \circ')$.

**Example 2.1.** Let $(X, d)$ be a metric space and let $f$ be an isometry from $(X, d)$ onto a dense subspace $(\tilde{X}, \tilde{d})$. The standard intuitionistic fuzzy 2-metric $(M, N)_t$ of $d$ is given by

$$M_d(\tilde{x}, \tilde{y}, \tilde{z}; t) = \frac{t}{t + d(\tilde{x}, \tilde{y}, \tilde{z})}, \quad N_d(\tilde{x}, \tilde{y}, \tilde{z}; t) = \frac{d(\tilde{x}, \tilde{y}, \tilde{z})}{t + d(\tilde{x}, \tilde{y}, \tilde{z})}$$
for all \( x, y, z \in X \) and \( t > 0 \) where \( a \ast b = \min\{a, b\} \) and \( a \circ b = \max\{a, b\} \) for all \( a, b \in [0, 1] \). Hence, we have \( M_d(x, y, z; t) = M_d(f(x), f(y), f(z); t) \) and \( N_d(x, y, z; t) = N_d(f(x), f(y), f(z); t) \) for all \( x, y, z \in X \) and \( t > 0 \).

Here, we display an example to show that there exists an intuitionistic fuzzy 2-metric space that does not admit any intuitionistic fuzzy 2-metric completion in the sense of Definition 2.11.

**Example 2.2.** Let \( a \ast b = \max\{0, a + b - 1\} \) and \( a \circ b = \min\{1, a + b\} \) for all \( a, b \in [0, 1] \). Now let \( (x_n)_{n=3}^{\infty} \) and \( (y_n)_{n=3}^{\infty} \) be two sequences of distinct units such that \( A \cap B = \emptyset \), where \( A = \{x_n : n \geq 3\} \) and \( B = \{y_n : n > 3\} \). Put \( X = A \cup B \). Define two real valued functions \( M \) and \( N \) on \( X^2 \times [0, \infty) \) as follows:

\[
M(x_n, x_m, x_l; t) = M(y_n, y_m, y_l; t) = \max\left\{1 - \left[\frac{1}{n \wedge m} - \frac{1}{n \vee m}\right], 1 - \left[\frac{1}{m \wedge l} - \frac{1}{m \vee l}\right], 1 - \left[\frac{1}{n \wedge l} - \frac{1}{n \vee l}\right]\right\}
\]

and

\[
N(x_n, x_m, x_l; t) = N(y_n, y_m, y_l; t) = \min\left\{\frac{1}{n \wedge m} - \frac{1}{n \vee m}, \frac{1}{m \wedge l} - \frac{1}{m \vee l}, \frac{1}{n \wedge l} - \frac{1}{n \vee l}\right\}
\]

and

\[
M(x_n, y_m, x_l; t) = M(y_n, x_m, y_l; t) = M(x_n, y_m, x_l; t) = M(y_n, y_m, y_l; t)
\]

and

\[
N(x_n, y_m, x_l; t) = N(y_n, x_m, y_l; t) = N(x_n, y_m, x_l; t) = N(y_n, y_m, y_l; t)
\]

for all \( l, m \geq 3 \). Then \((X, M, N, \ast, \circ)\) is an intuitionistic fuzzy 2-metric space.

Now, we show that \( (x_n)_{n=3}^{\infty} \) is a Cauchy sequence in the intuitionistic fuzzy 2-metric space \((X, M, N, \ast, \circ)\). Fix \( \varepsilon \in (0, 1) \) and \( t > 0 \). Therefore, there exists \( n_0 \geq 3 \) such that \( \left|\frac{1}{n} - \frac{1}{m}\right| < \varepsilon \) for all \( n, m \geq n_0 \). Suppose, without loss of generality, that \( m > n \). Then

\[
M(x_n, x_m, x_l; t) = \max\left\{1 - \left[\frac{1}{n \wedge m} - \frac{1}{n \vee m}\right], 1 - \left[\frac{1}{m \wedge l} - \frac{1}{m \vee l}\right], 1 - \left[\frac{1}{n \wedge l} - \frac{1}{n \vee l}\right]\right\} \geq 1 - \frac{1}{n \wedge m} < \varepsilon,
\]

for \( n, m \geq n_0 \) and for all \( l \geq 3 \).

\[
N(x_n, x_m, x_l; t) = \min\left\{\frac{1}{n \wedge m} - \frac{1}{n \vee m}, \frac{1}{m \wedge l} - \frac{1}{m \vee l}, \frac{1}{n \wedge l} - \frac{1}{n \vee l}\right\} \leq \frac{1}{n \wedge m} - \frac{1}{n \vee m} = \frac{1}{n} \leq \varepsilon,
\]

for \( n, m \geq n_0 \) and for all \( l \geq 3 \). Hence \( (x_n)_{n=3}^{\infty} \) is a Cauchy sequence in the intuitionistic fuzzy 2-metric space \((X, M, N, \ast, \circ)\).

However, \((x_n)_{n=3}^{\infty} \) and \((y_n)_{n=3}^{\infty} \) do not converge in \( X \) with respect to the topology \( \tau_{(M,N)} \) generated by \((M,N)\). In fact \( \tau_{(M,N)} \) is the discrete topology on \( X \) because for each \( n \geq 3 \) and each \( t > 0 \), we have

\[
\left\{x_n : \frac{1}{n(n+1)} < t\right\} = \{x_n\} \quad \text{and} \quad \left\{y_n : \frac{1}{n(n+1)} < t\right\} = \{y_n\}
\]

In order to prove the preceding equalities it suffices to observe that for each \( n \geq 3 \) we obtain \( m \geq 3 \) such that for all \( l \geq 3 \) and \( t > 0 \), we have

\[
M(x_n, x_m, x_l; t) > 1 - \frac{1}{n(n+1)} \quad \text{and} \quad N(x_n, x_m, x_l; t) < \frac{1}{n(n+1)} \quad \text{for all} \ x_l \in A,
\]

where

\[
1 - \left[\frac{1}{n \wedge m} - \frac{1}{n \vee m}\right] \leq 1 - \frac{1}{n(n+1)} = 1 - \frac{1}{n(n+1)}.
\]

Therefore

\[
M(x_n, x_m, x_l; t) = \max\left\{1 - \left[\frac{1}{n \wedge m} - \frac{1}{n \vee m}\right], 1 - \left[\frac{1}{m \wedge l} - \frac{1}{m \vee l}\right], 1 - \left[\frac{1}{n \wedge l} - \frac{1}{n \vee l}\right]\right\}
\]

\[
= 1 - \min\left\{\frac{1}{n \wedge m} - \frac{1}{n \vee m}, \frac{1}{m \wedge l} - \frac{1}{m \vee l}, \frac{1}{n \wedge l} - \frac{1}{n \vee l}\right\}.
\]

Now for \( l = m \) or \( l = n \), we get \( M(x_n, x_m, x_l; t) = 1 \). Hence \( M(x_n, x_m, x_l; t) = 1 \) for all \( x_l \in A \) if and only if \( x_n = x_m \) that is when \( n = m \). Hence \( M(x_n, x_m, x_l; t) > 1 - \frac{1}{n(n+1)} \) for each \( n \geq 3 \), \( m = n \) and for all \( l \geq 3 \).
and
\[ N(x_n, x_m; x_i; t) = \min \left\{ \frac{1}{n \land m}, \frac{1}{n \lor m}, \frac{1}{m \land l}, \frac{1}{m \lor l}, \frac{1}{n \land l}, \frac{1}{n \lor l} \right\}. \]

Now for \( l = m \) or \( l = n \), we get \( N(x_n, x_m; x_i; t) = 0 \). Hence \( N(x_n, x_m; x_i; t) = 0 \) for all \( x_i \in A \) if and only if \( x_n = x_m \) that is when \( n = m \). Hence \( N(x_n, x_m; x_i; t) < \frac{1}{n \lor m} \) for each \( n \geq 3, n = m \) and for all \( l \geq 3 \).

And, similarly, \( M(y_n, y_m; y_i; t) > 1 - \frac{1}{n \lor m} \) for each \( n \geq 3, n = m \) and for all \( l \geq 3 \) and \( N(y_n, y_m; y_i; t) < \frac{1}{n \lor m} \) for each \( n \geq 3, n = m \) and for all \( l \geq 3 \).

Now for \( n, m \geq 3, l \geq 3 \) and \( t > 0 \), we have
\[ M(y_n, y_m; x_i; t) = 1 - \left( 1 - \frac{1}{n + m} \right) \leq 1 - \left( 1 - \frac{1}{n + 1} \right) \]
and
\[ N(y_n, y_m; x_i; t) = \frac{1}{n + m} \geq \frac{1}{n + 1}. \]

Similarly,
\[ M(x_n, x_m; y_i; t) = 1 - \left( 1 - \frac{1}{n + m} \right) \leq 1 - \left( 1 - \frac{1}{n + 1} \right) \]
and
\[ N(x_n, x_m; y_i; t) = \frac{1}{n + m} \geq \frac{1}{n + 1}. \]

Therefore, \((X, M, N, *, \circ)\) is not complete.

Suppose that \((X, M, N, * , \circ)\) admits an intuitionistic fuzzy 2-metric completion \((\bar{X}, \bar{M}, \bar{N}, \cdot, \circ)\). Then, there exists an isometry \( i \) from \( X \) onto a dense subspace of \( \bar{X} \). Since \((i(x_n))_{n=3}^{\infty}\) and \((i(y_n))_{n=3}^{\infty}\) are Cauchy sequences in \((\bar{X}, \bar{M}, \bar{N}, \cdot, \circ)\), there exist \( \bar{x}, \bar{y} \in \bar{X} \) such that for each \( t > 0 \),
\[ \bar{M}(\bar{x}, i(x_n); \bar{z}, t) \to 1 \quad \text{and} \quad \bar{N}(\bar{x}, i(x_n); \bar{z}, t) \to 0 \]
\[ \bar{M}(\bar{y}, i(y_n); \bar{z}, t) \to 1 \quad \text{and} \quad \bar{N}(\bar{y}, i(y_n); \bar{z}, t) \to 0, \]
as \( n \to \infty \) and for all \( \bar{z} \in \bar{X} \).

Fix \( t > 0 \) and put \( \bar{M}(\bar{x}, i(x_n); \bar{z}, t) = r \) and \( \bar{N}(\bar{x}, i(x_n); \bar{z}, t) = 1 - r \), with \( 0 < r < 1 \). For each \( k \geq 3 \) there exists \( n_k \geq k \) such that
\[ \bar{M}(\bar{x}, i(x_{n_k}); \bar{z}, t) > 1 - \frac{1}{k}, \quad \bar{N}(\bar{x}, i(x_{n_k}); \bar{z}, t) < \frac{1}{k} \quad \text{and} \quad \bar{M}(\bar{y}, i(y_{n_k}); \bar{z}, t) > 1 - \frac{1}{k}, \quad \bar{N}(\bar{y}, i(y_{n_k}); \bar{z}, t) < \frac{1}{k} \quad \text{for all} \ \bar{z} \in \bar{X}. \]
Since \( \bar{M}(\bar{x}, i(x_{n_k}); i(y_{n_k}); \bar{z}, 3t) = 1 - \frac{1}{n_k} \) and \( \bar{N}(\bar{i}, i(x_{n_k}); i(y_{n_k}); \bar{z}, 3t) = \frac{1}{n_k} \), it follows from the relation
\[ \bar{M}(i(x_{n_k}), i(y_{n_k}); \bar{z}, 9t) \geq \bar{M}(\bar{x}, i(x_{n_k}); \bar{z}, 3t) \ast \bar{M}(\bar{y}, i(y_{n_k}); \bar{z}, 3t) \ast \bar{M}(\bar{i}, i(x_{n_k}), \bar{z}, 3t) \ast \bar{M}(\bar{i}, i(y_{n_k}), \bar{z}, 3t) \ast \bar{M}(\bar{i}, i(x_{n_k}), i(y_{n_k}), \bar{z}, 3t). \]

Now
\[ \bar{M}(\bar{x}, i(y_{n_k}); \bar{z}, 3t) \geq \bar{M}(\bar{y}, i(y_{n_k}); \bar{z}, t) \ast \bar{M}(\bar{x}, i(y_{n_k}); i(y_{n_k}); \bar{y}, t) \ast \bar{M}(\bar{i}, i(y_{n_k}), \bar{y}, t) \ast \bar{M}(\bar{i}, i(x_{n_k}); \bar{y}, t) \ast \bar{M}(\bar{i}, i(x_{n_k}), i(y_{n_k}), \bar{y}, t) \ast \bar{M}(\bar{i}, i(x_{n_k}), i(y_{n_k}), \bar{z}, 3t) \ast \bar{M}(\bar{i}, i(x_{n_k}), i(y_{n_k}), \bar{z}, 3t). \]
\[
\left( 1 - \frac{1}{K} \right) \ast r \ast \left( 1 - \frac{1}{K} \right) \ast \left( 1 - \frac{1}{K} \right) \ast \left( 1 - \frac{2}{n_k} \right) \leq \left( 1 - \frac{2}{n_k} \right)
\]
and
\[ \frac{1}{K} \circ (1 - r) \circ \frac{1}{K} \circ \frac{1}{n_k} \geq \frac{2}{n_k} \]
for each \( k \geq 3 \). On the other hand, since \( * \) is a continuous \( t \)-norm and \( \circ \) is a continuous \( t \)-conorm,
\[ \left( 1 - \frac{1}{K} \right) \ast r \ast \left( 1 - \frac{1}{K} \right) \ast \left( 1 - \frac{1}{K} \right) \ast \left( 1 - \frac{2}{n_k} \right) \to r \quad \text{and} \quad \frac{1}{K} \circ (1 - r) \circ \frac{1}{K} \circ \frac{1}{n_k} \to 1 - r \]
as \( k \to \infty \). So, by (3), we deduce that \( r = 0 \), a contradiction.

Hence \((X, M, N, *, \circ)\) has no intuitionistic fuzzy 2-metric completion.

### 3. Main Results

The following lemma is trivial.
**Lemma 3.1.** Let \((X, M, N, +, \cdot, 0)\) and \((Y, M', N', +', \cdot', 0')\) be two intuitionistic fuzzy 2-metric spaces. A mapping \(f : X \rightarrow Y\) is uniformly continuous if and only if \(f\) is uniformly continuous as a mapping from the uniform space \((Y, U_{(M,N)})\) to the uniform space \((X, U_{(M',N')})\).

Now we prove the following:

**Lemma 3.2.** Let \((X, M, N, +, \cdot, 0)\) be an intuitionistic fuzzy 2-metric space and \((Y, M', N', +', \cdot', 0')\) a complete intuitionistic fuzzy 2-metric space. If there is a uniformly continuous mapping \(f\) from a dense subspace \((X, M, N, +, \cdot, 0)\) to \((Y, M', N', +', \cdot', 0')\), then \(f\) has a unique uniformly continuous extension \(F : (X, M, N, +, \cdot, 0) \rightarrow (Y, M', N', +', \cdot', 0')\). In particular, \(F\) is an isometry whenever \(f\) so.

**Proof.** By **Lemma 3.1**, \(f\) is uniformly continuous mapping from the uniform space \((X, U_{(M,N)})\) to the uniform space \((Y, U_{(M',N')})\). Since \(A\) is dense in \(X\), it follows from classical results that \(f\) has a unique uniformly continuous extension \(f^* : (X, U_{(M,N)}) \rightarrow (Y, U_{(M',N')})\). Thus \(f^*\) is uniformly continuous mapping from the intuitionistic fuzzy 2-metric space \((X, M, N, +, \cdot, 0)\) to the intuitionistic fuzzy 2-metric space \((Y, M', N', +', \cdot', 0')\), by **Lemma 3.1**.

Now suppose that \(f\) is an isometry from \((X, M, N, +, \cdot, 0)\) to \((Y, M', N', +', \cdot', 0')\). Let \(x, y \in X\) and \(t > 0\). Then, there exist two sequences \((x_n)_{n \in \mathbb{N}}\) and \((y_n)_{n \in \mathbb{N}}\) in \(A\) such that \(x_n \rightarrow x\) and \(y_n \rightarrow y\) with respect to \(\tau_{(M,N)}\). Choose an arbitrary \(\epsilon \in (0, 1)\). By continuity of \(M(x,y,z)\) and \(N(x,y,z)\), there is \(k \in \mathbb{N}\) such that

\[
\epsilon + M(x,y,z; t) > M(x,y,z; t + \frac{8}{k}) \quad \text{and} \quad N(x,y,z; t) - \epsilon < N(x,y,z; t + \frac{8}{k}) \quad \text{for all} \quad z \in X.
\]

Furthermore, it follows from continuity of \(f^*\), that \(f^*(x_n) \rightarrow f^*(x)\) and \(f^*(y_n) \rightarrow f^*(y)\) with respect to \(\tau_{(M,N)}\). So by [19, Theorem 3.11], there is \(n_0\), such that

\[
M(x_n, x_0; z; \frac{1}{k}) > 1 - \epsilon \quad \text{and} \quad N(x_n, x_0; z; \frac{1}{k}) < \epsilon \quad \text{for all} \quad z \in X,
\]

\[
M(y_n, y_0; z; \frac{1}{k}) > 1 - \epsilon \quad \text{and} \quad N(y_n, y_0; z; \frac{1}{k}) < \epsilon \quad \text{for all} \quad z \in X,
\]

\[
M(f^*(x_n), f^*(x_0); z; \frac{1}{k}) > 1 - \epsilon \quad \text{and} \quad N(f^*(x_n), f^*(x_0); z; \frac{1}{k}) < \epsilon \quad \text{for all} \quad z \in X,
\]

\[
M(f^*(y_n), f^*(y_0); z; \frac{1}{k}) > 1 - \epsilon \quad \text{and} \quad N(f^*(y_n), f^*(y_0); z; \frac{1}{k}) < \epsilon \quad \text{for all} \quad z \in X,
\]

for all \(n \geq n_0\). Thus we have

\[
\epsilon + M(x,y,z; t) > M(x,y,z; t + \frac{8}{k}) \geq M(x_n,y_0; z; \frac{1}{k}) * M(y_0, x_0; t + \frac{6}{k}) * M(x_n, x_0; z; \frac{1}{k}) * M(y_n, y_0; t + \frac{4}{k} + \frac{6}{k}) \geq (1 - \epsilon) * (1 - \epsilon) * (1 - \epsilon),
\]

and

\[
N(x,y,z; t) - \epsilon < N(x,y,z; t + \frac{8}{k}) \leq N(x_n,y_0; z; \frac{1}{k}) * N(y_0, x_0; t + \frac{6}{k}) * N(x_n, x_0; z; \frac{1}{k}) * N(y_n, y_0; t + \frac{4}{k} + \frac{6}{k}) * N(x_n, y_n; z; \frac{1}{k}) \leq \epsilon \leq \epsilon \quad \text{for all} \quad z \in X.
\]

\[
M(f^*(x_n), f^*(y_0); z; t + \frac{2}{k}) \geq M(f^*(x_n), f^*(x_0); z; t + \frac{2}{k}) \cdot M(f^*(x_0), f^*(y_0); z; \frac{1}{k}) \cdot M(f^*(y_n), f^*(y_0); z; t + \frac{2}{k}) \cdot (1 - \epsilon)^3 (1 - \epsilon) \cdot (1 - \epsilon),
\]

\[
M(f^*(x_n), f^*(y_0); z; t + \frac{2}{k}) \geq M(f^*(y_n), f^*(y_0); z; t + \frac{2}{k}) \cdot M(f^*(x_n), f^*(y_0); z; t + \frac{2}{k}) \cdot (1 - \epsilon)^3 (1 - \epsilon).
\]
Hence
\[
M'(f(x_0), f'(y_0), f'(z_0); t + \frac{4}{\kappa}) \geq (1 - \epsilon)M(f(x), f'(y), f'(z); t) + (1 - \epsilon)\epsilon'(1 - \epsilon)\epsilon',
\]
\[
N'(f(x_0), f'(y_0), f'(z_0); t + \frac{2}{\kappa}) \leq N(f'(x_0), f'(y_0), f'(z_0); t + \frac{2}{\kappa}) + N(f'(x_0), f'(y_0), f'(z_0); t + \frac{2}{\kappa} + N(f'(x_0), f'(y_0), f'(z_0); t + \frac{2}{\kappa}).
\]
\[
\leq N'(f'(x), f'(y_0), f'(z); t + \frac{2}{\kappa}) + N'(f'(x), f'(y_0), f'(z); t + \frac{2}{\kappa}) = N'(f'(x), f'(y_0), f'(z); t + \frac{2}{\kappa})
\]
\[
\leq \epsilon\epsilon' N'(f'(x), f'(y_0), f'(z); t + \frac{2}{\kappa}) = \epsilon\epsilon' N(f'(x), f'(y_0), f'(z); t + \frac{2}{\kappa})
\]
\[
\leq \epsilon\epsilon' N(f'(x), f'(y_0), f'(z); t + \frac{2}{\kappa})
\]

for all \( n \gg n_0 \).

Therefore,
\[
\epsilon + M(x, y, z; t) > [(1 - \epsilon)M(f(x), f'(y), f'(z); t) + (1 - \epsilon)\epsilon'(1 - \epsilon)\epsilon'] > (1 - \epsilon) \cdot (1 - \epsilon)
\]
and
\[
N(x, y, z; t) - \epsilon < [\epsilon\epsilon' N'(f'(x), f'(y), f'(z); t) + \epsilon\epsilon' \epsilon'] < \epsilon\epsilon' \epsilon'.
\]

By the continuity of * and \( \epsilon' \), it follows that \( M(x, y, z; t) \geq N(f(x), f'(y), f'(z); t) \) and by continuity of \( \epsilon' \) and \( \epsilon \), it follows that \( N(x, y, z; t) \leq N(f(x), f'(y), f'(z); t) \). Similar arguments show that \( M(x, y, z; t) \leq M(f(x), f'(y), f'(z); t) \) and \( N(x, y, z; t) \geq N'(f(x), f'(y), f'(z); t) \) for all \( x, y, z \in X \) and \( t > 0 \). We conclude that \( f^* \) is an isometry from \( \langle X, M, N, *, \epsilon \rangle \) to \( \langle Y, M^*, N^*, \epsilon^* \rangle \).

**Theorem 3.1.** Suppose that \( \langle Y_1, M_1', N_1', \epsilon_1', \epsilon_1' \rangle \) and \( \langle Y_2, M_2', N_2', \epsilon_2', \epsilon_2' \rangle \) are two intuitionistic fuzzy 2-metric completions of \( \langle X, M, N, *, \epsilon \rangle \). Then \( Y_1 \) and \( Y_2 \) are isometric. Thus, if an intuitionistic fuzzy 2-metric space has an intuitionistic fuzzy 2-metric completion, it is unique up to isometry.

**Proof.** Since \( \langle Y_2, M_2', N_2', \epsilon_2', \epsilon_2' \rangle \) is an intuitionistic fuzzy 2-metric completion of \( \langle X, M, N, *, \epsilon \rangle \) there is isometry \( f \) from \( \langle X, M, N, *, \epsilon \rangle \) onto a dense subspace of \( \langle Y_2, M_2', N_2', \epsilon_2', \epsilon_2' \rangle \). By Lemma 3.2, \( f \) admits a (unique) extension \( f^* \) to \( \langle Y_1, M_1', N_1', \epsilon_1', \epsilon_1' \rangle \) which is also an isometry. So, it remains to see that \( f^* \) is onto. But this fact follows from standard arguments. Indeed, given \( y_2 \in Y_2 \), there is a sequence \( (y_n)_{n \in \mathbb{N}} \) in \( X \) such that \( f(y_n) \rightarrow y_2 \). Since \( f^* \) is an isometry, \( (y_n)_{n \in \mathbb{N}} \) is a Cauchy sequence, so it converges to some point \( y_1 \in Y_1 \). Consequently \( f^*(y_1) = y_2 \).

**Theorem 3.2.** Let \( (X, d) \) be a 2-metric space. Then, standard intuitionistic fuzzy 2-metric space \( (X, M_d, N_d, *, \epsilon) \) admits an (up the isometry) unique intuitionistic fuzzy 2-metric completion, which is exactly the standard intuitionistic fuzzy 2-metric space of the completion of \( (X, d) \).

**Proof.** Let \( (X, M_d, N_d, *, \epsilon) \) be the standard intuitionistic fuzzy 2-metric space of the completion \( (\tilde{X}, \tilde{d}) \) of \( (X, d) \). Then \( (X, M_d, N_d, *, \epsilon) \) is a complete intuitionistic fuzzy 2-metric space and it is the unique intuitionistic fuzzy 2-metric completion of \( (X, M_d, N_d, *, \epsilon) \) (up to isometry). Indeed, since there is an isometry \( f \) from \( (X, d) \) onto a dense subspace of \( (X, d) \) and the topologies generated by \( d \) and \( (M_d, N_d) \) coincide, \( f(X) \) is dense in \( (X, M_d, N_d, *, \epsilon) \). Furthermore, \( M_d(f(x), f(x), f(z), t) = M_d(x, y, t) \) and \( N_d(f(x), f(y), f(z), t) = N_d(x, y, t) \) for all \( x, y, z \in X \) and \( t > 0 \) (see Example). So \( (X, M_d, N_d, *, \epsilon) \) is an intuitionistic fuzzy 2-metric completion of \( (X, M_d, N_d, *, \epsilon) \), and, by Theorem 3.1, it is unique up to isometry.

**Remark 3.1.** If \( (X, d) \) is a 2-metric space and
\[
M_{d_{\text{kmn}}}(x, y, z; t) = \frac{kd}{t^2} + m_{d_{\text{kmn}}}(x, y, z) \quad \text{and} \quad N_{d_{\text{kmn}}}(x, y, z; t) = \frac{md}{t^2} + m_{d_{\text{kmn}}}(x, y, z),
\]
with \( k, m, n > 0 \), then \( (X, M_{d_{\text{kmn}}}, N_{d_{\text{kmn}}}, *, \epsilon) \) is an intuitionistic fuzzy 2-metric space where \( a * b = ab \) and \( a \circ b = \min\{1, a + b\} \) for all \( a, b \in [0, 1] \). In particular, we have the standard intuitionistic fuzzy 2-metric space when \( k = m = n = 1 \). Note that the proof of Theorem 3.2 also shows that \( (X, M_{d_{\text{kmn}}}, N_{d_{\text{kmn}}}, *, \epsilon) \) is (up to isometry) the unique intuitionistic fuzzy 2-metric completion of \( (X, M_{d_{\text{kmn}}}, N_{d_{\text{kmn}}}, *, \epsilon) \).

At the end of the paper, we show that it is still possible to obtain a general solution to the problem of intuitionistic fuzzy 2-metric completion by using uniform isomorphism instead of isometries. The next result is an immediate consequence of Lemma 3.1 and Definition 2.1.
Lemma 3.3. Let \((X, M, N, *, \odot)\) and \((Y, M', N', *, \odot')\) be two intuitionistic fuzzy 2-metric spaces. Then, \(X\) and \(Y\) are uniformly isomorphic if and only if the uniform spaces \((X, \tilde{U}_{M,N})\) and \((Y, \tilde{U}_{M',N'})\) are uniformly isomorphic.

If \((X, d)\) is a 2-metric space, \(\tilde{U}_d\) will denote the uniformity induced by \(d\) on \(X\).

Lemma 3.4. For any 2-metric space \((X, d)\), the uniformities \(\tilde{U}_d\) and \(\tilde{U}_{M,N,\odot}\) coincide on \(X\).

Proof. It is routine to check that for each \(x, y, z \in X\) and \(n \in \mathbb{N}\), we have \(d(x, y, z) < \frac{1}{n}\) for all \(z \in X\) if and only if \(M_d(x, y, z, \frac{1}{n}) > 1 - \frac{1}{n}\) and \(N_d(x, y, z, \frac{1}{n}) < \frac{1}{n}\) for all \(z \in X\). Consequently, \(\tilde{U}_d = \tilde{U}_{M,N,\odot}\) on \(X\).

Theorem 3.3. Let \((X, M, N, *, \odot)\) be an intuitionistic fuzzy 2-metric space. Then, there is an (up to uniform isomorphism) unique complete intuitionistic fuzzy 2-metric space \((Y, M', N', *, \odot')\) such that \((X, M, N, *, \odot)\) is uniformly isomorphic to a dense subspace of \(Y\). Furthermore, \((Y, M', N', *, \odot')\) is a standard intuitionistic fuzzy 2-metric space and the uniform space \((Y, \tilde{U}_{M',N'})\) is uniformly isomorphic to the completion of \((X, \tilde{U}_{M,N})\).

Proof. Let \(d\) be a 2-metric \(X\) such that the uniformity \(\tilde{U}_d\) induced by \(d\) coincides with the uniformity \(\tilde{U}_{M,N,\odot}\) induced by \((M, N, \odot)\) (such a metric if one exists). Consider the completion \((X, d)\) of \((X, d)\). Then, the standard intuitionistic fuzzy 2-metric space \((X, \tilde{M}_d, \tilde{N}_d, *, \odot)\) is complete. Since, by Lemma 3.4, \(\tilde{U}_d = \tilde{U}_{M,N,\odot}\) on \(X\), it follows that \((X, \tilde{U}_{M,N,\odot})\) is uniformly isomorphic to the completion of \((X, \tilde{U}_{M,N,\odot})\). So \((X, M, N, *, \odot)\) is uniformly isomorphic to a dense subspace of \((X, \tilde{M}_d, \tilde{N}_d, *, \odot)\) by Lemma 3.3. Uniqueness of \((X, \tilde{M}_d, \tilde{N}_d, *, \odot)\) (up to uniform isomorphism) of the completion of any uniform space.

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